SEMISTABILITY, THE BÖKSTEDT SMASH PRODUCT, AND CLASSICAL FIBRANT REPLACEMENT FOR DIAGRAM SPECTRA

CARY MALKIEWICH

In these notes we discuss the most classical method for replacing a prespectrum by an Ω-spectrum, and how it interacts with the more modern framework of diagram spectra. We also give a brief summary of the concept of semistability. As a corollary, we arrive at a modern understanding of Bökstedt's original model for the smash product of spectra.

The short story is that one has to be careful, to avoid running afoul of certain impossibility theorems. The additional structure of symmetric and orthogonal spectra does not interact as nicely as one would hope with the process of taking a colimit of loopspaces of the levels.

1. Semistability

I'm going to assume the definitions of prespectra, symmetric spectra, and orthogonal spectra. A π_*-isomorphism is a map of any such spectra that induces an isomorphism on the naïve homotopy groups $\pi_k(X) = \operatorname*{colim}_n \pi_{k+n}(X_n)$. For the purposes of this section, when I say "homotopy category" I mean that we invert the π_*-isomorphisms. It is well-known that for orthogonal spectra and prespectra this gives the usual stable homotopy category, while for symmetric spectra it does not. (At least, the forgetful functor from symmetric spectra to prespectra does not induce an equivalence of homotopy categories.) (see Schwede's book project)

Let's try to explain why that goes wrong, in an easy-to-remember way, without getting bogged down in technical details. Here is the one-paragraph explanation: there is a shift functor $\mathrm{sh}(-)$ which commutes with all the forgetful functors from orthogonal to symmetric to prespectra, and is always a right adjoint. In orthogonal spectra and prespectra, it has two additional properties: it (1) is an equivalence on the homotopy category, and (2) is equivalent to the suspension functor. In suspension spectra with π_*-isomorphisms, both (1) and (2) fail to hold: it is not an equivalence and therefore not equivalent to the suspension functor. This is what prevents the homotopy categories from being equivalent. And, in fact, it is the only issue. Indeed, we fix it by making every spectrum equivalent to a semistable one, and as soon as we do this, the above properties hold and the homotopy category of symmetric spectra becomes equivalent to the others.

Now for a few more details. Recall that there are two ways to "shift a prespectrum up:" you take the reduced suspension ΣX, or the shift $\operatorname{sh} X$. The prespectrum $\operatorname{sh} X$ is defined by $(\operatorname{sh} X)_n := X_{n+1}$ with the obvious structure maps coming from those of X. On the homotopy category of prespectra, these two functors are isomorphic. The proof is quite messy; on the point-set level there is an obvious map $\Sigma X_n \to (\operatorname{sh} X)_n$, but they don't fit into a map of spectra because they don't commute with the structure maps. If you flip the Σ coordinate for all the odd values of n, then it commutes up to homotopy, and there is a "cylinder construction" that lets you rectify this into a zig-zag of equivalences of prespectra. To see that the map on homotopy groups is an isomorphism we take the map of colimit systems and define the inverse by the dotted maps as shown; the minus signs indicate that we take the negative of the obvious map, in order to make all the triangles commute (after suspension), and their pattern repeats every two rectangles:

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_{k+2n}(\Sigma X_{2n}) \arrow[r] \arrow[d] & \pi_{k+1+2n}(\Sigma X_{1+2n}) \arrow[r] \arrow[d, "(-)"] & \pi_{k+2+2n}(\Sigma X_{2+2n}) \arrow[r] \arrow[d] & \ldots \arrow[r] & \pi_k(\Sigma X) \arrow[d, shift right] \\
\ldots \arrow[r] & \pi_{k+2n}(X_{1+2n}) \arrow[r] \arrow[ur, dashed, "(-)"'] & \pi_{k+1+2n}(X_{2+2n}) \arrow[r] \arrow[ur, dashed] & \pi_{k+2+2n}(\Sigma X_{3+2n}) \arrow[r] & \ldots \arrow[r] & \pi_k(\operatorname{sh} X) \arrow[u, dashed, shift right]
\end{tikzcd}$$

Therefore suspension and shift are naturally isomorphic in the homotopy category. Once we know this, the inverses ΩX and $\operatorname{sh}^{-1} X$ are also equivalent on the homotopy category. The latter functor is defined by $(\operatorname{sh}^{-1} X)_0 = *$ and $(\operatorname{sh}^{-1} X)_n := X_{n-1}$ for $n \geq 1$; it is easy to check that it is a left adjoint to $\operatorname{sh}(-)$ on the point-set level, and an equivalence on the homotopy category.

