BICATEGORIES, PSEUDOFUNCTORS, SHADOWS: A CHEAT SHEET

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We give in tabular form the definition of a symmetric monoidal category, a bicategory, a pseudofunctor, a shadowed bicategory, and other related concepts. There is something of a general pattern to these definitions, that is easiest to see when they're written out this way. This point of view was helpful when the author was trying to learn these concepts, so hopefully it will be helpful for other people as well.

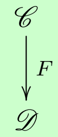
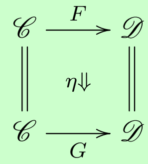
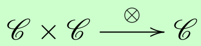
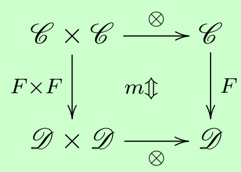
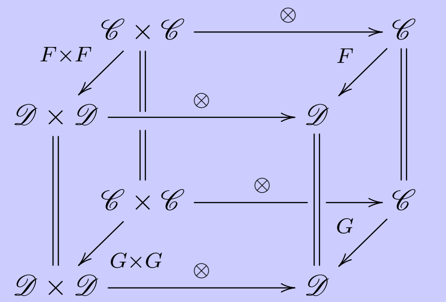
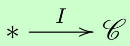
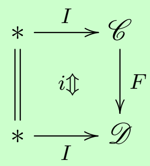
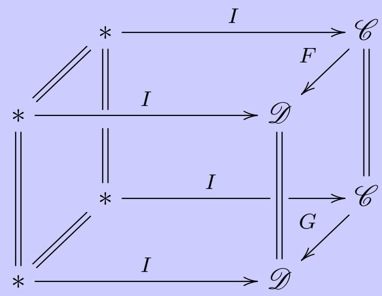
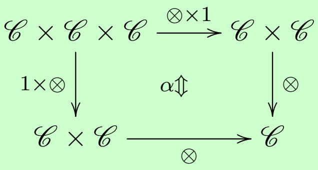
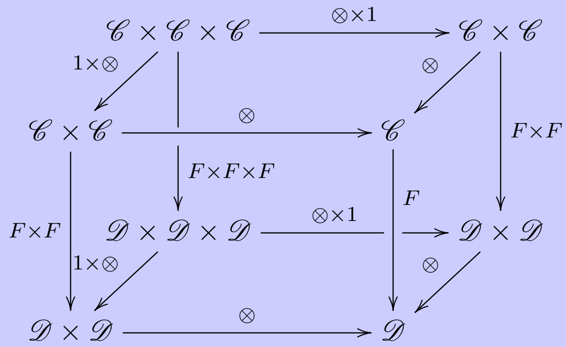
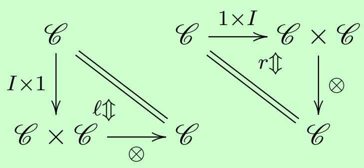
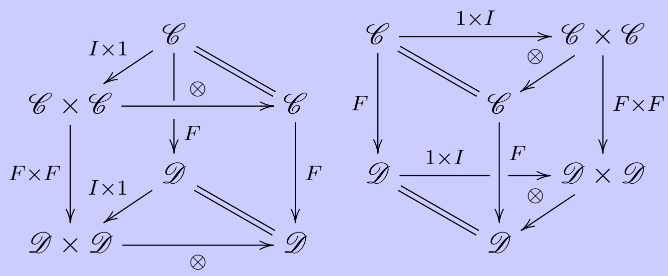
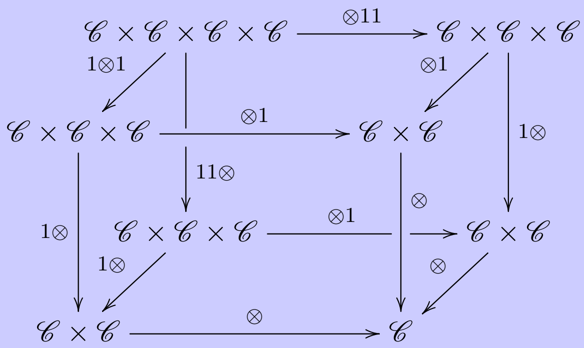
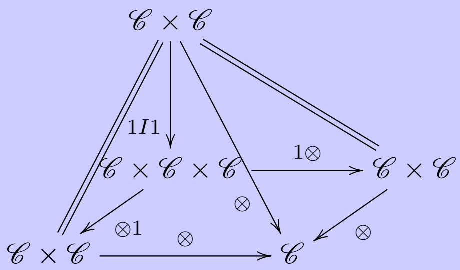
Categories

