The Bar Construction

Let k be a commutative ring and let A be a k-algebra. Let X be a right module over A and let Y be a left module over A. Then we can construct a simplicial k-module $\{B_n(X, A, Y)\}_{n=0}^{\infty}$ whose nth level is $X \otimes A^{\otimes n} \otimes Y$. The degeneracy maps insert a new copy of A and set that coordinate to the identity element of A. The face maps multiply two adjacent things - either two copies of A, or X with A, or A with Y. We can turn this simplicial k-module into a chain complex $B(X, A, Y)$ of k-modules in the usual way, by taking the alternating sum of the face maps. This yields the *bar complex*, the usual chain complex for computing $\mathrm{Tor}_A(X, Y)$.

Notice that this construction works in any monoidal category C, where A is a monoid in that category, X is a right module over A, Y is a left module over A, and the final result $\{B_n(X, A, Y)\}_{n=0}^{\infty}$ is a simplicial object of C. If C has a reasonable notion of geometric realization, then we can form an object $B(X, A, Y)$; this is the generalized *bar construction*.

Let's consider the case where C is the category of topological spaces. Let G be a topological monoid, and choose $*$ and $*$ as our right and left G-modules. Then the above construction yields

$$B_n(*, G, *) = * \times G^n \times * = G^n$$

$$BG = \coprod_{n=0}^{\infty} G^n \times \Delta^n / \left\{ \begin{array}{c} (g_0, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_{n+1}, t_0, \ldots, t_{n+1}) \\ = (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n, t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1}) \\ (g_0, g_1, \ldots, g_n, 0, t_0, \ldots, t_{n-1}) = (g_1, \ldots, g_n, t_0, \ldots, t_{n-1}) \\ (g_0, \ldots, g_n, t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \\ = (g_0, \ldots, g_i g_{i+1}, \ldots, g_n, t_0, \ldots, t_{n-1}) \\ (g_0, \ldots, g_{n-1}, g_n, t_0, \ldots, t_{n-1}, 0) = (g_0, \ldots, g_{n-1}, t_0, \ldots, t_{n-1}) \end{array} \right\}$$

The first equation comes from degeneracies, and the last three come from faces. Under the assumption that G is discrete, we can give this a more geometric description. There is one n-simplex for each n-tuple of elements of G. If the n-tuple contains an identity element $1 \in G$, then the n-simplex collapses onto a simplex of lower dimension. So we can think of one n-simplex for each n-tuple of non-identity elements of G. Under this description, the 0th face of the $(n+1)$-simplex corresponding to $(g_0, g_1, \ldots, g_n)$ is the n-simplex corresponding to $(g_1, \ldots, g_n)$. The ith face is the n-simplex corresponding to $(g_0, \ldots, g_i g_{i+1}, \ldots, g_n)$, which if $g_i g_{i+1} = 1$ is further collapsed to the $(n-1)$-simplex $(g_0, \ldots, g_{i-1}, g_{i+2}, \ldots, g_n)$ by mapping the ith and $i+1$st vertices of the n-simplex to the ith vertex of the $(n-1)$-simplex. Finally, the $(n+1)$st face of $(g_0, g_1, \ldots, g_n)$ is the n-simplex $(g_0, \ldots, g_{n-1})$.

Let $G \longrightarrow H$ be a map of monoids. Then this clearly gives a map $G^n \times \Delta^n \longrightarrow H^n \times \Delta^n$. It agrees with the face and degeneracy identifications because it preserves multiplications and the identity element; therefore we get a map $BG \longrightarrow BH$. The identity map $G \longrightarrow G$ yields the identity $BG \longrightarrow BG$, and a composition of maps $G \longrightarrow H \longrightarrow K$ yields the composition $BG \longrightarrow BH \longrightarrow BK$, which can easily be checked by seeing that it works for the simplices themselves before we quotient anything down. So B gives a functor from topological monoids to topological spaces. More generally, each map of triples (X, G, Y) that preserves all the multiplications induces a map between their bar complexes.

More properties of BG:

- BG always has a canonical basepoint $G^0 \times \Delta^0$.

- If G is grouplike and has nondegenerate basepoint, there is a natural weak homotopy equivalence $G \longrightarrow \Omega BG$, given by the formula

  $$(g, t) \mapsto (g, t, 1 - t) \in G \times \Delta^1$$

  So we call BG a "delooping" of G.

- B is a strong monoidal functor. In other words, $B(G \times H)$ is naturally homeomorphic to $BG \times BH$. This follows from Milnor's Theorem $|X \times Y| \cong |X| \times |Y|$. The homeomorphism is given on the simplicial spaces by

  $$(G \times H)^n \times \Delta^n \longrightarrow (G^n \times \Delta^n) \times (H^n \times \Delta^n)$$

  by the projections onto each factor.

