# FINITE SPECTRA

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These notes were written in 2014-2015 to help me understand how the different notions of finiteness for spectra are related. I am usually surprised that the basics are not more widely known, so I have decided to make these "private" notes public. I apologize in advance for any errors or sloppiness, in particular the references are given inline and not collected at the end. Please contact me if you find a mistake or have trouble chasing down a reference.

Our attention is limited here to spectra, module spectra, and $G$-spectra for finite groups $G$. Finiteness is a condition that is detected in the homotopy category, so the choice of model is not really so important here. We will be biased toward orthogonal spectra. In an appendix we give a careful treatment of how finite cell complexes and filtered colimits of orthogonal spectra interact. I don't believe any of these results are new.

## 1. Spectra

An orthogonal spectrum $X$ is

(1) *finite* if it is stably equivalent to a CW-spectrum with finitely many stable cells.
(2) *homology-finite* if its total homology $H_*(X)$ is finitely generated.
(3) *perfect* if it is in the smallest thick subcategory containing the sphere spectrum.
(4) *dualizable* if $DX \wedge X \longrightarrow F(X, X)$ is a stable equivalence.
(5) *naively dualizable* if $X \longrightarrow D(DX)$ is a stable equivalence.
(6) *triangulated-compact* if $F(X, -)$ commutes with all sums (even uncountable ones) up to stable equivalence.
(7) *model-compact* if $F(X, -)$ commutes with all sequential homotopy colimits up to stable equivalence.

We prove that these are all equivalent, except for (5) and (7) which seem to be weaker. Simple induction arguments, along with the equivalence of cofiber and fiber sequences, are enough to prove that (1) implies all of the others. So we aim to work our way back.

(2) $\Rightarrow$ (1)
By the usual Hurewicz argument, there is a CW spectrum $Y$ and an equivalence $Y \xrightarrow{\sim} X$; since $H_*(X)$ is finitely generated it suffices to attach finitely many cells at each stage, and one we get to the point where the homology is zero, we see the leftover relative

homology group must be free and so it can be killed with finitely many additional cells. $\square$

(3) $\Rightarrow$ (1)
This thick subcategory must (by induction) contain all finite spectra. These are clearly closed under cofibers and fibers; they are also closed under retracts because (1) $\Leftrightarrow$ (2) and (2) is obviously closed under retracts. $\square$

(4) $\Leftrightarrow$ (6)
It is easy to show that (6) is true iff the natural map $DX \wedge Y \longrightarrow F(X, Y)$ is an equivalence for all $Y$. Then (6) $\Rightarrow$ (4) is by specializing to $Y = X$, and (4) $\Rightarrow$ (6) is done by using the equivalence in (4) to exhibit $- \wedge X$ as a left adjoint to $- \wedge DX$, giving by the Yoneda lemma

$$F(\mathbb{S}, DX \wedge Y) \cong F(X, Y) \quad \square$$

(4) $\Rightarrow$ (5)
is found in LMS, III.i.

(4) $\Rightarrow$ (1)
The coevaluation map $\mathbb{S} \longrightarrow X \wedge DX$ must factor in the homotopy category through $X' \wedge DX$ for some finite spectrum $X'$. This is because $\mathbb{S}$ is compact and $X \wedge DX$ is the filtered homotopy colimit of $X' \wedge DX$ for all finite subcomplexes $X' \longrightarrow X$. (See the appendix below for more details.) Then the commuting diagram in the homotopy category

$$\begin{array}{ccccc} & & X' \wedge DX \wedge X & \xrightarrow{1\wedge \mathrm{ev}} & X' \\ & \nearrow^{\mathrm{coev}'\wedge 1} & \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{coev}\wedge 1} & X \wedge DX \wedge X & \xrightarrow{1\wedge \mathrm{ev}} & X \end{array}$$

shows that $X$ is a retract of $X'$, therefore finite. $\square$

(6) $\Rightarrow$ (7)
$F(X, -)$ preserves cofibers, and a hocolim is a cofiber of a map between two infinite sums. In fact this shows that (6) implies $F(X, -)$ commutes with all hocolims up to equivalence. $\square$

In total, (2), (3), (4), and (6) are all equivalent to (1). (5) appears to be weaker but the author does not have a counterexample. (7) also appears to be weaker, unless we generalize "sequence" to allow transfinite things, which would make it equivalent.