category $\mathscr{C}$	functor $F\colon \mathscr{C} \to \mathscr{D}$	equality of functors $F = G$	natural transformation $\eta\colon F \Rightarrow G$
set $\mathrm{ob}\mathscr{C}$	function $\begin{array}{c} \mathrm{ob}\mathscr{C} \\ \downarrow{\scriptstyle F} \\ \mathrm{ob}\mathscr{D} \end{array}$	equality $\begin{array}{ccc} \mathrm{ob}\mathscr{C} & \xrightarrow{F} & \mathrm{ob}\mathscr{D} \\ \Vert & & \Vert \\ \mathrm{ob}\mathscr{C} & \xrightarrow[G]{} & \mathrm{ob}\mathscr{D} \end{array}$	elements $* \xrightarrow{\eta(a)} \mathscr{D}(Fa, Ga)$
sets $\mathscr{C}(a,b)$	functions $\begin{array}{c} \mathscr{C}(a,b) \\ \downarrow{\scriptstyle F} \\ \mathscr{D}(Fa,Fb) \end{array}$	equalities $\begin{array}{ccc} \mathscr{C}(a,b) & \xrightarrow{F} & \mathscr{D}(Fa,Fb) \\ \Vert & & \Vert \\ \mathscr{C}(a,b) & \xrightarrow[G]{} & \mathscr{D}(Ga,Gb) \end{array}$	commuting squares $\begin{array}{ccc} \mathscr{C}(a,b) & \xrightarrow{F \times \eta(b)} & \mathscr{D}(Fa,Fb) \times \mathscr{D}(Fb,Gb) \\ {\scriptstyle \eta(a) \times G}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathscr{D}(Fa,Ga) \times \mathscr{D}(Ga,Gb) & \xrightarrow[\circ]{} & \mathscr{D}(Fa,Gb) \end{array}$
composition functions $\mathscr{C}(a,b) \times \mathscr{C}(b,c) \xrightarrow{\circ} \mathscr{C}(a,c)$	preserves composition $\begin{array}{ccc} \mathscr{C}(a,b) \times \mathscr{C}(b,c) & \xrightarrow{\circ} & \mathscr{C}(a,c) \\ {\scriptstyle F \times F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathscr{D}(Fa,Fb) \times \mathscr{D}(Fb,Fc) & \xrightarrow[\circ]{} & \mathscr{D}(Fa,Fc) \end{array}$		
units $* \xrightarrow{\mathrm{id}_a} \mathscr{C}(a,a)$	preserves units $\begin{array}{ccc} * & \xrightarrow{\mathrm{id}_a} & \mathscr{C}(a,a) \\ \Vert & & \downarrow{\scriptstyle F} \\ * & \xrightarrow[\mathrm{id}_{Fa}]{} & \mathscr{D}(Fa,Fa) \end{array}$		
associativity $\begin{array}{ccc} \mathscr{C}(a,b) \times \mathscr{C}(b,c) \times \mathscr{C}(c,d) & \xrightarrow{\circ \times 1} & \mathscr{C}(a,c) \times \mathscr{C}(c,d) \\ {\scriptstyle 1 \times \circ}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathscr{C}(a,b) \times \mathscr{C}(b,d) & \xrightarrow[\circ]{} & \mathscr{C}(a,d) \end{array}$			
unitality $\begin{array}{ccc} \mathscr{C}(a,b) & & \\ {\scriptstyle \mathrm{id}_a \times 1}\downarrow & \searrow\!\!\!\!= & \\ \mathscr{C}(a,a) \times \mathscr{C}(a,b) & \xrightarrow[\circ]{} & \mathscr{C}(a,b) \end{array}$ $\begin{array}{ccc} \mathscr{C}(a,b) & \xrightarrow{1 \times \mathrm{id}_b} & \mathscr{C}(a,b) \times \mathscr{C}(b,b) \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \circ} \\ & & \mathscr{C}(a,b) \end{array}$			

Light green cells are data, light blue cells are conditions.

Notice the “extruding” pattern that brings us from each column to the next. The dimension increases by one each time. When the diagram becomes 2-dimensional, it becomes a condition, and then disappears from the subsequent column (because the obvious next diagram is actually a vacuous condition). Unfortunately this pattern does not continue to the last column (natural transformations).

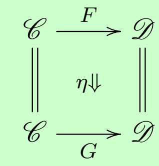
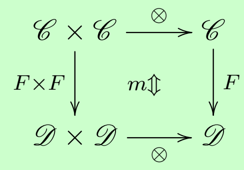
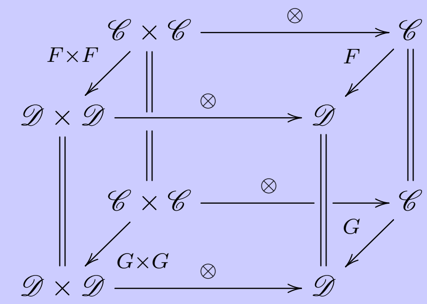
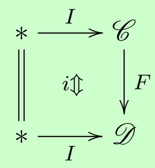
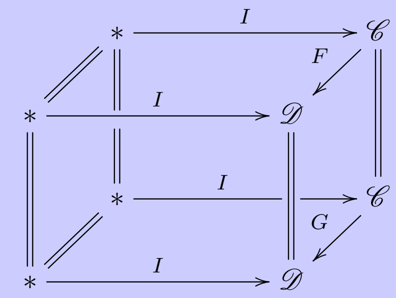
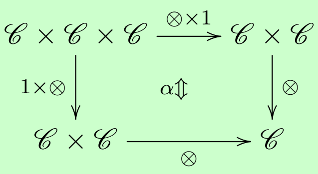
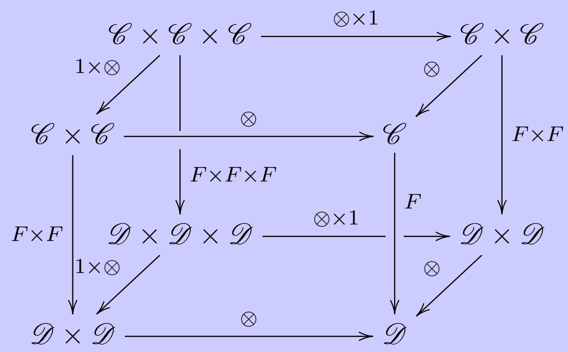
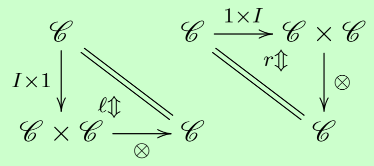
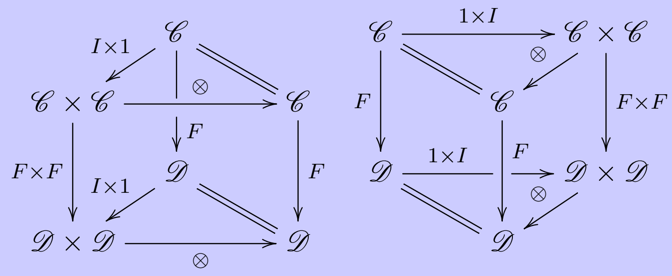
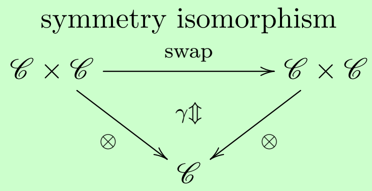
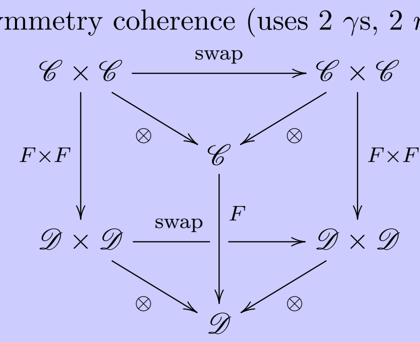
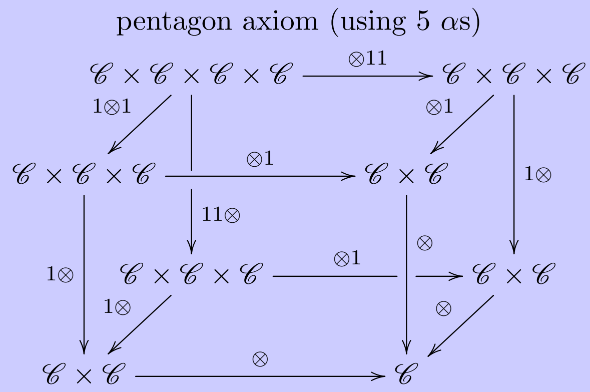
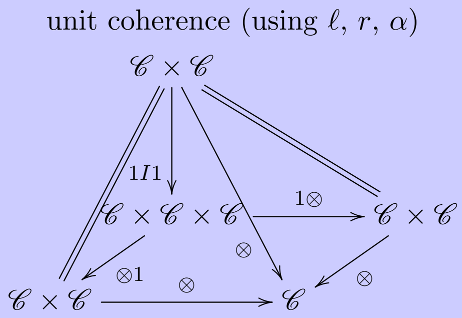
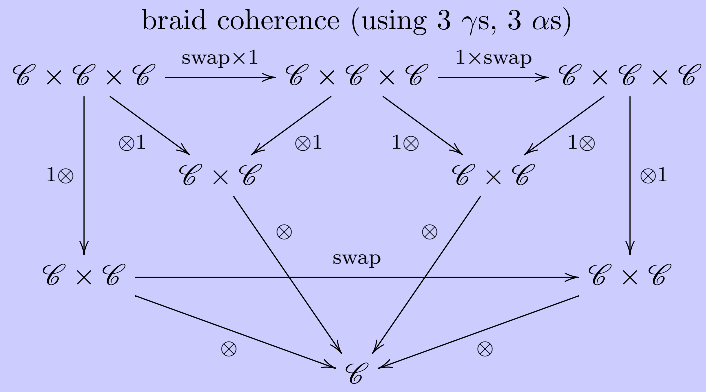
Monoidal categories

