BICATEGORIES, PSEUDOFUNCTORS, SHADOWS: A CHEAT SHEET

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Contents

This document is a quick-and-dirty set of definitions for bicategories, pseudofunctors, and pseudonatural transformations. The accompanying document organizes these definitions in table form.

Think of the definitions here as "zoomed-in:" they focus on multiplying individual objects and 1-cells. The ones in the table are "zoomed-out:" they focus on the entire category of 1-cells and 2-cells. In the zoomed-in definitions, the coherences are polygonal regions whose vertices are functors and whose edges are natural transformations. In the zoomed-out definitions, the coherences are given by three-dimensional polyhedra whose vertices are categories, edges are functors, and faces are natural transformations.

The "zoomed in" description in this document tends to be more useful when you want to *use* a bicategory or a pseudofunctor. The "zoomed out" description in the table tends to be more useful when you want to *construct or verify* a particular example of a bicategory or a pseudofunctor.

1. Category

A **category C** consists of

- a collection of objects a,
- a set of morphisms $b \xleftarrow{f} a$ between any two objects a, b,
- a composition law that assigns every pair of composable morphisms to a composition:

$$c \xleftarrow{g} b \xleftarrow{f} a \qquad \rightsquigarrow \qquad c \xleftarrow{g\circ f} a$$

- an a choice of unit morphism $a \xleftarrow{\mathrm{id}_a} a$ for each object a.

These must satisfy the following conditions.

- The composition is associative. For every triple of composable morphisms
$$d \xleftarrow{h} c \xleftarrow{g} b \xleftarrow{f} a$$
we have the equality $(h \circ g) \circ f = h \circ (g \circ f)$ of morphisms $d \leftarrow a$.
- The composition is unital. The two strings of arrows
$$b \xleftarrow{f} a \xleftarrow{\mathrm{id}_a} a, \qquad b \xleftarrow{\mathrm{id}_b} b \xleftarrow{f} a$$
both compose to f. In other words $f \circ \mathrm{id}_a = f = \mathrm{id}_b \circ f$.

2. Functor

A **functor** F from a category $\mathbf{C}$ to a category $\mathbf{D}$ consists of

- a function of objects assigning each $a \in \mathrm{ob}\,\mathbf{C}$ to $Fa \in \mathrm{ob}\,\mathbf{D}$,
- and a function of morphisms
$$b \xleftarrow{f} a \qquad \rightsquigarrow \qquad Fb \xleftarrow{F(f)} Fa.$$

These must satisfy the following conditions.

- F respects composition:
$$\begin{array}{ccc} c \xleftarrow{g} b \xleftarrow{f} a & \rightsquigarrow & Fc \xleftarrow{F(g)} Fb \xleftarrow{F(f)} Fa \\ \downarrow & & \downarrow \\ c \xleftarrow{g \circ f} a & \rightsquigarrow & Fc \xleftarrow{F(g\circ f) = F(g) \circ F(f)} Fa \end{array}$$
- F respects identity morphisms:
$$a \xleftarrow{\mathrm{id}_a} a \qquad \rightsquigarrow \qquad Fa \xleftarrow{F(\mathrm{id}_a) = \mathrm{id}_{Fa}} Fa.$$

3. Natural transformation

A **natural transformation** η from a functor $F : \mathbf{C} \to \mathbf{D}$ to a functor $G : \mathbf{C} \to \mathbf{D}$ consists of

- an assignment of each object a of $\mathbf{C}$ to a morphism $Ga \xleftarrow{\eta(a)} Fa$ in $\mathbf{C}D$.

These must satisfy the following conditions.

- η commutes with morphisms. For each morphism f in $\mathbf{C}$ the following square commutes.
$$\begin{tikzcd} Fb \arrow[d, "\eta(a)"] & Fa \arrow[l, "F(f)"'] \arrow[d, "\eta(b)"] \\ Gb & Ga \arrow[l, "G(f)"] \end{tikzcd}$$

η is a **natural isomorphism** if each $\eta(a)$ is an isomorphism in $\mathbf{D}$.