Now when we get to symmetric spectra, $\Sigma(-)$ and $\Omega(-)$ are still inverse equivalences that shift the homotopy groups up and down. And $\operatorname{sh}(-)$ still makes sense, even though $\operatorname{sh}^{-1}(-)$ does not. But something changes: there is no longer a natural isomorphism between $\Sigma(-)$ and $\operatorname{sh}(-)$ in the homotopy category. There are two ways we might try to make such an equivalence. First there is the cylinder trick above, but that fails because it doesn't give a map of symmetric spectra. Second, we can map $\Sigma X_n \to (\operatorname{sh} X)_{n+1}$ using the structure map of X, and then can compose with a permutation in Σ_n that shuffles the first letter past the other letters all the way to the right-hand side:

$$\Sigma X_n \longrightarrow X_{1+n} \xrightarrow{\tau_{1,n}} X_{n+1}$$

This permutation $\tau_{1,n}$ is just what we need so that the result is a map of symmetric spectra! And, this map is an equivalence if the permutation acts on the homotopy groups through the sign representation. But if it doesn't... the argument we did in the diagram above no longer works, and we cannot construct an inverse to show that our map is an isomorphism on π_*. All you get is that the map is an injection, that is in general not an isomorphism.

More generally, there is no natural zig-zag of equivalences of symmetric spectra between ΣX and $\operatorname{sh} X$. If there were, then $\operatorname{sh}(-)$ would be an equivalence on the homotopy category. We know however that $\operatorname{sh} X \cong F(F_1 S^0, X)$ is a right adjoint in symmetric spectra, and its left adjoint is $F_1 S^0 \wedge -$. And if we take $\operatorname{sh}(F_1 S^0 \wedge \mathbb{S})$, we get a spectrum whose homotopy groups are an infinite sum of copies of $\pi_*(\mathbb{S})$. In particular they are not isomorphic to a single copy of $\pi_*(\mathbb{S})$, so $\mathrm{sh}(-)$ cannot be an equivalence.

In other words, there is a natural equivalence $\Sigma X \simeq \mathrm{sh}\, X$ in the homotopy category of prespectra, but there is no natural lift of this to the homotopy category of symmetric spectra (with the π_*-isomorphisms).

This should sound crazy. After all, their homotopy groups are always abstractly isomorphic, how could they not be equivalent? Unfortunately, they just aren't. We do not have enough maps between them that agree with the symmetric spectrum structure. The combinatorial structure of symmetric spectra is somehow creating an obstruction to making these two functors agree in the homotopy category.

At any rate, we can conclude that the forgetful functor from symmetric spectra to prespectra fails to give an equivalence of homotopy categories. If it did, the lift of $\mathrm{sh}(-)$ to the homotopy category of symmetric spectra would also be a self-equivalence of categories.

And of course not all is lost, we just have to change the category somehow until it becomes the right one again. The most obvious way to fix this is to restrict to symmetric spectra with the property that the natural map $\Sigma X \to \mathrm{sh}\, X$ is a π_*-isomorphism. These are the *semistable* symmetric spectra. And while it is not totally obvious that this will work, it does indeed work; the homotopy category of semistable spectra and π_*-isomorphisms is equivalent to the homotopy category of prespectra.

The less obvious way to fix this is to define a broader notion of weak equivalence of symmetric spectra, such that $\Sigma X \simeq \mathrm{sh}\, X$ is always one of these broader equivalences. You can find more details in Hovey-Shipley-Smith and Mandell-May-Schwede-Shipley.

Finally, we recall that orthogonal spectra are all semistable. So when X ranges over orthogonal spectra, the maps $\Sigma X \to \mathrm{sh}\, X$ are all equivalences of orthogonal spectra. The proof of this uses in an essential way the growing connectivity of the quotient space $O(n)/O(n-1)$ as $n \to \infty$, which of course does not happen for the quotient sets Σ_n/Σ_{n-1}.