monoidal category $\mathscr{C}$	strong monoidal functor $F\colon \mathscr{C} \to \mathscr{D}$	monoidal natural transformation $\eta\colon F \Rightarrow G$	nothing
nothing	nothing	nothing	
category $\mathscr{C}$	functor 	natural transformation 	
tensor product 	tensor isomorphism 	respects m (uses η, $\eta \times \eta$, and 2 ms) 	
unit 	unit isomorphism 	respects i (uses η and 2 is) 	
associator 	associator coherence (uses 2 αs, 4 ms) 		
unitors 	unitor coherence (each uses i, m, and 2 ℓs or 2 rs) 		
pentagon axiom (using 5 αs) 			
unit coherence (using ℓ, r, α) 			

This looks like the definition of a category, except that the dimension has increased, because we are using categories in the place where we used sets before. Every time we had a condition that a square commutes, we now get a natural isomorphism making that square commute. So the old conditions have become data, and the coherence of that data is captured by new conditions.

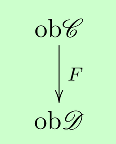
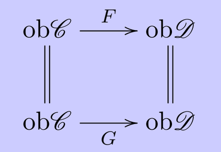
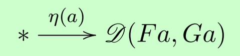
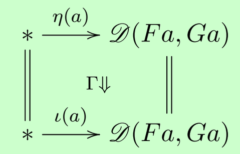