4. Coherence

From the zoomed-out view, the conditions in each definition are really a minimal sufficient set from an infinite list of coherences. The full list of coherences for a category says that there is a unique composition function

$$\mathbf{C}(a_0,a_1)\times\ldots\times\mathbf{C}(a_{n-1},a_n)\to\mathbf{C}(a_0,a_n)$$

that is equal to any function obtained by repeated application of $\circ$ and id_a. The coherence theorem for functors says that there is a unique composition function

$$\mathbf{C}(a_0,a_1)\times\ldots\times\mathbf{C}(a_{n-1},a_n)\to\mathbf{D}(Fa_0,Fa_n)$$

that is equal to any function obtained by applying $\circ$ and id_a as many times as desired, applying F to everything, and then applying $\circ$ and id_a as many times more as desired, until we arrive at $\mathbf{D}(Fa_0,Fa_n)$. Finally a natural transformation's coherence theorem says that there is a unique composition function

$$\mathbf{C}(a_0,a_1)\times\ldots\times\mathbf{C}(a_{n-1},a_n)\to\mathbf{D}(Fa_0,Ga_n)$$

defined just as before, but when we apply F, instead of applying it to everything, we pick a slot a_i in between two of the sets on the left-hand side, apply F to everything to the left of a_i, G to everything to the right of a_i, and put $\eta(a_i)$ in the middle. The strong parallel between this coherence theorem and the one for functors is explained by the fact that the data of two functors and a natural transformation is nothing more than the data of a functor $\mathbf{C}\times[0\to 1]\to\mathbf{D}$ where $[0\to 1]$ is the non-trivial poset with two elements.

The zoomed-in definitions also reveal the close parallel with group theory. A category is like a group but the objects have extra labels that govern when they can be composed, and a functor is just like a group homomorphism. Just as with words in a group, any string of composable morphisms

$$a_n \xleftarrow{f_n} \ldots \xleftarrow{f_2} a_1 \xleftarrow{f_1} a_0$$

has a unique (well-defined) composite, any functor F applied to such a string gives a well-defined composite in the target category from $F(a_0)$ to $F(a_n)$, and any natural transformation $F\Rightarrow G$ gives a well-defined composite from $F(a_0)$ to $G(a_n)$.

5. Monoidal category

A monoidal category $\mathbf{C}$ consists of

- a category $\mathbf{C}$,
- a tensor product functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,
- a unit object $I \in \operatorname{ob} \mathbf{C}$,
- an associator natural isomorphism

$$\alpha(a,b,c) : (a \otimes b) \otimes c \xrightarrow{\cong} a \otimes (b \otimes c),$$

- and a left unit and a right unit natural isomorphism

$$l(a) : I \otimes a \xrightarrow{\cong} a, \quad r(a) : a \otimes I \xrightarrow{\cong} a.$$

These must satisfy the following conditions.

- The following pentagon diagram commutes.

$$\begin{tikzcd}
 & & (a \otimes (b \otimes c)) \otimes d \arrow[drr, "\alpha(a,b\otimes c,d)"] & & \\
((a \otimes b) \otimes c) \otimes d \arrow[urr, "\alpha(a,b,c)\otimes \mathrm{id}_d"] \arrow[dr, "\alpha(a\otimes b,c,d)"'] & & & & a \otimes ((b \otimes c) \otimes d) \arrow[dl, "\mathrm{id}_a \otimes \alpha(b,c,d)"] \\
 & (a \otimes b) \otimes (c \otimes d) \arrow[rr, "\alpha(a,b,c\otimes d)"] & & a \otimes (b \otimes (c \otimes d)) &
\end{tikzcd}$$

- The following triangle diagram commutes.

$$\begin{tikzcd}
(a \otimes I) \otimes b \arrow[rr, equal, "\alpha(a,I,b)"] \arrow[dr, "r_a \otimes \mathrm{id}_b"'] & & a \otimes (I \otimes b) \arrow[dl, "\mathrm{id}_a \otimes l_b"] \\
 & a \otimes b &
\end{tikzcd}$$

Equivalently, every diagram made from α, l, r, and tensor products, commutes.

Discussion: The point is that given any string of objects $a_1, \ldots, a_n$ with $n \geq 0$, we can take an n-fold tensor product $a_1 \otimes \ldots \otimes a_n$, and this has a well-defined meaning, up to canonical isomorphism. The isomorphism between any two recipes for building $a_1 \otimes \ldots \otimes a_n$ is given by composing the associators an unit maps, in any combination desired. The coherence theorem guarantees that all such isomorphisms will coincide, so that we can say the isomorphism is canonical. (It may or may not be given by a universal property. At the very least, it is determined in a canonical way from the chosen monoidal category structure on $\mathbf{C}$.)