2. Fibrant replacment by taking a colimit of loops

Now instead of considering the behavior of shifts, we will consider the process of taking "fibrant replacements" of spectra X by taking a colimit of loops of the levels of X. Although the subject matter is different, this section has the same feel as the previous one. We make a construction of an Ω-spectrum replacement of X, that works at the level of prespectra but does not lift to symmetric spectra. In symmetric spectra a different construction arises, but then it does not always give an Ω-spectrum.

We say that a prespectrum X is a *strong* Ω*-spectrum* or just Ω*-spectrum* if the adjoint structure maps $X_n \to \Omega X_{1+n}$ are homeomorphisms. We say that it is a *weak* Ω*-spectrum* if the adjoint structure maps are weak homotopy equivalences. For prespectra, symmetric spectra, and orthogonal spectra, it is by now common practice to replace X by a weak Ω-spectrum, by using the small-object argument to build a fibrant replacement in a model structure where the fibrant objects are the weak Ω-spectra. Here, we would like to instead consider a much more classical construction.

If X is a prespectrum, and the structure maps are all closed inclusions, then for each n, let

$$(RX)_n = \operatorname*{colim}_r \Omega^r X_{r+n}$$

where the colimit is taken along the adjoints of the structure maps. The homotopy groups of $(RX)_n$ capture homotopy groups of X in a range:

$$\pi_k((RX)_n) \cong \pi_{k-n}(X), \quad k \geq 0$$

Furthermore, the spaces $(RX)_n$ assemble together into a prespectrum $R'X$, by the following "diagonal" structure maps:

$$\begin{tikzcd}
X_n \arrow[r] & \underline{\Omega} X_{1+n} \arrow[r] \arrow[dl, leftrightarrow] & \Omega\underline{\Omega} X_{2+n} \arrow[r] \arrow[dl, leftrightarrow] & \Omega^2\underline{\Omega} X_{3+n} \arrow[r] \arrow[dl, leftrightarrow] & \dots \arrow[r] & RX_n \arrow[d, leftrightarrow] \\
\underline{\Omega}(X_{1+n}) \arrow[r] & \underline{\Omega}(\Omega X_{2+n}) \arrow[r] & \underline{\Omega}(\Omega^2 X_{3+n}) \arrow[r] & \underline{\Omega}(\Omega^3 X_{4+n}) \arrow[r] & \dots \arrow[r] & \underline{\Omega}(RX_{1+n})
\end{tikzcd}$$

These maps are all homeomorphisms, and so $R'X$ is a strong Ω-spectrum. It is easy to check that it receives a map of prespectra from X that is a stable equivalence.

We could also take a homotopy colimit instead of a strict colimit. This obviates the need to assume that the structure maps are closed inclusions, but then of course we get a weak Ω-spectrum instead of a strong one.

Now for the first pitfall. If X is a symmetric or orthogonal spectrum, not isomorphic to $*$, then the prespectrum $R'X$ is *not* symmetric or orthogonal. This can be checked directly – the structure maps are each equivariant but their two-fold composites are not Σ_2-equivariant. However it is known that more is true, as we now explicitly show:

Proposition 2.1. *The only symmetric spectrum that is also a strong* Ω*-spectrum is the zero spectrum* $*$*.*

Proof. Assume X is a symmetric Ω-spectrum. The symmetric spectrum structure forces $X_n \to \Omega^2 X_{2+n}$ to be Σ_2-equivariant. Therefore $\Omega^2 X_{2+n}$ is entirely fixed by Σ_2. It follows that the path component of the basepoint of X_{2+n} must be entirely Σ_2-fixed. If X_{2+n} has any nontrivial continuous path γ starting at the basepoint, then we can easily construct a map $S^2 \to X_{2+n}$ which is not Σ_2-equivariant, just by sending each ray from the origin in $\mathbb{R}^2 \cup \{\infty\}$ to a path that travels some distance along γ before traveling back to the basepoint, and arranging the parametrizations so that one of these paths is nontrivial, while the path opposite it in S^2 stays at the basepoint. Therefore all paths in X_{2+n} at the basepoint must be trivial, so $X_n \cong \Omega^2 X_{2+n} \cong *$. $\square$