- If G is an abelian topological group, then multiplication $G \times G \longrightarrow G$ and inversion $G \longrightarrow G$ are homomorphisms. By the above, this implies that there is a multiplication map $BG \times BG \cong B(G \times G) \longrightarrow BG$ and an inversion map $BG \longrightarrow BG$ turning BG into a topological group. Therefore we can take $B^2G = B(BG)$. In fact, we can drop the assumption that G has inverses. If G is just a commutative topological monoid, then we still have the multiplication map $G \times G \longrightarrow G$ and it turns BG into a topological monoid.

- If G is commutative, then BG is commutative as well. (Just check that the reverse of the above map on the diagonal in $\Delta^n \times \Delta^n$ is commutative.) So we can take $B^nG = B(B(\ldots(B(G))\ldots))$ for any nonnegative integer n. This turns any topological commutative group or monoid G into a topological commutative group or monoid B^nG. This generalizes from commutative monoids to E^n spaces; we take B^n of an E^n space using a different construction found in loopspace theory.

- If H acts on X on the left, and this commutes with G acting on the right, then H acts on $B(X, G, Y)$ on the left. Same for the right-hand side.

- Define $EG = B(*, G, G)$; then when G is a group, it acts freely on EG on the right, EG is contractible, and $EG/G \cong BG$. So $EG \longrightarrow BG$ is a universal principal G-bundle. That is, if X is homotopy equivalent to a paracompact space, then there is a natural bijection between $[X, BG]$ and isomorphism classes of principal G-bundles over X. (Recall that a principal G-bundle is a locally trivial fibration with a fiberwise right G-action giving homeomorphisms between G and each fiber; it could also be described as a fiber bundle with fiber G and structure group G acting on the left.)

- We can generalize the last bullet point to G any grouplike monoid using [2]. In this case, $EG \longrightarrow BG$ is only a quasifibration with a right G-action; we can apply the functor Γ to replace it by an equivalent "$G\mathcal{U}$-fibration." Then over any space X homotopy equivalent to a CW complex, $[X, BG]$ is in natural bijection with equivalence classes of "principal G-fibrations." This last notion refers to maps $E \longrightarrow X$ that are quasifibrations with a fiberwise right G-action giving weak equivalences $G \longrightarrow E_x$ by $g \mapsto yg$ for any point $y \in E$ over $x \in X$. Equivalences are generated by the equivariant fiberwise maps. We can strengthen from quasifibrations to Serre fibrations, Hurewicz fibrations, or "$G\mathcal{U}$ fibrations" and get the same result. If G has the homotopy type of a CW complex, we can also restrict to spaces such that the maps $G \longrightarrow E_x$ are strong homotopy equivalences.

- If X is a space, then $X \cong B(X, *, *) \cong B(*, *, X)$. If X and Y are spaces, then $X \times Y \cong B(X, *, Y)$. If X has a right G-action and Y has a left G-action, then $B(X, G, Y)$ is a homotopy-theoretic version of $X \times_G Y$.

- For every inclusion of groups $H \hookrightarrow G$ (not necessarily normal) we can form quotient spaces of left cosets G/H and right cosets $H\backslash G$. Then $G/H \cong B(G, H, *)$ and $H\backslash G \cong B(*, H, G)$.

- If $H \longrightarrow G$ is any homomorphism, not necessarily an injection, than we can use the bar complexes $B(G, H, *)$ and $B(*, H, G)$ as the definition of the generalized homotopy quotients G/H and $H\backslash G$. Then the two rows of this diagram are equivalent fibration sequences:

$$\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "\sim"] & G \arrow[r] \arrow[d, "\sim"] & G/H \arrow[r] \arrow[d] & BH \arrow[r, "Bf"] \arrow[d, equal] & BG \arrow[d, equal] \\
\Omega BH \arrow[r] & \Omega BG \arrow[r] & F(Bf) \arrow[r] & BH \arrow[r, "Bf"] & BG
\end{tikzcd}$$

- We can carry out a two-sided bar construction $B(X, G, Y)$ anytime we're in a context where we have associative multiplications between copies of G, X, and Y. For example, if G is a monad (a functor **Top** $\longrightarrow$ **Top** that behaves like a monoid) on spaces, Y is a space that is a left G-module, and X is a functor that is a right G-module, we can define $B_n(X, G, Y)$ in a similar way and get a simplicial space. Taking G to be the little n-cubes operad, $X = \Omega^n$, and Y an algebra over the little n-cubes, this construction yields an n-fold delooping of y.

- We can also generalize the reduced bar construction $B(*, G, *)$ from topological monoids G to topological categories. Instead of points of G, we consider morphisms in such a category. The nth space is defined as above, though we must require that each n-tuple of arrows is composable, i.e. the target of each arrow is the source of the next. In the case of a one-object category, this gives exactly the same construction as above.

References

[1] J.F. Adams, *Infinite Loop Spaces.*

[2] J.P. May, *Classifying Spaces and Fibrations*.

[3] J.P. May, *The Geometry of Iterated Loop Spaces*.