The same discussion applies to tensoring together morphisms an n-tuple of morphisms $f_i : a_i \to b_i$. This gives a well-defined morphism up to canonical isomorphism of morphisms (i.e. up to commuting squares in which two parallel legs are isomorphisms).

A point to make here which recurs in the definitions below. The tensoring $f_1 \otimes f_2$ of two morphisms can always be re-expressed by tensoring each with an identity and then composing them:

$$f_1 \otimes f_2 = (f_1 \otimes \mathrm{id}_{b_2}) \circ (\mathrm{id}_{a_1} \otimes f_2) = (\mathrm{id}_{b_1} \otimes f_2) \circ (f_1 \otimes \mathrm{id}_{a_2})$$

Thus "horizontal composition of morphisms can be defined using vertical composition." This might feel strange in examples, but the above structure forces it to happen, so we might as well get used to it.

6. Strong monoidal functor

A strong monoidal functor F from a monoidal category $\mathbf{C}$ to a monoidal category $\mathbf{D}$ consists of

- a functor $F\colon \mathbf{C} \to \mathbf{D}$,
- a natural isomorphism $m(x,y)\colon F(x) \otimes F(y) \xrightarrow{\cong} F(x \otimes y)$ of functors $\mathbf{C} \times \mathbf{C} \to \mathbf{D}$,
- and a single isomorphism $i\colon I_{\mathbf{D}} \xrightarrow{\cong} F(I_{\mathbf{C}})$.

These must satisfy the following conditions.

- m is associative. The following hexagon diagram commutes.

$$\begin{tikzcd}
(F(x)\otimes F(y))\otimes F(z) \arrow[r, "m(x,y)\otimes \mathrm{id}_{F(z)}"] \arrow[d, "\alpha(F(x),F(y),F(z))"'] & F(x\otimes y)\otimes F(z) \arrow[r, "m(x\otimes y,z)"] & F((x\otimes y)\otimes z) \arrow[d, "F(\alpha(x,y,z))"] \\
F(x)\otimes(F(y)\otimes F(z)) \arrow[r, "\mathrm{id}_{F(x)}\otimes m(y,z)"] & F(x)\otimes F(y\otimes z) \arrow[r, "m(x,y\otimes z)"] & F(x\otimes(y\otimes z))
\end{tikzcd}$$

- m respects units. The following square diagrams commute.

$$\begin{tikzcd}
I_{\mathbf{D}}\otimes F(x) \arrow[r, "i\otimes \mathrm{id}_{F(x)}"] \arrow[d, "l_{F(x)}"'] & F(I_{\mathbf{C}})\otimes F(x) \arrow[d, "m(I_{\mathbf{C}},x)"] \\
F(x) & F(I_{\mathbf{C}}\otimes x) \arrow[l, "F(l_x)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(x)\otimes F(I_{\mathbf{C}}) \arrow[d, "m(x,I_{\mathbf{C}})"'] & F(x)\otimes I_{\mathbf{D}} \arrow[l, "\mathrm{id}_{F(x)}\otimes i"'] \arrow[d, "r_{F(x)}"] \\
F(x\otimes I_{\mathbf{C}}) \arrow[r, "F(r_x)"] & F(x)
\end{tikzcd}$$

Discussion: The point is that a strong monoidal functor is acting like a homomorphism, both with respect to the composition of morphisms, and with respect to the tensoring operation on objects and on morphisms. The homomorphism condition $f(ab) = f(a)f(b)$ or more generally $f(a_1 \dots a_n) = f(a_1)\dots f(a_n)$ for $n \geq 0$ is relaxed from an equality to an isomorphism, (as it must be, because the two sides are only defined up to canonical isomorphism anyway,) and again we make this isomorphism canonical by insisting that any two recipes for going from $f(a_1 \otimes \dots \otimes a_n)$ to $f(a_1) \otimes \dots \otimes f(a_n)$ must in fact give the same isomorphism.

Again, we must also remember that you can tensor morphisms. For any n-tuple of morphisms $f_i\colon a_i \to b_i$, we can tensor their sources in an essentially unique way, e.g. $f(a_1 \otimes \dots \otimes a_n)$ or $f(a_1) \otimes \dots \otimes f(a_n)$, and same for the targets. Each model for these tensor products comes with its own model for how to tensor the maps f_i, e.g. $f(f_1 \otimes \dots \otimes f_n)$ or $f(f_1) \otimes \dots \otimes f(f_n)$. The coherence conditions are just enough to guarantee that along our canonical isomorphisms on the objects, all of the morphisms so constructed agree with each other.