Fortunately enough, there is a second way of defining structure maps between the same spaces $(RX)_n$ to give a symmetric or orthogonal spectrum RX receiving a stable equivalence from X. It will turn out that RX is a weak Ω-spectrum precisely when X is semistable. (Of course, this is the best we can do. When X is not semistable, it is known that it is impossible to give a π_*-isomorphism between X and a weak Ω-spectrum.) We will describe the symmetric case, but the orthogonal case is argued in exactly the same way.

The slick way to say this is that there is a functor $\mathcal{I}_S \wedge \mathcal{I}_S \to \mathcal{I}_S$ which takes the disjoint union of the two sets. Then we may pull back X along this functor to get a bi-symmetric spectrum, or a symmetric spectrum object in symmetric spectra. (This move doesn't work for prespectra because the disjoint union operation doesn't define a functor.) Then, since the above construction $(R-)_0$ from spectra to spaces (using colim or hocolim) is natural with respect to maps of symmetric spectra, this bi-symmetric spectrum gives a symmetric spectrum RX whose nth level is $(R \operatorname{sh}^n X)_0 \cong (RX)_n$ for all n.

More explicitly, tracing through the above definition, we see that the structure map $RX_n \to RX_{1+n}$ is on the rth term in the colimit system the composite

$$\Omega^r X_{r+n} \longrightarrow \Omega\Omega^r X_{1+r+n} \longrightarrow \Omega\Omega^r X_{r+1+n}$$

of the structure map and the action of the permutation $\tau_{1,r}$ that shuffles the first element past the next r elements. We can also confirm directly that these commute with the maps of the colimit system, and that a composition of two of them is $\Sigma_2 \times \Sigma_n$-equivariant. It is easy to see that the inclusion of the 0th term in the colimit system defines a map of symmetric spectra $X \to RX$.

To summarize, for each symmetric spectrum X, we have constructed a map of symmetric spectra $X \to RX$ and a map of prespectra $X \to R'X$. The spectra RX and $R'X$ have homeomorphic levels, but different structure maps. In general, the structure maps do not even agree up to homotopy! Instead, at level $r+1$ of the colimit system, the "vertical" structure map of RX and the "diagonal" structure map of $R'X$ give the following diagram, where the upper-left triangle commutes but the remaining region does not.

$$\begin{tikzcd}
\Omega^{r+1} X_{r+1+n} \arrow[rr] \arrow[d] & & \Omega^{1+r+\underline{1}} X_{1+r+1+n} \arrow[ddl, "\text{swap loops}"] \\
\Omega^{1+r+1} X_{1+r+1+n} \arrow[urr, equal] \arrow[d, "\Omega^{1+r+1}\tau_{1,r+1}"] & & \\
\Omega^{1+r+1} X_{r+2+n} \arrow[r, equal] & \Omega^{\underline{1}+1+r} X_{1+r+1+n} &
\end{tikzcd}$$

The best we can do is analyze the difference on homotopy groups. We assume for this last part that either the structure maps are closed inclusions or we took homotopy colimits when defining the spaces $(RX)_n$.

We begin by remarking that the maps of the spectrum RX aren't inducing equivalences on each level so there's no guarantee they give an equivalence on the colimits. That being said, in the quadrilateral region of the above diagram, all maps are isomorphisms on homotopy groups. Of course, they are different isomorphisms. The one for the structure map of RX induces $\pi_*(\tau_{1,r+1})$, while the structure map of $R'X$ composed with one map of the colimit system induces $(-1)^{r+1}$. Stabilizing to the colimit, the structure maps of RX give some map $\pi_k(RX_n) \to \pi_k(RX_{1+n})$ and the structure maps of $R'X$ give an isomorphism between the same two groups. We may compose the structure map of RX with the inverse of this isomorphism and get a self-map of

$$\pi_k(RX_n) \cong \pi_{k-n}(X).$$

Which map is it? Comparing with Schwede's notes (and flipping the order of our sphere coordinates to match his convention), we conclude that it is the action of the "shift operator" from the injection monoid $\mathcal{M}$. As a consequence, this map is an isomorphism precisely when X is semistable (using Schwede's notes I.8.8.iii and I.8.25). In conclusion:

Proposition 2.2. *The symmetric spectrum RX is a weak Ω-spectrum precisely when X is semistable.*

Of course, if X is an orthogonal spectrum then it is always semistable, so in that case RX is always an orthogonal weak Ω-spectrum. On the other hand, if X is symmetric and not semistable, then I don't know much about the behavior of RX. I am not even sure how worthwhile it is to try to find out!

