FIBRATION SEQUENCES AND PULLBACK SQUARES

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ABSTRACT. We lay out some foundational facts about fibration sequences and pullback squares of topological spaces. We pay careful attention to connectivity ranges and to basepoint issues. All of these results are probably well-known.

Contents

Fibration sequences and pullback squares are some of the most basic and frequently used tools of homotopy theory, but their behavior is somewhat tricky to nail down when you're trying to be careful with basepoints, or when you are only studying maps that are an equivalence in a certain range. In this note we'll attempt to give a complete treatment of the core technical results for these constructions.

1. Fibration sequences: definitions and warnings.

Recall the following definition. Given a map of unbased spaces $E \longrightarrow B$ and a choice of point $b \in B$, the *homotopy fiber* hF_b is the pullback

$$\begin{tikzcd} hF_b \arrow[r] \arrow[d] & E \arrow[d] \\ (B,b)^{(I,0)} \arrow[r, "\mathrm{ev}_1"] & B \end{tikzcd}$$

Definition 1.1. A *fibration sequence* or *fiber sequence* is a sequence of maps of based spaces

$$F \xrightarrow{i} E \xrightarrow{p} B$$

together with one of the following extra sets of data:

- A zig-zag of weak equivalences to some other sequence in which the map p is a Serre fibration and i is the inclusion of the fiber over the basepoint.
- A choice of equivalence in the homotopy category of spaces over E between F and the homotopy fiber of p at the basepoint.
- A choice of nullhomotopy for $p \circ i$, possibly after cofibrant replacement of F and fibrant replacement of E and B, yielding a map $F \longrightarrow hF$ which is a weak equivalence.

A *map of fibration sequences* is a commuting diagram

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r, "i'"] & E' \arrow[r, "p'"] & B' \end{tikzcd}$$

which agrees with the extra data described above.

It is not hard to see that these conditions are equivalent, and by standard homotopy theory they each yield the long exact sequence of homotopy sets

$$\ldots \longrightarrow \pi_{i+1}(B,*) \longrightarrow \pi_i(F,*) \longrightarrow \pi_i(E,*) \longrightarrow \pi_i(B,*) \longrightarrow \pi_{i-1}(F,*) \longrightarrow \ldots$$

At the end of this sequence we get the map of sets $\pi_0 E \longrightarrow \pi_0 B$ which need not be surjective.

If p actually is a Serre fibration, and i is actually the inclusion of the fiber, there is a canonical choice for each of these three kinds of data. Furthermore, any commuting diagram as above forms a map between two such fibration sequences. In this lucky situation, one can speak of the "fibration sequence" property as a condition, not extra data. However this condition is not invariant under weak equivalence. Once we move away from point-set fibrations, it is essential to remember the extra data.

As a cautionary example, the well-known "fiber sequence"

$$\Omega X \longrightarrow * \longrightarrow X$$

is not actually a fiber sequence in our sense, unless we supply extra data. It seems sloppy to call this a fiber sequence. Rather, one should use the path space

$$\Omega X \longrightarrow PX \longrightarrow X$$

or remember a choice of equivalence between ΩX and the homotopy fiber of $* \longrightarrow X$. This is more than just pendantry. In the following diagram

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d, "0"] & * \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\ \Omega X \arrow[r] & PX \arrow[r] & X \end{tikzcd}$$

every square commutes, the rows are fiber sequences, and the middle and right verticals are equivalences; however the left vertical is NOT an equivalence. This is why a map of fiber sequences has to be more than just a couple of commuting squares!

Remark. In the *stable* setting, the extra data can be rewritten as the third map in a distinguished triangle. Of course, a map of triangles has to give three commuting squares, not just two. The homotopy fiber is also characterized by the fact that it fits into such a triangle, so any two candidates that do this are equivalent, but the choice of equivalence between them is not unique.

So far, we have assumed that all of our spaces are based. If B has multiple path components, then each one can have a different homotopy fiber. Unfortunately, our fiber sequence only tells us about the behavior of E over the basepoint component of B. The other components could have very different behavior. Some of them may even have empty preimage in E. This makes it hard to use F to go from B to E, or from E to B. As a second cautionary example, if X is any unbased space, the sequence $* \longrightarrow * \longrightarrow X_+$ is a fibration sequence!