7. Monoidal natural transformation

A monoidal natural transformation η from a monoidal functor $F: \mathbf{C} \to \mathbf{D}$ to a monoidal functor $G: \mathbf{C} \to \mathbf{D}$ consists of

- a natural transformation η from F to G.

These must satisfy the following conditions.

- η commutes with m. The following square commutes.

$$\begin{tikzcd}
F(x)\otimes F(y) \arrow[r, "m(x,y)"] \arrow[d, "\eta(x)\otimes\eta(y)"] & F(x\otimes y) \arrow[d, "\eta(x\otimes y)"] \\
G(x)\otimes G(y) \arrow[r, "m(x,y)"] & G(x\otimes y)
\end{tikzcd}$$

- η commutes with i. The following triangle commutes.

$$\begin{tikzcd} & I_{\mathbf{D}} \arrow[dl, "i"'] \arrow[dr, "i"] & \\ F(I_{\mathbf{C}}) \arrow[rr, "\eta(I_{\mathbf{C}})"] & & G(I_{\mathbf{C}}) \end{tikzcd}$$

Discussion: A monoidal natural transformation is giving a well-defined morphism from all the different models for $f(a_1 \otimes \ldots \otimes a_n)$ to all the different models for $g(a_1 \otimes \ldots \otimes a_n)$, forming a commuting square with the morphisms $f(f_1 \otimes \ldots \otimes f_n)$ and $g(f_1 \otimes \ldots \otimes f_n)$ for any tuple of morphisms $f_i : a_i \to b_i$. Equivalently, it sends this data to a morphism $f(f_1 \otimes \ldots \otimes f_n) \to g(f_1 \otimes \ldots \otimes f_n)$ that is unique up to canonical isomorphism.

8. Bicategory

A bicategory $\mathbf{C}$ consists of

- a collection of objects (0-cells) ob $\mathbf{C}$,
- a category $\mathbf{C}(a,b)$ for each pair of 0-cells a,b (whose objects are called 1-cells, morphisms are called 2-cells, composition called "vertical composition"),
- a "tensoring" or "horizontal composition" functor
$$\odot : \mathbf{C}(a,b) \times \mathbf{C}(b,c) \to \mathbf{C}(a,c)$$
for each triple of 0-cells a,b,c,
- a unit 1-cell $I_a \in \text{ob}\,\mathbf{C}(a,a)$ for each 0-cell a,
- an associator natural isomorphism
$$\alpha(X,Y,Z) : (X \odot Y) \odot Z \xrightarrow{\cong} X \odot (Y \odot Z)$$
of functors
$$\mathbf{C}(a,b) \times \mathbf{C}(b,c) \times \mathbf{C}(c,d) \to \mathbf{C}(a,d),$$
for each quadruple of 0-cells a,b,c,d,
- and a left unit and a right unit natural isomorphism
$$l(X) : I_a \odot X \xrightarrow{\cong} X, \quad r(Y) : Y \odot I_a \xrightarrow{\cong} Y,$$
of functors $\mathbf{C}(a,b) \to \mathbf{C}(a,b)$ and $\mathbf{C}(b,a) \to \mathbf{C}(b,a)$ respectively, for each pair of 0-cells a,b.

These must satisfy the following conditions.

- For each quintuple of 0-cells a,b,c,d,e and quadruple of 1-cells $W \in \mathbf{C}(a,b)$, $X \in \mathbf{C}(b,c)$, $Y \in \mathbf{C}(c,d)$, and $Z \in \mathbf{C}(d,e)$, the following pentagon diagram (whose morphisms are 2-cells) commutes.

$$\begin{tikzcd}
& (W \odot (X \odot Y)) \odot Z \arrow[dr, "\alpha(W,X\odot Y,Z)"] & \\
((W \odot X) \odot Y) \odot Z \arrow[ur, "\alpha(W,X,Y)\odot \mathrm{id}_Z"] \arrow[d, "\alpha(W\odot X,Y,Z)"] & & W \odot ((X \odot Y) \odot Z) \arrow[d, "\mathrm{id}_W \odot \alpha(X,Y,Z)"] \\
(W \odot X) \odot (Y \odot Z) \arrow[rr, "\alpha(W,X,Y\odot Z)"] & & W \odot (X \odot (Y \odot Z))
\end{tikzcd}$$