If X is a semistable symmetric spectrum then RX can alternatively be defined using a homotopy colimit over the category $\mathcal{I}$ of finite sets and injections. Semistability implies that the homotopy colimit over $\mathcal{I}$ agrees with the smaller homotopy colimit over $\mathcal{N}$, so this produces another fibrant replacement functor for semistable symmetric spectra. If X is orthogonal then we could also do a homotopy colimit over $\mathcal{I}$, or even over the category $\mathcal{O}$ of finite-dimensional innner product spaces and isometric linear maps, and we would get different model for RX that is level-equivalent to the model we considered above.

Finally, it's worth remarking that the other expected result goes through without a hitch:

Proposition 2.3. *The maps from X to RX and $R'X$ are always stable equivalences.*

For $R'X$ this is true because we can build it as a (homotopy) colimit of spectra where everything below some level X_n has been replaced by loopspaces of X_n, and the maps between these spectra are clearly stable equivalences. For RX we think of the bispectrum perspective and build the homotopy colimit up one term at a time (for the whole spectrum), observing that at each stage we are getting the standard map $X \to \Omega^r \operatorname{sh}^r X$, which is a stable equivalence.

3. The Bökstedt smash product

The Bökstedt smash product is a funny construction from the modern point of view. (Though of course it is all the more impressive because it predates modern smash products of spectra.) It takes as input an k-tuple $X^1, \ldots, X^k$ of symmetric spectra, but its output is a *space*, which in under very mild assumptions (the levels are well-pointed) is equivalent to the derived 0th space of the smash product $X^1 \wedge \ldots \wedge X^k$. The essential idea is to take a homotopy colimit of the spaces $\Omega^{n_1+\ldots+n_k}(X^1_{n_1} \wedge \ldots \wedge X^k_{n_k})$ for varying k-tuples of nonnegative integers $(n_1, \ldots, n_k)$. We have a choice between taking this homotopy colimit over $\mathcal{I}^n$ or $\mathcal{N}^n$. In the official definition, it is taken over $\mathcal{I}^n$, so that the smash product has good functoriality properties needed to form (for instance) cyclic bar constructions of ring spectra and FSPs. On the other hand, we usually want this homotopy colimit to agree with the corresponding homotopy colimit over $\mathcal{N}^n$, in order to have a better handle on its homotopy type.

As a result of using $\mathcal{I}$, the Bökstedt smash product has a certain functoriality with respect to maps between smash products of symmetric spectra. It cannot be as simple as we would like, of course. For instance, since the Bökstedt smash product takes an k-tuple of symmetric spectra to a space, it does not even make sense to ask whether it is associative! (Just as strangely, though associativity does not make sense, commutativity does make sense.)

We also want to consider the Bökstedt smash product as taking k-tuples of symmetric spectrum *objects* in some reasonable category $\mathcal{C}$, and giving an output that is an object of $\mathcal{C}$. This is because we want the output to be a spectrum. This point of view tells us that, if we want a spectrum to come out, we ought to feed in a collection of symmetric spectrum objects in spectra. Of course, this is not hard to arrange – every symmetric spectrum of spaces X produces a symmetric spectrum object in symmetric spectra, in fact in two ways: one by taking suspension spectra of the levels of X, and the other by taking all the shifts of X. The first method will actually recover Bökstedt's original trick for making his space into a spectrum, by inserting an extra sphere in the smash product on the inside.