## 2. $R$-module spectra

Let $R$ be an associative orthogonal ring spectrum and let $X$ be an orthogonal $R$-module spectrum. By an $R$-cell complex we mean a spectrum built as a sequential colimit of pushouts of maps of the form

$$R \wedge F_k(S^{n-1}_+) \longrightarrow R \wedge F_k(D^n_+)$$

Every module is stably equivalent to an $R$-cell complex. One may use the small-object argument, or an argument that works one spectrum level at a time (though that technically gives a colimit along the ordinal $\omega^2$ instead of $\omega$). If $R$ is connective and $X$ is bounded below we may even argue using homotopy or homology.

The module spectrum $X$ is *finite cell module* if it is stably equivalent to a $R$-cell complex with finitely many stable cells. (See the appendix.) In general this seems a little too strong. We will prove that the following are equivalent: $X$ is

(1) *semi-finite* if it is a retract of a finite cell module in the homotopy category.
(2) *perfect* if it is in the smallest thick subcategory containing $R$.
(3) *dualizable* if $F_R(X,R) \wedge_R X \longrightarrow F_R(X,X)$ is an equivalence of $\mathbb{S}$-modules.
(4) *triangulated-compact* if $F_R(X,-)$ commutes with all sums (even uncountable ones) up to stable equivalence.

Before we begin, we should remark that our notion of dualizability here is for noncommutative ring spectra and so it differs a bit from the presentation in EKMM, III.7. In fact, it fits into the framework of duality in a bicategory as expounded in May and Sigurdsson's Parametrized Homotopy Theory, Chapter 16. We consider the dual of a left $R$-module $X$ to be a right $R$-module $X^*$, and it will turn out to be canonically equivalent to $F_R(X,R)$. Our evaluation map is a map of $R-R$ bimodules

$$X \wedge_{\mathbb{S}} X^* \longrightarrow R$$

and coevaluation is a map of $\mathbb{S}-\mathbb{S}$ bimodules (i.e. ordinary spectra)

$$\mathbb{S} \longrightarrow X^* \wedge_R X$$

These are chosen so that the composite

$$X \cong X \wedge_{\mathbb{S}} \mathbb{S} \longrightarrow X \wedge_{\mathbb{S}} X^* \wedge_R X \longrightarrow R \wedge_R X \cong X$$

is the identity in the homotopy category of left $R$-modules, and the composite

$$X^* \cong \mathbb{S} \wedge_{\mathbb{S}} X^* \longrightarrow X^* \wedge_R X \wedge_{\mathbb{S}} X^* \longrightarrow X^* \wedge_R R \cong X^*$$

is the identity in the homotopy category of right $R$-modules. We can let $X$ be any left $R$-module and try to see if $X^* = F_R(X,R)$ is its dual. The evaluation map is canonical. If a suitable coevaluation map exists, it can be composed with a canonical (different) evaluation map to give

$$\mathbb{S} \longrightarrow F_R(X,R) \wedge_R X \longrightarrow F_R(X,X)$$

and the composite must pick out the identity map of $X$. It turns out that dualizability is equivalent to

$$F_R(X,R) \wedge_R X \longrightarrow F_R(X,X)$$

being an equivalence of ordinary spectra (no $R$-actions).