- For each triple of 0-cells a,b,c and pair of 1-cells $X \in \mathbf{C}(a,b)$ and $Y \in \mathbf{C}(b,c)$, the following triangle diagram of 2-cells commutes.

$$\begin{tikzcd}
(X \odot I_b) \odot Y \arrow[rr, "\alpha(X,I_b,Y)"] \arrow[dr, "r_X \odot \mathrm{id}_Y"'] & & X \odot (I_b \odot Y) \arrow[dl, "\mathrm{id}_X \odot l_Y"] \\
& X \odot Y &
\end{tikzcd}$$

Equivalently, every diagram made from α, l, r, and $\odot$, commutes.

Discussion: This is a monoidal category on many objects. As before, the point is that given any string of objects $a_0,\ldots,a_n$ with $n \geq 0$, we have a tensoring functor $\mathbf{C}(a_0,a_1) \times \ldots \times \mathbf{C}(a_{n-1},a_n) \to \mathbf{C}(a_0,a_n)$ that is well-defined up to canonical isomorphism. In particular, any string of composable 1-cells has a well-defined composition up to canonical isomorphism. The associator and unit maps again generate the isomorphism, and it is well-defined because of the coherence axioms.

Furthermore, these tensorings and isomorphisms between them are natural in 2-cells. So, given any string of 2-cells between two composable strings of 1-cells, they can also be horizontally composed to get a 2-cell between the resulting composite 1-cells. As in the case of a monoidal category, the definition of this composite 2-cell is invariant under the canonical isomorphisms between the different definitions for the composite of the 1-cells.

Theorem: Endomorphisms of I_a always form a commutative monoid. (Proof: Eckmann-Hilton argument.) Special case of R as an $R-R$ bimodule has the endomorphism ring $Z(R)$, the center of R (which is a subring).

9. Pseudofunctor

A pseudofunctor (= strong functor of bicategories) F from a bicategory $\mathbf{C}$ to a bicategory $\mathbf{D}$ consists of

- a function of 0-cells $F\colon \mathrm{ob}\,\mathbf{C} \to \mathrm{ob}\,\mathbf{D}$,
- a functor $F\colon \mathbf{C}(a,b) \to \mathbf{D}(Fa,Fb)$ for each pair of 0-cells a,b,
- a natural isomorphism

$$m(X,Y)\colon F(X)\odot F(Y) \xrightarrow{\cong} F(X\odot Y)$$

  of functors $\mathbf{C}(a,b)\times\mathbf{C}(b,c) \to \mathbf{D}(Fa,Fc)$ for each triple of 0-cells a,b,c,
- and an isomorphism 2-cell

$$i\colon I_{Fa} \xrightarrow{\cong} F(I_a)$$

  for every 0-cell a.

These must satisfy the following conditions.

- m is associative. The following hexagon diagram commutes.

$$\begin{tikzcd}
(F(X)\odot F(Y))\odot F(Z) \arrow[r, "m(X,Y)\odot\mathrm{id}_{F(Z)}"] \arrow[d, "\alpha(F(X),F(Y),F(Z))"'] & F(X\odot Y)\odot F(Z) \arrow[r, "m(X\odot Y,Z)"] & F((X\odot Y)\odot Z) \arrow[d, "F(\alpha(X,Y,Z))"] \\
F(X)\odot(F(Y)\odot F(Z)) \arrow[r, "\mathrm{id}_{F(X)}\odot m(Y,Z)"] & F(X)\odot F(Y\odot Z) \arrow[r, "m(X,Y\odot Z)"] & F(X\odot(Y\odot Z))
\end{tikzcd}$$

- m respects units. The following square diagrams commute for all 1-cells $X\in\mathbf{C}(a,b)$ and $Y\in\mathbf{C}(b,c)$.

$$\begin{tikzcd}
I_{Fb}\odot F(Y) \arrow[r, "i\odot\mathrm{id}_{F(Y)}"] \arrow[d, "l_{F(Y)}"'] & F(I_b)\odot F(Y) \arrow[d, "m(I_b,Y)"] \\
F(Y) & F(I_b\odot Y) \arrow[l, "F(l_Y)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X)\odot F(I_b) \arrow[d, "m(X,I_b)"'] & F(X)\odot I_{Fb} \arrow[l, "\mathrm{id}_{F(X)}\odot i"'] \arrow[d, "r_{F(X)}"] \\
F(X\odot I_b) \arrow[r, "F(r_X)"] & F(X)
\end{tikzcd}$$

Discussion: Bicategories generalize monoidal categories, and pseudofunctors generalize strong monoidal functors. So just as for strong monoidal functors, a pseudofunctor behaves like a homomorphism in two directions, both for the vertical composition of 2-cells (strictly) and for the horizontal composition of both 1-cells and 2-cells (up to canonical isomorphism).