It's tempting to amend the definition of fiber sequence, requiring further that $\pi_0 E \longrightarrow \pi_0 B$ is surjective. But that doesn't solve our problems. We still can't control the homotopy type of E over the non-basepoint components of B. It's also problematic because we want the dual of a cofibration sequence to be a fibration sequence, but very often the fibration sequence we get does not end with a π_0-surjection.

2. Connectivity and fiber sequences.

Next we recall the notion of connectedness, both for maps and for spaces.

Definition 2.1.

- A map of unbased spaces $A \longrightarrow X$ is 0-*connected* when it induces a surjection on π_0.
- The map $A \longrightarrow X$ is n-*connected* for some $n > 0$ when it induces an isomorphism on π_0, and on each component of A (or equivalently X) it induces an isomorphism on π_i when $1 \leq i < n$ and a surjection on π_n.
- A map that is n-connected for all n is a *weak equivalence.*
- A based or unbased space X is n-*connected* when it is path-connected and all homotopy groups vanish through degree n. Equivalently, any map $* \longrightarrow X$ is n-connected.
- By convention, all nonempty spaces are (-1)-connected, and the empty space $\emptyset$ is (-2)-connected. We do not define negative connectivity for maps. We say that $\pi_0(\emptyset) = \emptyset$, and all higher homotopy groups are undefined, since they rely on a choice of basepoint. Therefore $\emptyset \longrightarrow \emptyset$ is a weak equivalence. On the other hand, $\emptyset \longrightarrow X$ is not even 0-connected when X is nonempty.

Of course, n-connected is always stronger than $(n-1)$-connected, regardless of whether we are talking about maps or about spaces. We will also list the standard theorems on connectivity without proof:

Proposition 2.2. *A map $A \longrightarrow X$ is n-connected iff it can be factored into an inclusion $A \longrightarrow X'$ of a relative CW complex with cells in dimension $n+1$ and higher, followed by a weak equivalence $X' \longrightarrow X$.*

Proposition 2.3. *A map $A \longrightarrow X$ is n-connected iff every square*

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & A \arrow[d] \\ D^k \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

has a lift up to a homotopy of pairs $((D^k, S^{k-1}), (X, A))$, for all $0 \leq k \leq n$. If $A \longrightarrow X$ is a Serre fibration, then it is n-connected iff such lifts exist on the nose for all $0 \leq k \leq n$.

It is easy to relate the connectivity of a map to that of its homotopy fibers.

Proposition 2.4. *The following are equivalent for a map $E \xrightarrow{p} B$ and integer $n \geq 0$.*

- *p is n-connected.*
- *Every homotopy fiber of p is $(n-1)$-connected.*
- *In each path component of B, the homotopy fiber over some point b is $(n-1)$-connected.*

Proof. Use the long exact sequence of the fibration, for one point $b \in B$ in each path component of B. We see that 0-connectedness is equivalent to the fibers being nonempty (i.e. -1-connected), and that higher connectedness can be read off directly from the homotopy groups of the fibers. □

The next proposition has an easy proof, because if the space B is highly connected, we do not have to worry about multiple path components.

Proposition 2.5. *Given a fiber sequence*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

and integer $k \geq 0$,

- *If E is k-connected and B is $(k+1)$-connected then F is k-connected.*
- *If B is k-connected and F is k-connected then E is k-connected.*
- *If F is $(k-1)$-connected, E is k-connected, and B is 0-connected, then B is k-connected.*

Proof. Remember that for us "fiber sequence" means that all the spaces have basepoints, so they are nonempty. The proposition follows quickly from the long exact sequence of homotopy sets. If a pointed set in such a sequence is preceded and followed by the trivial one-point set, it must be the trivial set as well. $\square$

When we consider maps between fiber sequences, we are not so lucky. Similar results are true, but more work is needed because of multiple path components.