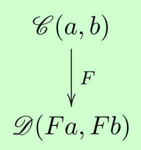
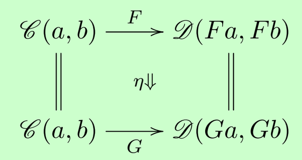
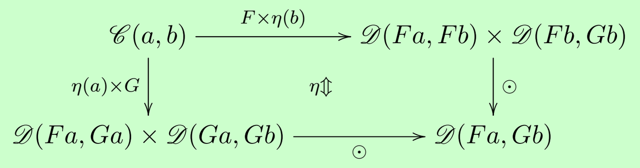
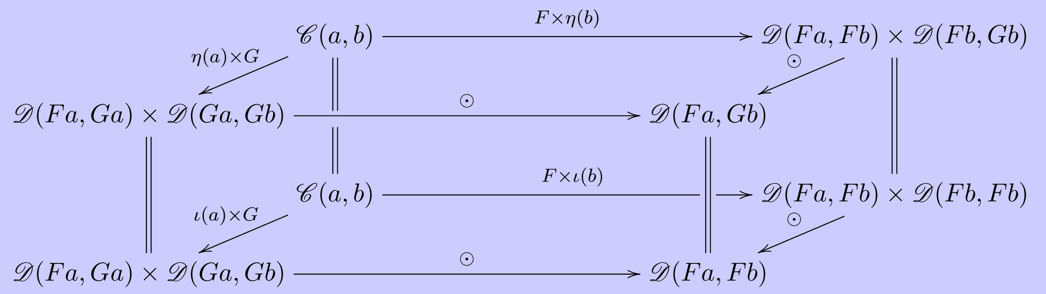
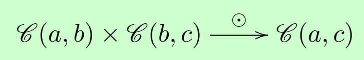
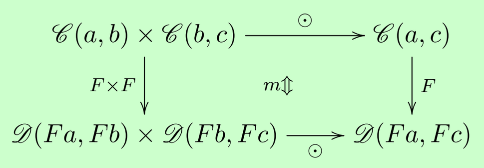
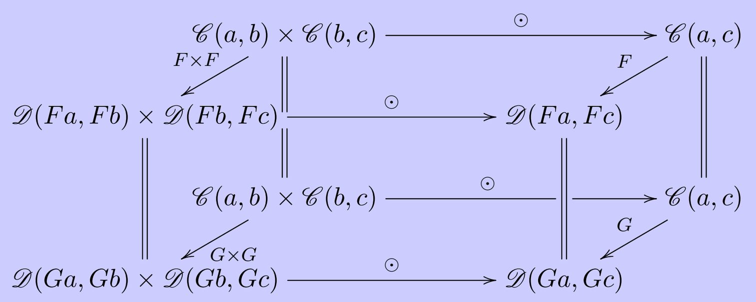
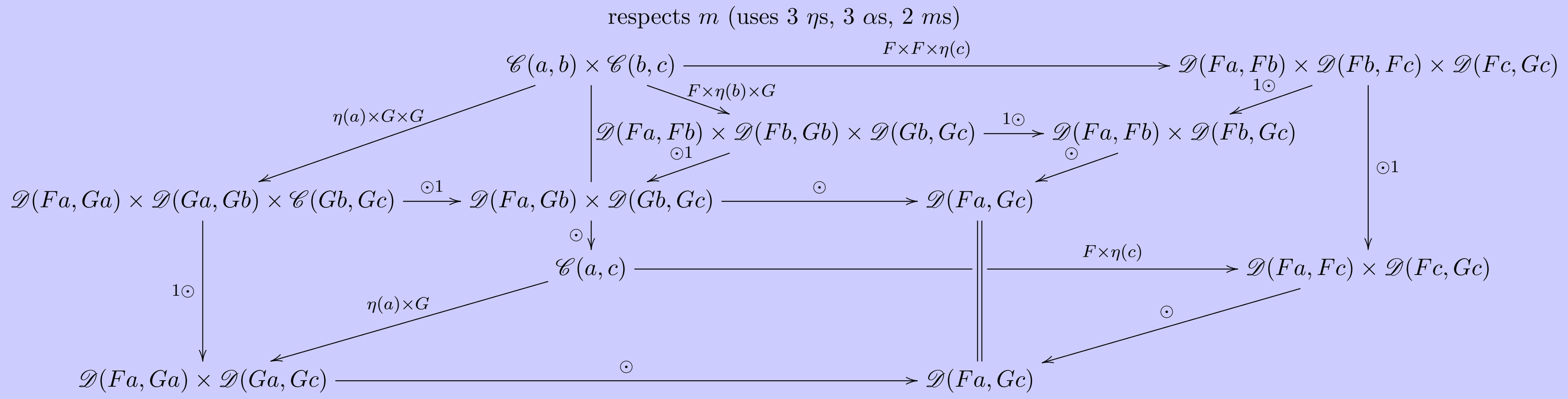
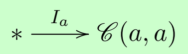
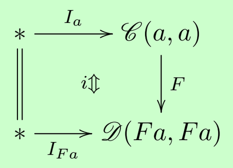
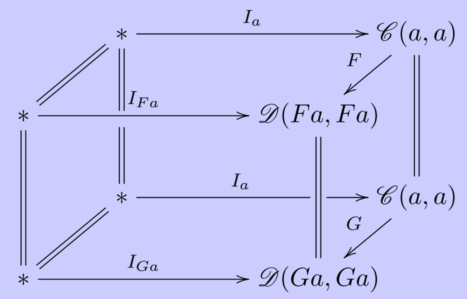
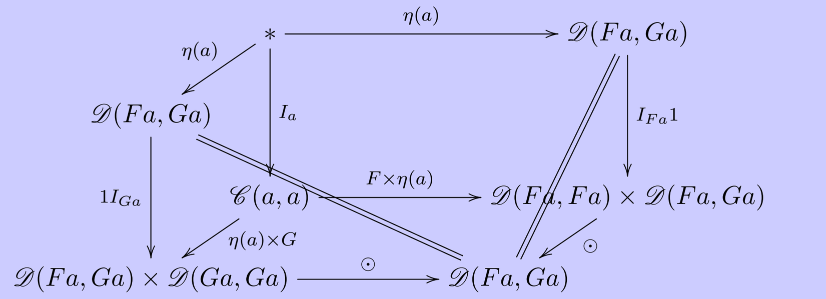
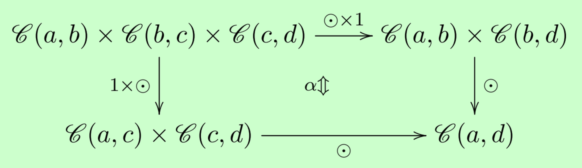
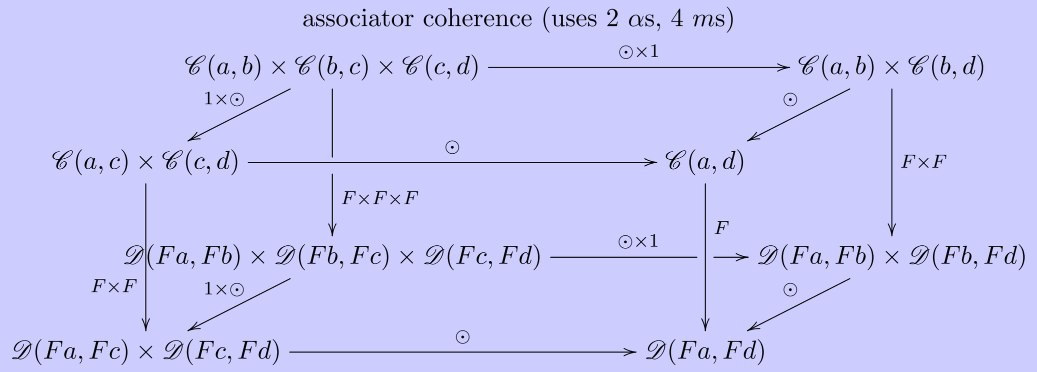
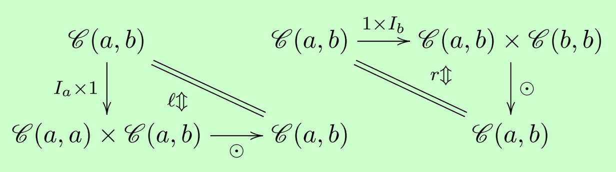
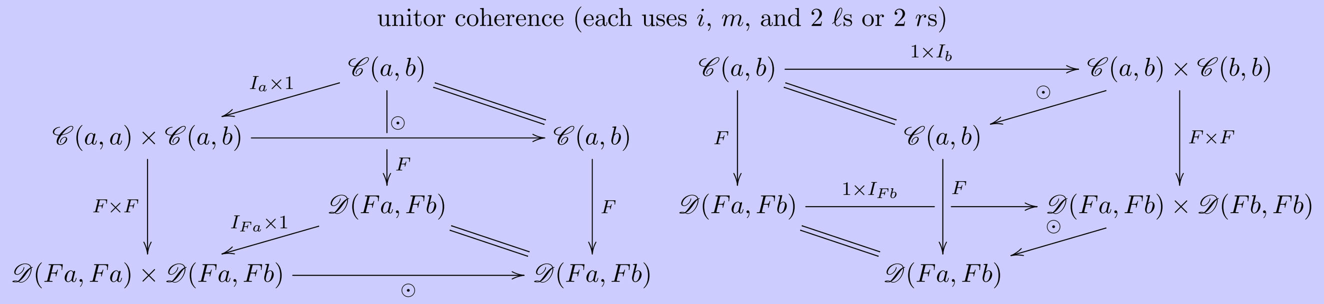
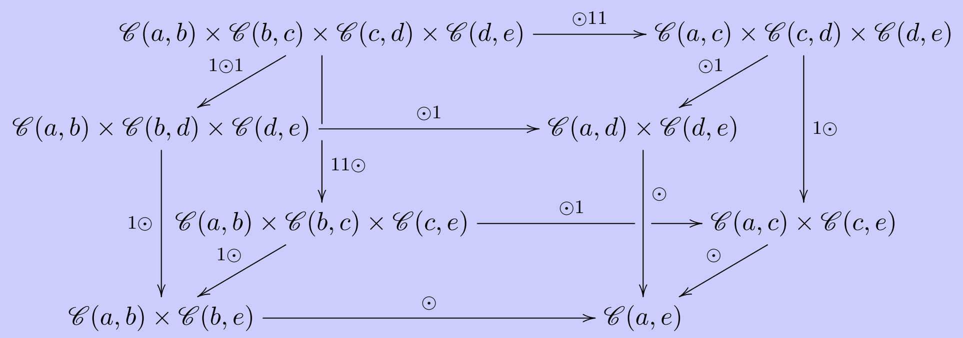
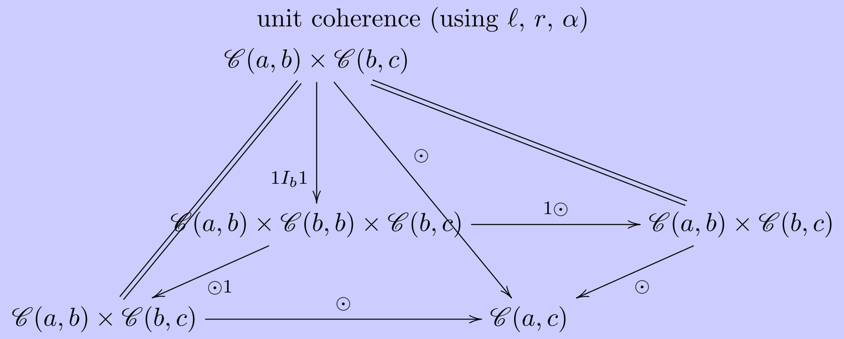
The "extruding" pattern is present throughout this table. This gives a helpful mnemonic: to get the definition of a monoidal functor from the definition of a monoidal category, just multiply by an interval. Another way of saying this, is that in each entry we get the product of a diagram from the left-hand column and a diagram from the top row.