Enough philosophy; here's the definition. Let $\mathcal{C}$ be a pointed closed symmetric monoidal category receiving a strong symmetric monoidal functor from simplicial sets. Let $\mathcal{S}p^\Sigma(\mathcal{C})$ denote symmetric spectrum objects in $\mathcal{C}$. To each k-tuple $X^1, \ldots, X^k$ of symmetric spectrum objects of $\mathcal{C}$ we create a diagram $\Omega^{\mathcal{I}}_k$ indexed by $\mathcal{I}^k$ as follows. The tuple

$(n_1,\dots,n_k)$ is assigned to the object of $\mathcal{C}$ given by

$$\mathrm{Hom}_{\mathcal{C}}(S^{n_1+\dots+n_k}, X^1_{n_1}\otimes\dots\otimes X^k_{n_k}).$$

Here we are blurring the distinction between the simplicial set $S^n = (\Delta[1]/\partial\Delta[1])^{\wedge n}$ and its image in $\mathcal{C}$, and of course the tensor products and the hom are given by the closed symmetric monoidal structure of $\mathcal{C}$. To define the morphism induced by a tuple of inclusion maps $\alpha_i : n_i \to m_i$, we let $S^{m_i-n_i}$ denote the sphere indexed by the complement set of n_i in m_i. The symmetric spectrum structure on X^i gives a map of the form

$$S^{m_i-n_i}\otimes X^i_{n_i} \to X^i_{m_i}$$

induced by the injection α_i, which is compatible with compositions in the obvious way. The tensor of these for $1 \le i \le k$ induces a map

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}}(S^{n_1+\dots+n_k}, X^1_{n_1}\otimes\dots\otimes X^k_{n_k})\\ \longrightarrow\ &\mathrm{Hom}_{\mathcal{C}}(S^{n_1+\dots+n_k}, \mathrm{Hom}_{\mathcal{C}}(S^{m_1-n_1}\otimes\dots\otimes S^{m_k-n_k}, X^1_{m_1}\otimes\dots\otimes X^k_{m_k}))\\ \cong\ &\mathrm{Hom}_{\mathcal{C}}(S^{m_1+\dots+m_k}, X^1_{m_1}\otimes\dots\otimes X^k_{m_k}).\end{aligned}$$

The adjoint of this is an evaluation followed by the symmetric spectrum structure:

$$\begin{aligned}&S^{m_1+\dots+m_k}\otimes \mathrm{Hom}_{\mathcal{C}}(S^{n_1+\dots+n_k}, X^1_{n_1}\otimes\dots\otimes X^k_{n_k})\\ \longrightarrow\ &S^{m_1-n_1}\otimes\dots\otimes S^{m_k-n_k}\otimes X^1_{n_1}\otimes\dots\otimes X^k_{n_k} \longrightarrow X^1_{m_1}\otimes\dots\otimes X^k_{m_k}.\end{aligned}$$

To check these are compatible with composition with $\beta_i : m_i \to p_i$ we check that two certain maps

$$S^{p_1+\dots+p_k}\otimes \mathrm{Hom}_{\mathcal{C}}(S^{n_1+\dots+n_k}, X^1_{n_1}\otimes\dots\otimes X^k_{n_k}) \longrightarrow X^1_{p_1}\otimes\dots\otimes X^k_{p_k}$$

are the same. By a long diagram-chase we identity them each with an evaluation, followed by either a single application of the symmetric spectrum structure or one after the other. By the assumption that the X^i are symmetric spectrum objects these agree, hence we have a diagram indexed by $\mathcal{I}^k$.

Now that we have defined a diagram $\Omega^{\mathcal{I}}_k(X^1,\dots,X^k)$ of objects of $\mathcal{C}$ indexed by $\mathcal{I}^k$, we take its homotopy colimit (using the Bousfield-Kan formula) and call the result the *Bökstedt smash product* of the spectrum objects $X^1,\dots,X^k$:

$$\mathcal{B}ok_k(X^1,\dots,X^k) := \underset{(n_1,\dots,n_k)\in\mathcal{I}^k}{\mathrm{hocolim}}\ \mathrm{Hom}_{\mathcal{C}}(S^{n_1+\dots+n_k}, X^1_{n_1}\otimes\dots\otimes X^k_{n_k})$$

Our functoriality statement for $\mathcal{B}ok_k$ follows. It is necessarily complicated because we need to vary the number of terms in the smash product, and can only take the smash product of k spectra all at once (not in stages).