Proposition 2.6. *Given a map of fiber sequences*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r, "i'"] & E' \arrow[r, "p'"] & B' \end{tikzcd}$$

and integer $k \geq 0$,

(1) *If g is k-connected and h is $(k+1)$-connected then f is k-connected.*
(2) *If h is k-connected, and for each $b \in B$ the map of homotopy fibers $F_b \longrightarrow F'_{h(b)}$ is k-connected, then g is k-connected.*
(3) *If g is k-connected, h is 0-connected, and for each $b \in B$ the map of homotopy fibers $F_b \longrightarrow F'_{h(b)}$ is a $(k-1)$-connected map of nonempty spaces, then h is k-connected.*

Proof. The idea is to take homotopy fibers in the vertical direction, creating a diagram in which every row and column is a fiber sequence and every map of rows and columns is a map of fiber sequences. That is almost enough to reduce this proposition to the previous one. To complete the proofs, however, we need to show that the desired map is at least 0-connected, which ensures that every path component of the target can fit meaningfully into such a square.

Let us describe the giant square in more detail first. We begin by deleting F and F'. This is fine because the map f is actually forced by the rest of the data, since a "map of fiber sequences" has to preserve that extra data. Next, we forget all basepoints, since we may need to work with multiple path-components in B and B'. We pick a point $e \in E$ and use its images $b \in B$, $e' \in E'$, and $b' \in B'$ as a temporary choice of "basepoint." With these choices we now form homotopy fibers in both directions:

$$\begin{tikzcd} F'' \arrow[r, "i''"] \arrow[d, "f'"] & E'' \arrow[r, "p''"] \arrow[d, "g'"] & B'' \arrow[d, "h'"] \\ F \arrow[r, "i"] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r, "i'"] & E' \arrow[r, "p'"] & B' \end{tikzcd}$$

Since the homotopy fiber construction commutes with itself, this gives four separate maps of fiber sequences in the above square.

Now we prove (1). First we demonstrate that the map $F \xrightarrow{f} F'$ is 0-connected. Without loss of generality F and F' are strict homotopy fibers. Each component of F' has some point x, which is a point $e' \in E'$ and path from its image in B' to the basepoint. Since $E \longrightarrow E'$ is 0-connected, we can modify e' by a homotopy (and the path by the image of that homotopy) so that e' is in the image of some point $e \in E$. The point $e \in E$ has image $b \in B$, and we just need to produce a path from b to the basepoint lying over our chosen path in B'. This is a question of lifting

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & B \arrow[d] \\ D^1 \arrow[r] \arrow[ur, dashed] & B' \end{tikzcd}$$

up to a homotopy of maps $((D^1, S^0), (B', B))$, which is possible because $B \longrightarrow B'$ is at least 1-connected. Therefore $x \in F'$ has a preimage in F up to homotopy, so $F \longrightarrow F'$ is 0-connected.

Now for each connected component of F', take some point in F in the preimage of that component and use that point's images in F, F', E, E', B, B' as a temporary choice of basepoint. Form the above square of fiber sequences. Since g is k-connected and h is $(k+1)$-connected, by Prop 2.4, E'' is $(k-1)$-connected and B'' is k-connected. By Prop 2.5, F'' is $(k-1)$-connected. Since this works for any component of F', we conclude by Prop 2.4 that f is k-connected.

Now we prove (2). Think of p and p' as strict fibrations. Then if g were not 0-connected, some component A' of E' would not be hit by E. But A' has image in B', which has preimage up to homotopy in B. Restricting to this path component of B and its image component in B', we know that the map of fibers $F_b \longrightarrow F_{h(b)}$ is 0-connected, and so this point in A' has preimage up to homotopy in F_b, which has image in E, contradiction. Therefore g is 0-connected.

Now for each component of E' we select some $e \in E$ hitting that component, and use the images of e as our basepoints. Construct the above square of fiber sequences. Since f and h are k-connected, by Prop 2.4, F'' and B'' are $(k-1)$-connected. By Prop 2.5, E'' is $(k-1)$-connected. Since this works for any component of E', we conclude by Prop 2.4 that g is k-connected.