Theorem 9.1. *If $F\mathbf{C}\to\mathbf{D}$ and $G\colon\mathbf{D}\to\mathcal{E}$ are pseudofunctors then there is a pseudofunctor $GF\colon\mathbf{C}\to\mathcal{E}$ defined by the obvious composition of F and G on the 0-cells and morphism categories, and by*

$$m_{GF}(X,Y)\colon GF(X)\odot GF(Y) \xrightarrow{\cong} G(F(X)\odot F(Y)) \xrightarrow{\cong} GF(X\odot Y)$$

$$i_{GF}(a)\colon I_{GFa} \xrightarrow{\cong} G(I_{Fa}) \xrightarrow{\cong} GF(I_a)$$

Moreover this composition is strictly associative.

Proof. Of course m_{GF} is natural because m_G and m_F are. The hexagon identity breaks apart into the hexagon identities for F and G, and two naturality squares for m_G. The unit coherence breaks apart into the unit coherences for i_F and i_G and another naturality square for m_G. □

The zoomed-out definitions make the following theorem intuitively clear (though there is still some careful checking in the proof):

Theorem 9.2. *Given a bicategory* $\mathbf{C}$*, a collection of categories* $\mathbf{D}(a,b)$ *for pairs of 0-cells* a,b *in* $\mathbf{C}$*, and equivalences of categories* $\mathbf{C}(a,b) \to \mathbf{D}(a,b)$*, then there is a bicategory* $\mathbf{D}$ *with these 1-cells and 2-cells such that the given equivalences extend to the data of a pseudofunctor* $F\colon \mathbf{C} \to \mathbf{D}$*. It is unique up to vertical (i.e. pointwise) equivalence of bicategories.*

Proof. (Sketch) Every datum for $\mathbf{D}$ extrudes to either a datum for F, in which case we can pick both simultaneously so that they agree, or to a condition for F, in which case there is a unique choice of datum for $\mathbf{D}$ making the condition for F hold. □

10. Vertical natural transformation (icon)

An icon (= vertical natural transformation) η from a pseudofunctor $F\colon \mathbf{C} \to \mathbf{D}$ to a pseudofunctor $G\colon \mathbf{C} \to \mathbf{D}$ consists of

- a statement that $F = G$ on 0-cells,
- and a natural transformation $\eta\colon F \Rightarrow G$ of functors $\mathbf{C}(a,b) \to D(Fa,Fb)$.

These must satisfy the following conditions.

- η commutes with m. For each pair of 1-cells $X \in \mathbf{C}(a,b)$ and $Y \in \mathbf{C}(b,c)$, the following square commutes.

$$\begin{tikzcd} F(X)\odot F(Y) \arrow[r, "m(X,Y)"] \arrow[d, "\eta(X)\odot\eta(Y)"] & F(X\odot Y) \arrow[d, "\eta(X\odot Y)"] \\ G(X)\odot G(Y) \arrow[r, "m(X,Y)"] & G(X\odot Y) \end{tikzcd}$$

- η commutes with i. For each 0-cell a the following square commutes.

$$\begin{tikzcd} I_{Fa} \arrow[r, equal] \arrow[d, "i"'] & I_{Ga} \arrow[d, "i"] \\ F(I_a) \arrow[r, "\eta(I_a)"] & G(I_a) \end{tikzcd}$$

Discussion: This generalizes the notion of a monoidal natural transformation. The coherence theorem for these says that if you have a rectangular grid of composable maps in the bicategory, any recipe you can think of gives you a map from F of the composite of the top row to G of the composite of the bottom row.