Proposition 3.1. *Let $X^1,\dots,X^k$ and $Y^1,\dots,Y^\ell$ be tuples of symmetric spectrum objects of $\mathcal{C}$. Let f denote a map of sets $\{1,\dots,k\}\to\{1,\dots,\ell\}$ equipped with a choice of total ordering on each preimage $f^{-1}(j)$. Given such an f along with maps*

$$\varphi_j : \bigotimes_{i\in f^{-1}(j)} X^i \to Y^j$$

where the $\otimes$ denotes tensor product of symmetric spectrum objects of $\mathcal{C}$, and the left-hand side is composed in the chosen total ordering on $f^{-1}(j)$, there is a map of Bökstedt smash products

$$\mathcal{B}ok(f, \{\varphi_j\}) : \mathcal{B}ok_k(X^1, \ldots, X^k) \longrightarrow \mathcal{B}ok_\ell(Y^1, \ldots, Y^\ell).$$

This construction respects identity maps and compositions (using the dictionary order to compose total orderings). It arises as a map of diagrams which respects the initial objects of $\mathcal{I}^k$ and $\mathcal{I}^\ell$, and on those initial objects it agrees with the tensor product $\bigotimes_{j\in\{1,\ldots,\ell\}}(\varphi_j)_0$.

Proof. We take the skeletal model of $\mathcal{I}$, in which the objects are the natural numbers. Define a functor $\alpha_f : \mathcal{I}^k \to \mathcal{I}^\ell$ taking $(n_1, \ldots, n_k)$ to the ℓ-tuple of finite sets whose jth term is the coproduct $\coprod_{i\in f^{-1}(j)} n_i$ using the chosen total ordering of $f^{-1}(j)$. The morphisms are similarly taken to a coproduct of morphisms. This construction clearly preserves compositions and the identity map.

To define $\mathcal{B}ok(f, \{\varphi_j\})$ it suffices to form a map of diagrams

$$\Omega^{\mathcal{I}}_k(X^1, \ldots, X^k) \to \Omega^{\mathcal{I}}_\ell(Y^1, \ldots, Y^\ell) \circ \alpha_f$$

so in particular for each $(n_1, \ldots, n_k)$ a map in $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(S^{\sum_i n_i}, \bigotimes_i X^i_{n_i}) \to \mathrm{Hom}_{\mathcal{C}}(S^{\sum_j \sum_{i\in f^{-1}(j)} n_i}, \bigotimes_j Y^j_{\sum_{i\in f^{-1}(j)} n_i})$$

using the maps $\{\phi_j\}$. We observe that the spheres on both sides are isomorphic, and we adopt the convention that when passing from one to the other we re-arrange the sphere coordinates so that each n_i on the left is sent to its sister n_i on the right. In this way both sides are $\mathcal{I}^k$ diagrams that arise from the $\Omega^{\mathcal{I}}_k$-construction on a k-multisymmetric spectrum, one of which is the external smash product of the spectra X^i, the other of which is obtained by pulling back the spectra Y^j along the direct sum maps $\mathcal{I}^{\wedge|f^{-1}(j)|} \to \mathcal{I}$. (In this direct sum map we also use the chosen total order on $f^{-1}(j)$.)

It therefore suffices to check that the ϕ_j define a map of k-multisymmetric spectra. But this is automatic from the definition of the smash product of symmetric spectra as a left Kan extension along this direct sum map. A map of symmetric spectra from the smash product of X^i for $i \in f^{-1}(j)$ into Y^j is equivalent to a map of multisymmetric spectra from the external smash product of the spectra X^i into the pullback of Y_j. Therefore the construction $\Omega^{\mathcal{I}}_k$ gives a map of $\mathcal{I}^k$-diagrams.