If $R$ is a commutative orthogonal ring spectrum, then the evaluation map factors through $X \wedge_R X^*$, and coevaluation extends to an $R$-linear map of $R$ into $X^* \wedge_R X$, giving $R$-linear maps

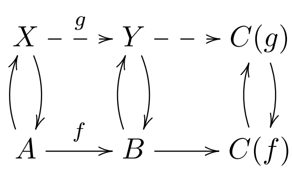
$$X \wedge_R X^* \longrightarrow R$$
$$R \longrightarrow X^* \wedge_R X$$

which make $X$ and $X^*$ $R$-dual in the more familiar sense. On the other hand, switching the roles of $\mathbb{S}$ and $R$ above gives an alternate form of duality, one in which the dualizable objects are determined by forgetting the $R$-action!

Now we'll prove that (1)−(4) are equivalent.

(1) ⇒ (2) $R \wedge S^n$ is easily shown to be in the thick subcategory by finitely many steps. Therefore by induction on the number of cells, all finite complexes are in the thick subcategory, and their retracts are in by definition. □

(2) ⇒ (1) It suffices to show that the category of semi-finite modules is thick. It is clearly closed under retracts, so we focus on taking cofibers. A cofiber of finite $R$-cell complexes is a finite $R$-cell complex by inspection. A cofiber of the retracts themselves $A \longrightarrow B$ extends to a cofiber of the whole retract diagrams



by simply defining $g$ to be the obvious composite $X \longrightarrow A \longrightarrow B \longrightarrow Y$. Therefore semi-finite modules form a thick subcategory. □

(1) ⇒ (4) This is easily shown to be true when $X = R$. Now just induct on the number of cells, using the fact that $F_R(-, Y)$ preserves cofiber sequences. Finally, if $X$ is a retract of a finite cell module, $F_R(X, -)$ is a retract of a functor that preserves coproducts, so it preserves coproducts too. □

(4) ⇒ (3) Consider the subcategory $\mathcal{C}$ of module spectra $Y$ for which

$$F_R(X, R) \wedge_R Y \longrightarrow F_R(X, Y)$$

is a weak equivalence of ordinary spectra. We always derive our mapping spectra so $\mathcal{C}$ is closed under weak equivalence.

Clearly $\mathcal{C}$ contains $R$, and is closed under fibers/cofibers/retracts. By assuming (4) we know that $\mathcal{C}$ is also closed under all (possibly infinite) coproducts. But every $R$-module is a sequential colimit along cofibrations for which the cofiber is an infinite wedge of objects of $\mathcal{C}$; inductively every level of the colimit is in $\mathcal{C}$. The colimit is also in $\mathcal{C}$ because mapping telescopes in this category can be expressed as a cofiber of a map between two countable wedges. Therefore all $R$-modules are in $\mathcal{C}$, and in particular $X \in \mathcal{C}$. □

(3) ⇒ (1) (Argument from EKMM III.7.9, p.79) Let $D_R X = F_R(X, R)$. The coevaluation map $\mathbb{S} \longrightarrow D_R X \wedge_R X$ must factor through $D_R X \wedge_R X'$ for some finite $R$-cell complex $X'$. Then the diagram which commutes in the homotopy category

$$\begin{array}{ccccc} & \overset{1\wedge \mathrm{coev}'}{\nearrow} & X \wedge_{\mathbb{S}} D_R X \wedge_R X' & \xrightarrow{\mathrm{ev}\wedge 1} & X' \\ & & \downarrow & & \downarrow \\ X & \xrightarrow{1\wedge \mathrm{coev}} & X \wedge_{\mathbb{S}} D_R X \wedge_R X & \xrightarrow{\mathrm{ev}\wedge 1} & X \end{array}$$

shows that $X$ is a retract of $X'$ in the homotopy category, so $X$ is semi-finite. □

## 3. Examples for $R$-modules

Let $X$ be a connected based space, and let $\Omega X$ denote an associative topological monoid with the homotopy type of loops in $X$ with the concatenation product, i.e. the Moore loop space. Let $G$ be a topological group weakly equivalent to $\Omega X$, i.e. the Kan loop group. Then $X \simeq BG$, and

$$R = \Sigma^\infty_+ G \simeq \Sigma^\infty_+ \Omega X$$

is an associative ring spectrum.