11. Horizontal natural transformation (pseudonatural transformation)

A pseudonatural transformation (= horizontal natural transformation) η from a pseudofunctor $F\colon \mathbf{C} \to \mathbf{D}$ to a pseudofunctor $G\colon \mathbf{C} \to \mathbf{D}$ consists of

- an assignment of each 0-cell a of $\mathbf{C}$ to a 1-cell $\eta(a) \in \mathbf{D}(Fa,Ga)$,
- and an assignment of each 1-cell $X \in \mathbf{C}(a,b)$ to an invertible 2-cell

$$\eta(X)\colon F(X)\odot\eta(b) \xrightarrow{\cong} \eta(a)\odot G(X)$$

in $\mathbf{D}(Fa,Gb)$.

Note: If this 2-cell weren't invertible then there would be four possible directions for a pseudonatural transformation between F and G, according to the side on which the 1-cells $\eta(a)$ and $\eta(b)$ are placed, and the direction of the 2-cell $\eta(X)$.

These must satisfy the following conditions.

- η commutes with 2-cells. For every 2-cell $f : X \to Y$ of 1-cells from a to b, the following square commutes.

$$\begin{tikzcd}
F(X) \odot \eta(b) \arrow[r, "F(f)\odot \mathrm{id}_{\eta(b)}"] \arrow[d, "\eta(X)"] & F(Y) \odot \eta(b) \arrow[d, "\eta(Y)"] \\
\eta(a) \odot G(X) \arrow[r, "\mathrm{id}_{\eta(a)} \odot G(f)"] & \eta(a) \odot G(Y)
\end{tikzcd}$$

- η commutes with m. For every triple of objects a, b, c and 1-cells $X \in \mathbf{C}(a,b)$ and $Y \in \mathbf{C}(b,c)$, the following octagon in the category $\mathbf{D}(Fa, Gc)$ commutes.

$$\begin{tikzcd}
F(X) \odot (F(Y) \odot \eta(c)) \arrow[d, "\mathrm{id}_{F(X)} \odot \eta(Y)"] & (F(X) \odot F(Y)) \odot \eta(c) \arrow[l, "\alpha(F(X),F(Y),\eta(c))"'] \arrow[r, "m \odot \mathrm{id}_{\eta(c)}"] & F(X \odot Y) \odot \eta(c) \arrow[ddd, "\eta(X \odot Y)"] \\
F(X) \odot (\eta(b) \odot G(Y)) & & \\
(F(X) \odot \eta(b)) \odot G(Y) \arrow[u, "\alpha(F(X),\eta(b),G(Y))"] \arrow[d, "\eta(X) \odot \mathrm{id}_{G(Y)}"] & & \\
(\eta(a) \odot G(X)) \odot G(Y) \arrow[r, "\alpha(\eta(a),G(X),G(Y))"] & \eta(a) \odot (G(X) \odot G(Y)) \arrow[r, "\mathrm{id}_{\eta(a)} \odot m"] & \eta(a) \odot G(X \odot Y)
\end{tikzcd}$$

- η commutes with i. For each a the following pentagon in the category $\mathbf{D}(Fa, Ga)$ commutes.

$$\begin{tikzcd}
I_{Fa} \odot \eta(a) \arrow[r, "l_{\eta(a)}"] \arrow[d, "i"] & \eta(a) & \eta(a) \odot I_{Ga} \arrow[l, "r_{\eta(a)}"'] \arrow[d, "i"'] \\
F(I_a) \odot \eta(a) \arrow[rr, "\eta(I_a)"] & & \eta(a) \odot G(I_a)
\end{tikzcd}$$

Discussion: The data here are actually rather different than that for a monoidal natural transformation, although the coherence axioms are the same. In a monoidal natural transformation there are no 1-cells chosen, and in addition the 2-cell from F to G need not be invertible. So in a pseudonatural transformation the "direction" is coming from the 1-cells, but in a monoidal natural transformation it's coming from the 2-cells.

This change reflects a shift in emphasis: before we had a single 0-cell, we focused on the functor's action on the 1-cells (which were just the objects), and we would relate these by a non-invertible 2-cell. Now, the focus is on the functor's action on the 0-cells, which need to be related by a 1-cell. In many examples the 2-cells in the definition are invertible, capturing the idea that the functor's action on 1-cells now commutes with this natural transformation on the 0-cells up to an isomorphism.