Now we prove (3). Without loss of generality, B' has just one path component, which contains the basepoint. When $k = 0$ there is nothing to prove. Assuming that $k \geq 1$, we can then prove that h is 1-connected. We have already assumed that h is 0-connected, so we prove that it is also injective on path components. If it is not, then we partition E into the preimage of the basepoint component of B, called P, and its complement Q. Since all homotopy fibers are nonempty, Q is nonempty. Let P' be the union of components of E' which are hit by the components of P. Then P' is not all of E', since $E \longrightarrow E'$ is a bijection on path components. Therefore the preimage of $i'^{-1}(P')$ in F' is not all of F'. But the entire image of F in F' lies inside the preimage of P', and $F \longrightarrow F'$ is therefore not surjective on path-components, violating the assumption that it is 0-connected. Therefore $B \xrightarrow{h} B'$ is a bijection on path components (and we may assume both have just one component).

Next we prove h is surjective on π_1. Replace the square

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ E' & \xrightarrow{p'} & B' \end{array}$$

by a square of fibrations. (One can do this by the usual construction on h and p', and by replacing E by the space of coherent maps of a square into B', a path into E' and B, and a point in E. One thinks of the point as the top-left of the square and the two paths as the top and left edges.) Now given a loop in B', it lifts to some path in E' rel F'. Since $F \longrightarrow F'$ is 0-connected the endpoints of the path can be deformed through F' to lie in the image of F. The image of this path is changed by a homotopy in B', but the endpoints never move because their lifts never left F'. Now the endpoints of this path have preimages in F. We show that we can lift this path on-the-nose to a path in E connecting those endpoints:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & E \arrow[d] \\ D^1 \arrow[r] \arrow[ur, dashed] & E' \end{tikzcd}$$

This is possible because $E \longrightarrow E'$ is a 1-connected fibration. The lift then projects down to a closed loop in B lying above our original closed loop in B'. Therefore $B \xrightarrow{h} B'$ is 1-connected.

Finally, this allows us to conclude that B'' is 0-connected. As before, we use Prop 2.4 to see that F'' is $(k-2)$-connected and E'' is $(k-1)$-connected. By Prop 2.5, B'' is $(k-1)$-connected. Since B' is connected, we conclude by Prop 2.4 that h is k-connected. $\square$

3. Pullback squares: definitions, warnings, and connectivity theorems.

Now we turn to pullbacks. Given a diagram of unbased spaces of the form

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ C & \longrightarrow & D \end{array}$$

the *homotopy pullback* P is defined as the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \times C \arrow[d] \\ D^I \arrow[r, "\mathrm{ev}_{\{0,1\}}"] & D \times D \end{tikzcd}$$

In other words, it is the space of all choices of path in D and lifts of the two endpoints of the path to B and C. Of course, the homotopy pullback of

$$\begin{tikzcd} & E \arrow[d] \\ \{b\} \arrow[r] & B \end{tikzcd}$$

is exactly the homotopy fiber of $E \longrightarrow B$ at $b \in B$.

Warnings.

- One may define a square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

  but it does *not* commute.
- The construction of P does not rely on a choice of basepoint, in contrast to the construction of the homotopy fiber.
- The homotopy pullback P may be empty even if B, C, and D are nonempty.

In the spirit of the first section, we might define a homotopy pullback square as a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

that commutes up to a choice of homotopy. That homotopy allows us to define a map $A \longrightarrow P$, which we require to be an equivalence. A map of homotopy pullback squares would then have to preserve this choice of homotopy.

For whatever reason, this is generally considered to be a bad idea. Instead, the common practice is to modify all of our squares so that they commute strictly. That way we can always choose our homotopy to be constant. So the map $A \longrightarrow P$ will always be the canonical map from A to the actual pullback, followed by the canonical map from the actual pullback to the homotopy pullback.[1]

[1]Paradoxically, this implies that in a homotopy pullback square, the strict pullback must contain the homotopy pullback as a retract in the homotopy category. They are usually equivalent in actual examples, but it would be interesting to give an example where they are not.