Suppose that $X$ is equivalent to a finite CW complex $\widetilde{X} \longrightarrow X$. Then we may form a contractible principal bundle

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & \widetilde{X} \end{array}$$

Each cell $D^n \longrightarrow \widetilde{X}$ lifts to a copy of $G \times D^n$, and this makes the weakly contractible space $EG$ into a finite $G$-CW complex. Therefore the suspension spectrum $\Sigma^\infty_+ EG$ is a finite module over $\Sigma^\infty_+ G$, and by the equivalence of modules

$$\Sigma^\infty_+ EG \longrightarrow \Sigma^\infty_+ * \cong \mathbb{S}$$

we conclude that $\mathbb{S}$ with the trivial action is a compact module over $\Sigma^\infty_+ \Omega X$.

## 4. Equivariant Spectra

Let $G$ be a *finite* group. An orthogonal $G$-spectrum $X$ is a *finite cell complex* if up to equivalence it is built out of finitely many cells of the form

$$F_V(G/H \times S^{n-1})_+ \hookrightarrow F_V(G/H \times D^n)_+$$

As was the case for modules over rings, this is too strong to be versatile. Instead we'll show that the following weaker conditions are all equivalent. An orthogonal $G$-spectrum $X$ is

(1) *semi-finite* if it is a retract of a finite cell complex spectrum in the homotopy category.
(2) *perfect* if it is in the smallest thick subcategory containing $\Sigma^\infty_+ G/H = F_0(G/H_+)$ for all subgroups $H \leq G$.
(3) *dualizable* if $F(X, \mathbb{S}) \wedge X \longrightarrow F(X, X)$ is a $G$-equivalence.
(4) *triangulated-compact* if $F^G(X, -)$ commutes with all sums (even uncountable ones) up to stable equivalence.
(5) *$G$-triangulated-compact* if $F(X, -)$ commutes with all sums (even uncountable ones) up to stable equivalence.

The reason we have two variants of "compact" is that the first one is more natural when you think of the equivariant stable homotopy category as just enriched over sets or over the homotopy category of spaces, but the second is more natural when you think of it as being enriched over the homotopy category of $G$-spaces.

(1) ⇒ (2) It's easy to see that if $A$ is any based finite G-CW complex, $\Sigma^\infty A$ is in this thick subcategory and so is $\Omega^n \Sigma^\infty A$. But to get all of the "cells" for our cell complexes, we have to show that $F_V A \simeq \Omega^V \Sigma^\infty A$ is in this thick subcategory for all finite-dimensional orthogonal $G$-representations V.

Now $S^V$ is a finite $G$-CW complex, so we can form finitely many cofiber sequences

$$\begin{gathered}
(G/(H_0) \times S^{n_0})_+ \longrightarrow S^0 \longrightarrow X_1 \\
(G/(H_1) \times S^{n_1})_+ \longrightarrow X_1 \longrightarrow X_2 \\
\vdots \\
(G/(H_k) \times S^{n_k})_+ \longrightarrow X_k \longrightarrow S^V
\end{gathered}$$

Taking derived mapping spectra out gives finitely many fiber sequences

$$\begin{gathered}
F((G/(H_0) \times S^{n_0})_+, \Sigma^\infty A) \longleftarrow A \longleftarrow F(X_1, A) \\
F((G/(H_1) \times S^{n_1})_+, \Sigma^\infty A) \longleftarrow F(X_1, A) \longleftarrow F(X_2, A) \\
\vdots \\
F((G/(H_k) \times S^{n_k})_+, \Sigma^\infty A) \longleftarrow F(X_k, A) \longleftarrow \Omega^V \Sigma^\infty A
\end{gathered}$$