Symmetric monoidal categories

symmetric monoidal category $\mathscr{C}$	strong symmetric monoidal functor $F\colon \mathscr{C} \to \mathscr{D}$	monoidal natural transformation $\eta\colon F \Rightarrow G$	nothing
nothing	nothing	nothing	
category $\mathscr{C}$	functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$	natural transformation 	
tensor product $\mathscr{C}\times\mathscr{C} \xrightarrow{\otimes} \mathscr{C}$	tensor isomorphism 	respects m (uses η, $\eta\times\eta$, and 2 ms) 	
unit $* \xrightarrow{I} \mathscr{C}$	unit isomorphism 	respects i (uses η and 2 is) 	
associator 	associator coherence (uses 2 αs, 4 ms) 		
unitors 	unitor coherence (each uses i, m, and 2 ℓs or 2 rs) 		
symmetry isomorphism 	symmetry coherence (uses 2 γs, 2 ms) 		
pentagon axiom (using 5 αs) 			
unit coherence (using ℓ, r, α) 			
braid coherence (using 3 γs, 3 αs) 			
reflexivity coherence: $\theta^2 = 1$			

This is identical to the table for monoidal categories, except there is one more row for the symmetry isomorphism, and two more coherences. We get braided versions of these definitions as well by dropping the condition $\theta^2 = 1$ and inserting the coherence that the braiding takes the left unit to the right unit.