The coherence theorem for this says if you have a grid of composable maps, you get a well-defined map from $F \odot \eta$ or $\eta \odot G$ of the top to the same two functors on the bottom. If you drop the condition that the 2-cell for η is invertible then you have to assume its direction agrees with the vertical direction of the maps in your grid, and you still get the same conclusion. In the step where you apply F and G together, you make a horizontal barrier that is allowed to step over by one step, and you apply F to the left (and above the step) and G to the right (and below the step). You can't take any staircase through this grid that you want, because it would require you to horizontally compose multiple objects at once (although if you first combine a bunch of columns together, and then apply the one-step staircase, you can always re-interpret the staircase move as having applied a more elaborate staircase to the objects you have already tensored together).

12. Modification

A modification Γ from a pseudonatural transformation $\eta\colon F \to G$ to another pseudonatural transformation $\iota\colon F \to G$ consists of

- an assignment of each 0-cell a of $\mathbf{C}$ to a 2-cell $\Gamma(a)\colon \eta(a) \to \iota(a)$.

These must satisfy the following conditions.

- Γ matches together the actions of η and ι on 1-cells. For every 1-cell $X \in \mathbf{C}(a,b)$, the following square of 2-cells commutes.

$$\begin{tikzcd}
F(X) \odot \eta(b) \arrow[r, "\mathrm{id}_{F(X)} \odot \Gamma(b)"] \arrow[d, "\eta(X)"] & F(X) \odot \iota(b) \arrow[d, "\iota(X)"] \\
\eta(a) \odot G(X) \arrow[r, "\Gamma(a) \odot \mathrm{id}_{G(X)}"] & \iota(a) \odot G(X)
\end{tikzcd}$$

Discussion: A horizontal natural transformation is invertible if it has another one that is its inverse up to invertible modification. If a pseudofunctor has an inverse up to invertible horizontal natural transformation we say it's an equivalence of bicategories. If a pseudofunctor has an inverse up to vertical natural transformation we say it's a vertical equivalence or pointwise equivalence of bicategories. The distinction between these two parallels the distinction between Dwyer-Kan equivalences and pointwise weak equivalences of categories enriched in simplicial sets, spaces, or spectra.

13. Symmetric monoidal category

A symmetric monoidal category **C** consists of

- a monoidal category **C**,
- and a commutator natural isomorphism
$$\gamma(a,b) : (a \otimes b) \xrightarrow{\cong} (b \otimes a).$$

These must satisfy the following conditions.

- The following hexagon diagram commutes.
$$\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "\gamma(a,b)\otimes \mathrm{id}_c"] \arrow[d, "\alpha(a,b,c)"] & (b \otimes a) \otimes c \arrow[r, "\alpha(b,a,c)"] & b \otimes (a \otimes c) \arrow[d, "\mathrm{id}_b \otimes \gamma(a,c)"] \\
a \otimes (b \otimes c) \arrow[r, "\gamma(a,b\otimes c)"] & (b \otimes c) \otimes a \arrow[r, "\alpha(b,c,a)"] & b \otimes (c \otimes a)
\end{tikzcd}$$
- The commutator is an involution in the sense that $\gamma(b,a) \circ \gamma(a,b) = \mathrm{id}_{a\otimes b}$.

Equivalently, every diagram made from α, γ, l, r, and tensor products, commutes. (Note: In many sources there is also an axiom that the left and right units agree along the commutator map. This follows from the coherences above, by a theorem of Kelly. But it is perhaps necessary when one considers the more general notion of a braided monoidal category.)

14. Strong symmetric monoidal functor

A strong symmetric monoidal functor F from a symmetric monoidal category **C** to a symmetric monoidal category **D** consists of

- a strong monoidal functor $F : \mathbf{C} \to \mathbf{D}$.

These must satisfy the following conditions.

- F respects the symmetry isomorphisms. The following square diagram commutes.
$$\begin{tikzcd}
F(x) \otimes F(y) \arrow[r, "\gamma(F(x),F(y))"] \arrow[d, "m(x,y)"] & F(y) \otimes F(x) \arrow[d, "m(y,x)"] \\
F(x \otimes y) \arrow[r, "F(\gamma(x,y))"] & F(y \otimes x)
\end{tikzcd}$$

15. Symmetric monoidal natural transformation

A symmetric monoidal natural transformation η from a symmetric monoidal functor $F : \mathbf{C} \to \mathbf{D}$ to a symmetric monoidal functor $G : \mathbf{C} \to \mathbf{D}$ consists of

- a monoidal natural transformation η from F to G

There are no extra conditions to satisfy.

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