It remains to check these respect compositions and identity maps in the f variable, but this is an almost trivial exercise. (The hard part was getting our conventions right in the first place; the choice of total ordering is important.) The functor α_f clearly preserves the zero object, and on that zero object our map of diagrams arises from the external smash product of the maps ϕ_j on multilevel $(0, \ldots, 0)$, where it is simply the smash product of the maps $(\phi_j)_0 : \otimes_{i\in f^{-1}(j)} X^i_0 \to Y^j_0$. $\square$

Remark. This functoriality statement would not hold if $\mathcal{B}ok_k(X^1, \dots, X^k)$ were defined as a homotopy colimit over $\mathcal{N}$, because the functor α_f does not preserve this subcategory. However it would hold if we replaced $\mathcal{I}$ with the smaller category of totally ordered sets and order-preserving injections. Furthermore a the colimit along this smaller category will agree with the colimit over $\mathcal{N}$ (and therefore $\mathcal{I}$) for semistable symmetric spectra. It seems therefore that the use of $\mathcal{I}$ is not strictly necessary to define the cyclic bar construction with the Bökstedt smash product! But of course we need something more than just $\mathcal{N}$. And $\mathcal{I}$ turns out to give the right homotopy type even if the spectra involved are not semistable, which is really useful.

Finally we check that if R is a symmetric ring spectrum object of $\mathcal{C}$ then we can use the above functoriality to form the cyclic bar construction of R.

Proposition 3.2. *If R is a (not necessarily commutative) symmetric ring spectrum object of $\mathcal{C}$, then the objects $\mathcal{B}ok_{k+1}(R, \dots, R)$ for $k \geq 0$ assemble into a cyclic object of $\mathcal{C}$.*

Proof. Given a morphism $h : [k-1] \to [\ell - 1]$ in the cyclic category $\mathbf{\Lambda}$, regarded as a degree-one functor from the necklace of k beads category to the necklace of ℓ beads category, we define $f : \{1, \dots, \ell\} \to \{1, \dots, k\}$ as the backwards map of arrows, with total ordering on the preimage of the arrow $i \to i+1$ given by the order of composition in the necklace category starting at $h(i)$ and ending at $h(i+1)$. We take all maps ϕ_j to be induced by the multiplication and unit maps of R. Then this rule for defining backwards maps of arrows is easily checked to respect compositions. The compositions of the maps ϕ_j agree by the assumption that R is an associative unital ring. $\square$

To arrive at the original Bökstedt smash product, we take $\mathcal{C}$ to be orthogonal spectra, and we apply $\mathcal{B}ok_k$ to the orthogonal suspension $\Sigma^\infty R$ of a symmetric ring spectrum R. Since orthogonal suspension spectra are preserved under smash product, the definition of $\mathcal{B}ok_k(\Sigma^\infty R, \dots, \Sigma^\infty R)$ rearranges into an orthogonal spectrum which is at level V the space

$$\operatorname*{hocolim}_{(n_1,\dots,n_k)\in\mathcal{I}^k} \Omega^{n_1+\dots+n_k}(S^V \wedge X^1_{n_1} \wedge \dots \wedge X^k_{n_k}).$$

We remark that by our conventions this homotopy colimit is taken in the based sense. Though the category $\mathcal{I}^k$ has an initial object and hence a contractible nerve, so as long as the spaces of the diagram are well-based the unbased homotopy colimit will have the same homotopy type.

I won't say much about semistability here, because the results are eerily strong. (I hope I am stating them correctly; this is almost entirely pulled from Shipley and Patchkoria-Sagave.) When $\mathcal{C}$ is spaces, $\mathcal{B}ok_k(X_1, \dots, X_k)$ behaves well all the time, and especially well when the X_i are semistable. In particular, it sends stable equivalences to weak equivalences, as soon as the levels of the X_i are well-pointed, and has the same homotopy type as the (derived) smash product $X_1 \wedge \dots \wedge X_k$ in symmetric spectra (Shipley's THH paper 4.2.3 and 4.2.9, see also Patchkoria-Sagave). And when the X_i are semistable, it becomes permissible to take the homotopy colimit along $\mathcal{N}^k$ instead of $\mathcal{I}^k$ without changing the homotopy type. So when our inputs are semistable everything we can think of is equivalent, and when they are not we should be wary about the hocolim over $\mathcal{N}^k$, but the hocolim over $\mathcal{I}^k$ and the derived smash product are still equivalent.

Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W Green St
Urbana, IL 61801
cmalkiew@illinois.edu