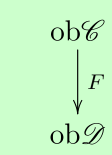
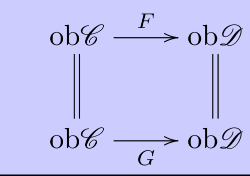
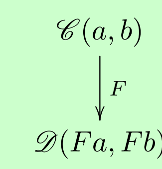
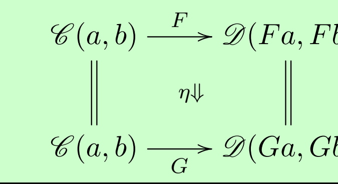
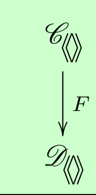
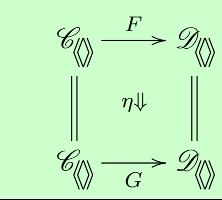
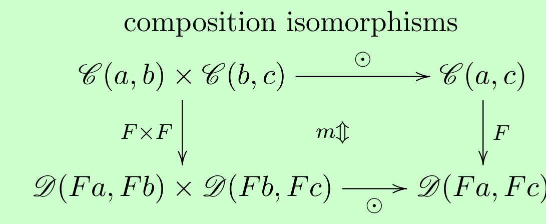
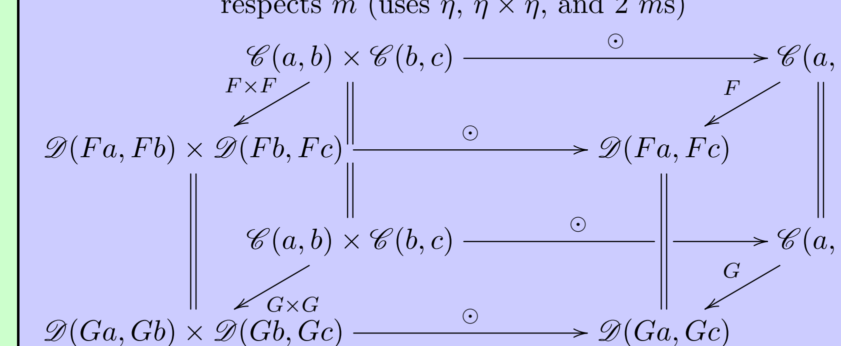
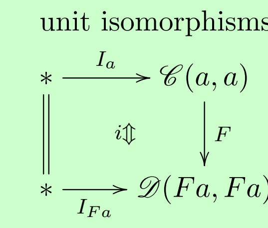
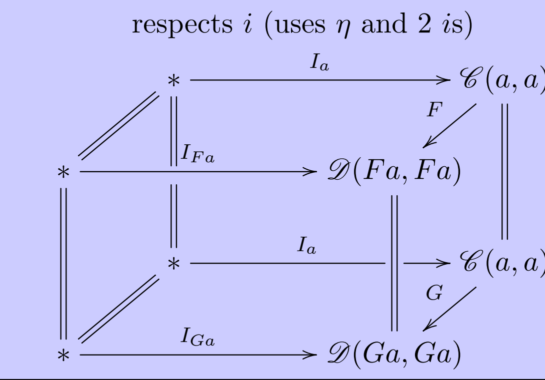
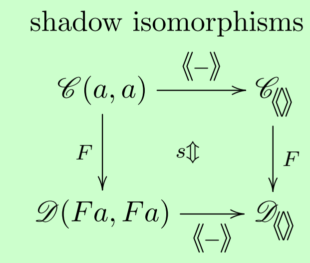
