COMPARING A CELL COMPLEX TO A COLIMIT OF SUBCOMPLEXES

CARY MALKIEWICH

Suppose X is a cell complex, and $\{X_i\}$ is a collection of subcomplexes indexed by a set $\mathbf{I}$. The set $\mathbf{I}$ has a partial ordering given by $i < j$ if $X_i \subseteq X_j$. We therefore have comparison maps from the colimit and homotopy colimit of these subcomplexes back to X:

$$\operatorname*{colim}_{i \in \mathbf{I}} X_i \to X, \tag{0.1}$$

$$\operatorname*{hocolim}_{i \in \mathbf{I}} X_i \to X. \tag{0.2}$$

In this note, we address the following question. When is (0.1) a homeomorphism, and when is (0.2) a homotopy equivalence?

For each cell $D_\alpha \to X$, let $\mathbf{I}_\alpha \subseteq \mathbf{I}$ be the subset of those subcomplexes X_i that contain D_α. This is also a poset, therefore a category, and we can ask whether it is connected or contractible.

Proposition 0.3. *The poset $\mathbf{I}_\alpha$ is connected for every α, iff (0.1) is a bijection, iff (0.1) is a homeomorphism.*

Proof. The map (0.1) is a bijection iff it's a homeomorphism, because the colimit of the X_i also has the cellular topology: a map out is continuous iff it is continuous when restricted to each cell in each X_i.

Clearly the map is surjective iff every point in the interior of each cell of X is in the image, which happens precisely when each $\mathbf{I}_\alpha$ is nonempty. On the other hand, injectivity happens precisely when each representative in the colimit of a point in X can be joined to each other representative. Since the maps of the diagram are the identity inside the cell D_α, this happens iff $\mathbf{I}_\alpha$ is connected. □

Proposition 0.4. *If the space $B\mathbf{I}_\alpha$ is contractible for every α, then (0.2) is a homotopy equivalence.*

Proof. By the Whitehead theorem it suffices to prove it is a weak equivalence. Regard X as a transfinite sequential colimit of maps that attach a single cell D_α. For each skeleton $X^{(\alpha)}$, the intersections

$$X_i^{(\alpha)} = X_i \cap X^{(\alpha)}$$

form a diagram indexed by $\mathbf{I}$. So we get a transfinite sequential filtration of both sides of (0.2):

$$\operatorname*{hocolim}_{i \in \mathbf{I}} X_i^{(\alpha)} \to X^{(\alpha)}. \tag{0.5}$$

We prove that (0.5) is an equivalence for all α by induction on α. When α is a successor ordinal, the left-hand side changes by the homotopy pushout

$$\begin{array}{ccc} (\partial D_\alpha) \times B\mathbf{I}_\alpha & \longrightarrow & D_\alpha \times B\mathbf{I}_\alpha \\ \downarrow & & \downarrow \\ \operatorname*{hocolim}_{i \in \mathbf{I}} X_i^{(\alpha-1)} & \longrightarrow & \operatorname*{hocolim}_{i \in \mathbf{I}} X_i^{(\alpha)} \end{array}$$

because we get a new cell in the homotopy colimit for every k-tuple of composable morphisms in $\mathbf{I}_\alpha$, which all together give $D_\alpha \times B\mathbf{I}_\alpha$. This maps to the corresponding pushout for the right-hand side

$$\begin{array}{ccc} (\partial D_\alpha) & \longrightarrow & D_\alpha \\ \downarrow & & \downarrow \\ X^{(\alpha-1)} & \longrightarrow & X^{(\alpha)} \end{array}$$

by collapsing the copies of $B\mathbf{I}_\alpha$ to a point. The map is an equivalence on the lower-left term of each square by inductive hypothesis, and an equivalence on the terms in the top row because $B\mathbf{I}_\alpha$ is contractible. Therefore it gives an equivalence on the lower-right terms as well.

When α is a limit ordinal, (0.5) is a transfinite sequential colimit along closed inclusions of a system of maps that are all equivalences, and therefore (0.5) is an equivalence as well. This completes the induction. □

For a poset to be contractible, it is enough for the poset (or its opposite) to be filtered. We therefore get a convenient corollary:

Corollary 0.6. *If the subcomplexes $\{X_i\}$ are closed under either pairwise intersection, or pairwise union, then (0.1) is a homeomorphism and (0.2) is a homotopy equivalence.*

Department of Mathematics, Binghamton University

Email address: `malkiewich@math.binghamton.edu`