In this way, the "homotopy pullback" condition becomes just a condition, with no extra data. Every homotopy pullback square can be modified to satisfy this stricter condition. In particular, the prototypical one above can be modified to the strictly commuting square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \times_D D^{[0,1/2]} \arrow[d] \\ D^{[1/2,1]} \times_D C \arrow[r] & D^{\{1/2\}} \end{tikzcd}$$

Definition 3.1. A homotopy pullback square is a commuting square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

such that the canonical map from A to the homotopy pullback P is a weak equivalence.

Since we have abandoned the idea of extra data, it is now completely false that this commuting square is a homotopy pullback:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

We begin with the most basic theorem for homotopy pullback squares. It gives a "Mayer-Vietoris" sequence for the homotopy groups of any such square.

Proposition 3.2. *If*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pullback square ($A \xrightarrow{\sim} P$) then for any choice of basepoint in A and its images in B, C, and D, there is a natural fiber sequence

$$\Omega D \longrightarrow P \longrightarrow B \times C$$

yielding a natural long exact sequence of homotopy sets

$$\ldots \longrightarrow \pi_i(A) \longrightarrow \pi_i(B) \times \pi_i(C) \longrightarrow \pi_i(D) \longrightarrow \pi_{i-1}(A) \longrightarrow \ldots$$

which ends with $\pi_0(B) \times \pi_0(C)$.

Proof. It is easy to check that the map $P \longrightarrow B \times C$ which forgets the path in D is a Hurewicz fibration, and its fiber over the basepoint is homeomorphic to ΩD. $\square$

The following result is the analogue of Prop 2.5 for pullback squares.

Proposition 3.3. *Given a homotopy pullback square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

and integer $k \geq -1$,

- *If B and C are k-connected and D is $(k+1)$-connected then A is k-connected.*
- *If A is k-connected and D is k-connected then B and C are k-connected.*
- *If A is $(k-1)$-connected, B and C are k-connected, and D is 0-connected, then D is k-connected.*

Proof. The case of $k = -1$ is checked directly. When $k \geq 0$, we check that basepoints can always be chosen, and then we apply Prop 2.5 to the fiber sequence

$$\Omega D \longrightarrow P \longrightarrow B \times C$$

Using the easy fact that the connectivity of $B \times C$ is the minimum of the connectivity of B and of C, this is enough to prove the proposition. A priori, for the second bullet point it seems we need to assume that B and C are 0-connected. However if that were not true, one easily checks (using that D is 0-connected) that the pullback $P \simeq A$ is not 0-connected, violating the assumption on A. $\square$

To give relative theorems, it will be helpful to note the following fact:

Proposition 3.4. *Take any commuting square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

Then it is a homotopy pullback square if and only if for any choice of basepoint in B, the homotopy fibers in the horizontal direction are equivalent:

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "\sim"] & A \arrow[r] \arrow[d] & B \arrow[d] \\ F_2 \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

Of course, the same applies to the homotopy fibers in the vertical direction.

Proof. The above diagram exists and commutes by naturality of the homotopy fiber. The only nontrivial thing is that the map of homotopy fibers is an equivalence iff the square is a homotopy pullback. First suppose that the square is homotopy pullback. Using the weakly equivalent square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \times_D D^{[0,1/2]} \arrow[d] \\ D^{[1/2,1]} \times_D C \arrow[r] & D^{\{1/2\}} \end{tikzcd}$$

one checks that the map of homotopy fibers is the projection map of a deformation retract and therefore a homotopy equivalence. So in our original square, the map of fibers is also a weak equivalence using Prop 2.6.

Going the other way, we assume the square simply commutes, and that for each choice of basepoint in B the map of homotopy fibers is an equivalence. For each such choice of basepoint, we examine the three fiber sequences

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] \arrow[dd, bend right, "\sim"'] & A \arrow[r] \arrow[d] & B \arrow[d, equal] \\ F_1' \arrow[r] \arrow[d, "\sim"] & P \arrow[r] \arrow[d] & B \arrow[d] \\ F_2 \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

and conclude that the map of fibers $F_1 \longrightarrow F_1'$ is an equivalence. So by Prop 2.6, the map $A \longrightarrow P$ is a weak equivalence, so our square is a homotopy pullback square. $\square$

Corollary 3.5. *If $C \longrightarrow D$ is a Serre fibration and*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a strict pullback square, then it is also a homotopy pullback square.