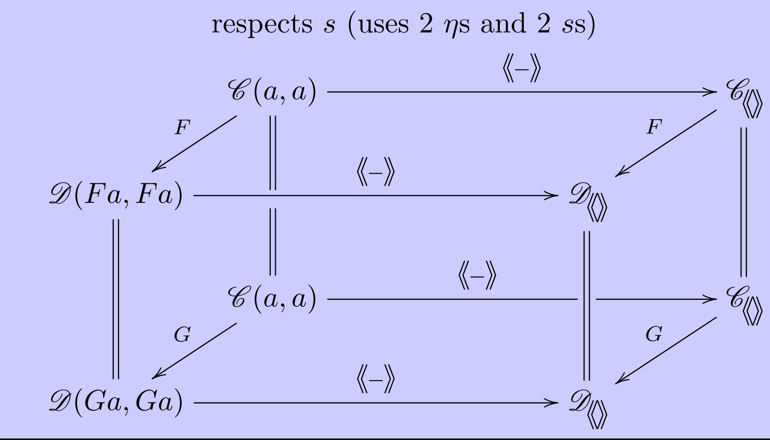
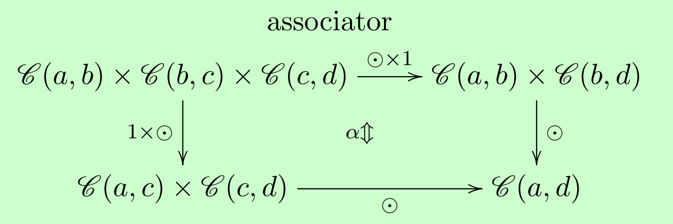
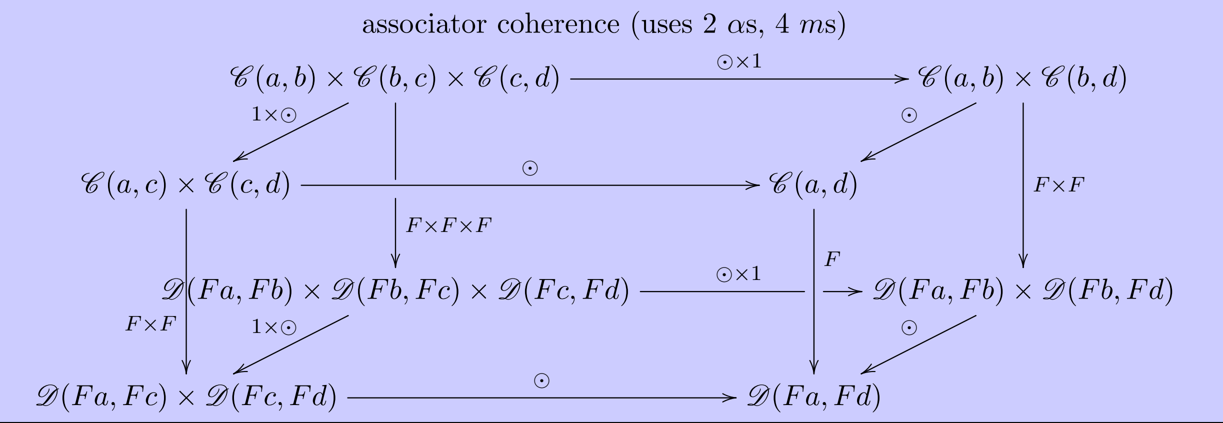
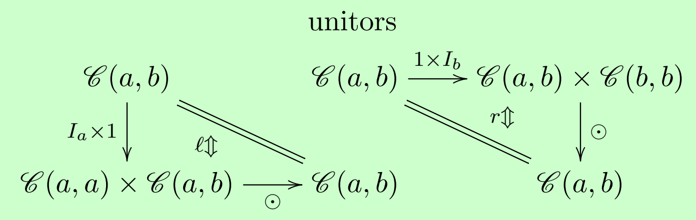
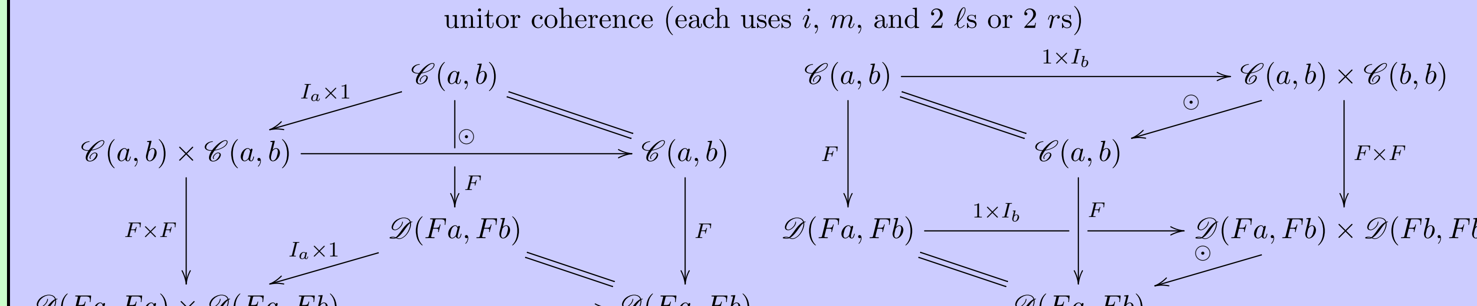
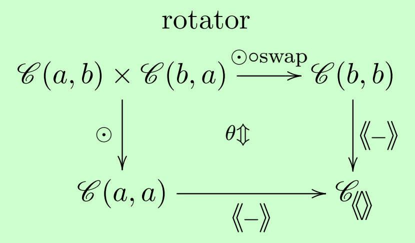
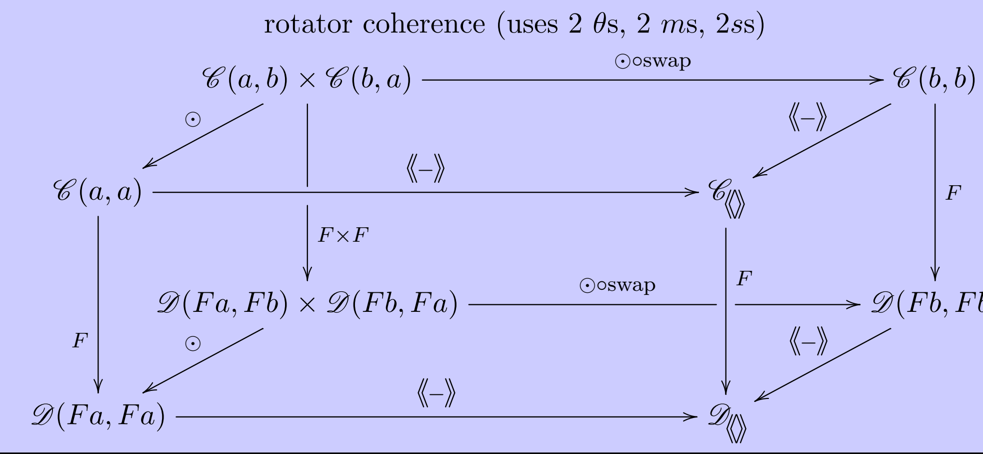
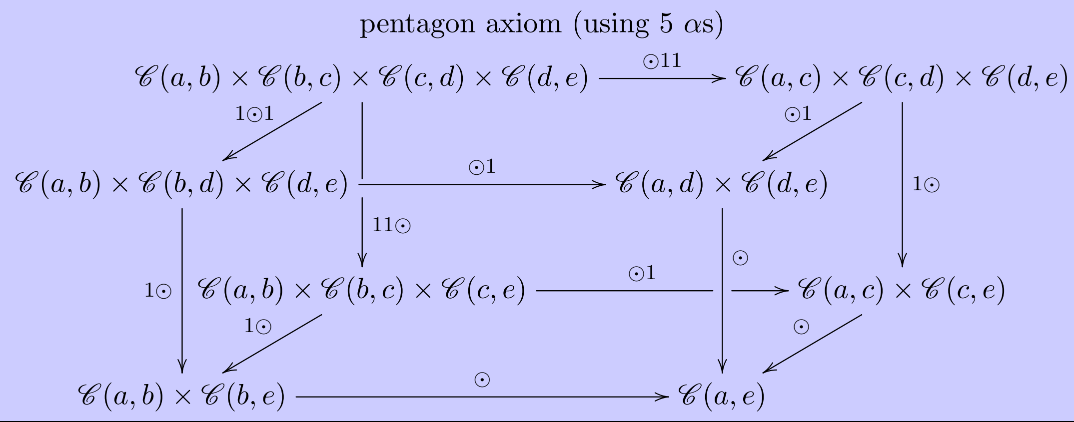
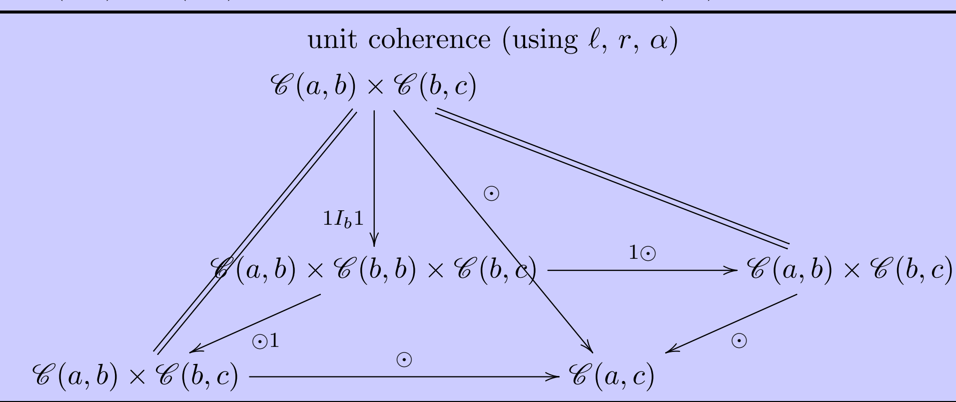
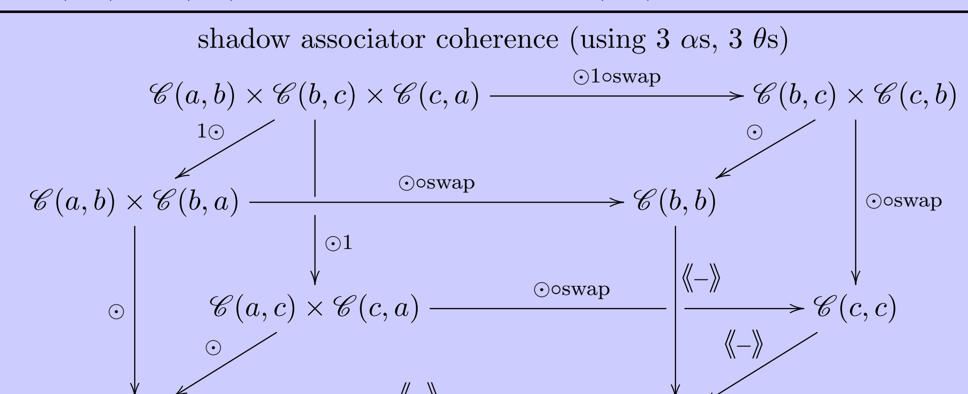
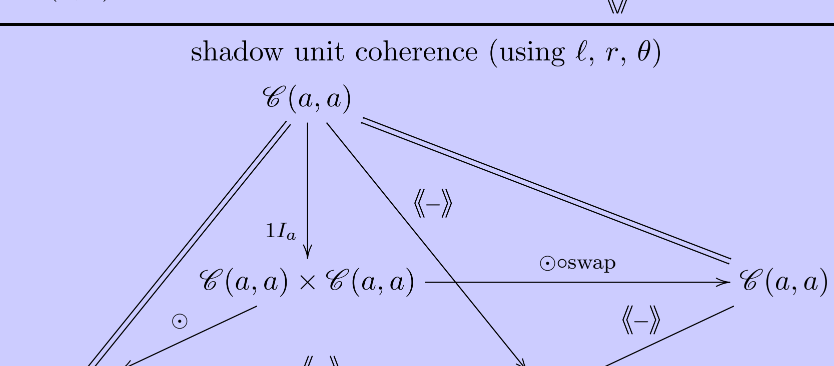
Bicategories

bicategory $\mathscr{C}$	pseudofunctor $F\colon \mathscr{C} \to \mathscr{D}$	icon (vertical natural transformation) $\eta\colon F \Rightarrow G$	pseudonatural transformation (horizontal natural transformation) $\eta\colon F \Rightarrow G$	modification $\Gamma\colon \eta \to \iota$
set $\mathrm{ob}\mathscr{C}$	function 	equality 	functors 	natural transformations 
categories $\mathscr{C}(a,b)$	functors 	natural transformations 	natural isomorphisms 	respects η (uses 2 ηs, 2 Γs) 
horizontal composition 	composition isomorphisms 	respects m (uses η, $\eta\times\eta$, and 2 ms) 	respects m (uses 3 ηs, 3 αs, 2 ms) 	
units 	unit isomorphisms 	respects i (uses η and 2 is) 	respects i (uses η, ℓ, r, and 2 is) 	
associator 	associator coherence (uses 2 αs, 4 ms) 			
unitors 	unitor coherence (each uses i, m, and 2 ℓs or 2 rs) 			
pentagon axiom (using 5 αs, the back face is the identity transformation) 				
unit coherence (using ℓ, r, α) 				

A bicategory is a monoidal category on many objects. You can see this by ignoring the top row of this table and comparing the rest of the first three columns to the table on monoidal categories. Therefore a vertical natural transformation is the natural generalization of a monoidal functor.

Horizontal natural transformations are different from these. They behave more like the natural transformations back in the table on categories, because they transform the 0-cells "along 1-cells." Once we have that definition, we can extrude it again and get the definition of a modification.

Bicategories with shadow

bicategory with shadow $\mathscr{C}$	strong shadow functor $F\colon \mathscr{C}\to\mathscr{D}$	shadow icon (vertical shadow natural transformation) $\eta\colon F\Rightarrow G$
set $\mathrm{ob}\mathscr{C}$	function 	equality 
categories $\mathscr{C}(a,b)$	functors 	natural transformations 
category $\mathscr{C}_{\langle\!\langle\rangle\!\rangle}$	functor 	natural transformation 
horizontal composition $\mathscr{C}(a,b)\times\mathscr{C}(b,c)\xrightarrow{\odot}\mathscr{C}(a,c)$	composition isomorphisms 	respects m (uses η, $\eta\times\eta$, and 2 ms) 
units $*\xrightarrow{I_a}\mathscr{C}(a,a)$	unit isomorphisms 	respects i (uses η and 2 is) 
shadows $\mathscr{C}(a,a)\xrightarrow{\langle\!\langle-\rangle\!\rangle}\mathscr{C}_{\langle\!\langle\rangle\!\rangle}$	shadow isomorphisms 	respects s (uses 2 ηs and 2 ss) 
associator 	associator coherence (uses 2 αs, 4 ms) 	
unitors 	unitor coherence (each uses i, m, and 2 ℓs or 2 rs) 	
rotator 	rotator coherence (uses 2 θs, 2 ms, 2ss) 	
pentagon axiom (using 5 αs) 		
unit coherence (using ℓ, r, α) 		
shadow associator coherence (using 3 αs, 3 θs) 		
shadow unit coherence (using ℓ, r, θ) 		

This is just the definition of a bicategory, except there is one more row for the shadow category, one for the shadow functor, one for the rotator isomorphism, and two more coherence conditions. The new coherence conditions mirror the original two, but with rotators in the place of associators. One would anticipate that there is an analogous definition for horizontal natural transformations and modifications here, but it doesn't seem to have been written down yet.