Proof. It is elementary to check that $A \longrightarrow B$ is also a Serre fibration, so the strict fibers in the horizontal direction are equivalent to the homotopy fibers. But the given map between these strict fibers is a homeomorphism, so on the homotopy fibers we must have a weak equivalence. $\square$

Corollary 3.6. *In a homotopy pullback square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

If $B \longrightarrow D$ is k-connected then $A \longrightarrow C$ is k-connected. If $A \longrightarrow C$ is k-connected and $B \longrightarrow D$ is 0-connected then $B \longrightarrow D$ is k-connected.

Proposition 3.7. *Given a map of a homotopy pullback squares*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}$$

and integer $k \geq 0$, if $B \longrightarrow B'$ and $C \longrightarrow C'$ are k-connected and $D \longrightarrow D'$ is $(k+1)$-connected then $A \longrightarrow A'$ is k-connected.

Proof. Take homotopy fibers in the horizontal direction

$$\begin{tikzcd}
F_1 \arrow[rr] \arrow[dr, "\sim"'] \arrow[dd] & & A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& F_2 \arrow[rr] \arrow[dd] & & C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
F_1' \arrow[rr] \arrow[dr, "\sim"'] & & A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& F_2' \arrow[rr] & & C' \arrow[rr] & & D'
\end{tikzcd}$$

and apply Prop 2.6 twice. □

Corollary 3.8 (Co-Gluing Lemma)**.** *A weak equivalence of pullback diagrams induces a weak equivalence on their homotopy pullbacks.*

Corollary 3.9. *The Quillen model structure on topological spaces is right proper; that is, a pullback of a weak equivalence along a fibration is a weak equivalence.*

These corollaries, in turn, can be used to establish the standard fact that homotopy limits preserve weak equivalences.

4. Generalization to Cartesian cubes and finite homotopy limits.

We will generalize Prop 3.3 and Prop 3.7 from pullback squares to any finite homotopy limit. Let $\mathbf{I}$ be small category with finitely many strings of composable, non-identity morphisms. In other words, the classifying space of $\mathbf{I}$ is a finite CW complex.

Theorem 4.1. *For each object $i \in \mathbf{I}$, let $d(i)$ be the length of the longest chain of nonidentity arrows terminating at i.*

- *Let $k \geq -1$. If X is a diagram indexed by $\mathbf{I}$, and for each object $i \in \mathbf{I}$ the space $X(i)$ is $k + d(i)$-connected, then the homotopy limit* $\operatorname{holim} X$ *is k-connected.*
- *Let $k \geq 0$. If $X \longrightarrow Y$ is a map of diagrams indexed by $\mathbf{I}$, and for each object $i \in \mathbf{I}$ the map $X(i) \longrightarrow Y(i)$ is $k + d(i)$-connected, then the map of homotopy limits* $\operatorname{holim} X \longrightarrow \operatorname{holim} Y$ *is k-connected.*

Proof. Filter the homotopy limit $\operatorname{holim} X$ by its coskeleta cosk_n, giving a tower of fibrations

$$\operatorname{holim} X \cong \operatorname{cosk}_N \longrightarrow \operatorname{cosk}_{N-1} \longrightarrow \ldots \longrightarrow \operatorname{cosk}_1 \longrightarrow \operatorname{cosk}_0 \cong \prod_{i \in \mathbf{I}} X(i)$$

At each stage we have a pullback square

$$\begin{tikzcd}
\operatorname{cosk}_n \arrow[r] \arrow[d] & \operatorname{cosk}_{n-1} \arrow[d] \\
\prod_{c_0 \to \ldots \to c_n} \operatorname{Map}(\Delta^n, X(c_n)) \arrow[r] & \prod_{c_0 \to \ldots \to c_n} \operatorname{Map}(\partial\Delta^n, X(c_n))
\end{tikzcd}$$

where the arrows $c_0 \to \ldots \to c_n$ must all be non-identity maps. (We have cancelled out the terms that arise from the "matching object" of the cosimplicial object giving $\operatorname{holim} X$.) The bottom map of our square is a Hurewicz fibration, so this is a homotopy pullback square by Cor 3.5.

Now for the first part of the theorem, we know that each $X(i)$ is at least k-connected, so $\operatorname{cosk}_0(X)$ is k-connected. Inductively, if $\operatorname{cosk}_{n-1}(X)$ is k-connected, then in the above square we check that $\operatorname{Map}(\Delta^n, X(c_n))$ has connectivity $k+n \geq k$, and $\operatorname{Map}(\partial\Delta^n, X(c_n))$ has connectivity $k + n - (n-1) = k+1$. By Prop 3.3, the pullback $\operatorname{cosk}_n(X)$ is k-connected and the induction is complete.

For the second part of the theorem, we observe that the map $\operatorname{cosk}_0(X) \longrightarrow \operatorname{cosk}_0(Y)$ is a product of k-connected maps and is therefore k-connected. Inductively, we assume that $\operatorname{cosk}_{n-1}(X) \longrightarrow \operatorname{cosk}_{n-1}(Y)$ is k-connected and draw the cube

$$\begin{tikzcd}
\operatorname{cosk}_n(X) \arrow[rr] \arrow[dr] \arrow[dd] & & \operatorname{cosk}_{n-1}(X) \arrow[dr] \arrow[dd] & \\
& \prod_{c_0 \to \ldots \to c_n} \operatorname{Map}(\Delta^n, X(c_n)) \arrow[rr] \arrow[dd] & & \prod_{c_0 \to \ldots \to c_n} \operatorname{Map}(\partial\Delta^n, X(c_n)) \arrow[dd] \\
\operatorname{cosk}_n(Y) \arrow[rr] \arrow[dr] & & \operatorname{cosk}_{n-1}(Y) \arrow[dr] & \\
& \prod_{c_0 \to \ldots \to c_n} \operatorname{Map}(\Delta^n, Y(c_n)) \arrow[rr] & & \prod_{c_0 \to \ldots \to c_n} \operatorname{Map}(\partial\Delta^n, Y(c_n))
\end{tikzcd}$$

The lower two vertical maps are at least k-connected and $(k+1)$-connected by assumption. Applying Prop 3.7, the map $\operatorname{cosk}_n(X) \longrightarrow \operatorname{cosk}_n(Y)$ is k-connected as well, which completes the induction. $\square$

Finally, we arrive at the corollary that motivated these notes. Let $\mathbf{I}$ be the poset of subsets of $\{1, 2, \ldots, n\}$. We regard $\mathbf{I}$ as a finite category, with one object for each subset and one morphism for each inclusion of subsets. A diagram indexed by $\mathbf{I}$ is typically called an n-cube.

Let $\mathbf{I}_0$ denote the full subcategory on the *nonempty* subsets of $\{1, 2, \ldots, n\}$; this is sometimes called a *punctured cube*. Given an n-cube X, the inclusion of categories $\mathbf{I}_0 \longrightarrow \mathbf{I}$ induces a map on homotopy limits

$$X(\emptyset) \simeq \operatorname*{holim}_{\mathbf{I}} X \longrightarrow \operatorname*{holim}_{\mathbf{I}_0} X$$

We say that X is *Cartesian* if this map is an equivalence. The above theorem implies

Corollary 4.2. *Given a map $X \longrightarrow Y$ of Cartesian n-cubes, if for each subset S of size $d > 0$ the map $X(S) \longrightarrow Y(S)$ is $k + (d-1)$-connected, then the map of initial vertices $X(\emptyset) \longrightarrow Y(\emptyset)$ is k-connected.*

This theorem can be used to prove connectivity estimates for the Taylor tower of a contravariant homotopy functor.

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