It suffices to show that each of the left-hand terms is in our thick subcategory. They each sit in a fiber sequence

$$\Omega^n F(G/H_+, \Sigma^\infty A) \longrightarrow F((G/H \times S^n)_+, \Sigma^\infty A) \longrightarrow F(G/H_+, \Sigma^\infty A)$$

Finally, using the fact that $\Sigma^\infty_+ G/H$ is self-dual, we conclude that

$$F(G/H_+, \Sigma^\infty A) \simeq \Sigma^\infty G/H_+ \wedge A$$

is in the thick subcategory and we are done showing that $F_V A$ is. (When $G$ is a compact Lie group I'm not sure how to show that the spectrum $F(G/H_+, \mathbb{S}) \simeq F_{T(G/H)} G/H_+$ is in this thick subcategory.)

Now all the domains and codomains of our cells are in the thick subcategory. Therefore by induction on the number of cells, all finite complexes are in the thick subcategory, and their retracts are in by definition. □

(2) ⇒ (1) Same as for $R$-modules. □

(1) ⇒ (5) We have natural isomorphisms of nonequiariant spectra

$$\begin{aligned} F^K(G/H_+, Y) &\cong F^K\left(\coprod_{[g]\in K\backslash G/H} KgH/H_+, Y\right) \\ &\cong \prod_{[g]\in K\backslash G/H} F^K\left(KgH/H_+, Y\right) \\ &\cong \prod_{[g]\in K\backslash G/H} Y^{gHg^{-1}} \end{aligned}$$

Under these isomorphisms, the commutation map with infinite wedges becomes

$$\begin{array}{ccc} \bigvee_\alpha F^K\left(G/H_+, Y_\alpha\right) & \longrightarrow & F^K\left(G/H_+, \bigvee_\alpha Y_\alpha\right) \\ \downarrow \cong & & \downarrow \cong \\ \bigvee_\alpha \prod_{[g]\in K\backslash G/H} Y_\alpha^{gHg^{-1}} & \longrightarrow & \prod_{[g]\in K\backslash G/H} \bigvee_\alpha Y_\alpha^{gHg^{-1}} \\ \uparrow \sim & & \uparrow \sim \\ \bigvee_\alpha \bigvee_{[g]\in K\backslash G/H} Y_\alpha^{gHg^{-1}} & \xrightarrow{\cong} & \bigvee_{[g]\in K\backslash G/H} \bigvee_\alpha Y_\alpha^{gHg^{-1}} \end{array}$$

which is therefore an equivalence. The derived commutation map is an equivalence as well:

$$\begin{array}{ccc} \bigvee_\alpha F^K\left(G/H_+, fY_\alpha\right) & \longrightarrow & F^K\left(G/H_+, f\bigvee_\alpha Y_\alpha\right) \\ \downarrow \cong & & \downarrow \cong \\ \bigvee_\alpha \prod_{[g]\in K\backslash G/H} fY_\alpha^{gHg^{-1}} & \longrightarrow & \prod_{[g]\in K\backslash G/H} f\bigvee_\alpha Y_\alpha^{gHg^{-1}} \\ \uparrow \sim & & \uparrow \sim \\ \bigvee_\alpha \bigvee_{[g]\in K\backslash G/H} fY_\alpha^{gHg^{-1}} & \xrightarrow{\sim} & \bigvee_{[g]\in K\backslash G/H} f\bigvee_\alpha Y_\alpha^{gHg^{-1}} \end{array}$$

with the bottom equivalence coming from the fact that homotopy groups commute with wedges, so wedges are always derived. Now that we know that maps out of $G/H_+$ commutes with coproducts up to equivalence, we just induct on the number of cells, using the fact that $F(-, Y)$ preserves cofiber sequences. Finally, if $X$ is a retract of a finite cell module, $F(X, -)$ is a retract of a functor that preserves coproducts, so it preserves coproducts too. □

(5) ⇒ (4) Just take $G$-fixed points. □

(4) ⇒ (5) We have the natural equivalences between derived functors

$$\begin{aligned} F^G(X, G/H_+ \wedge Y) &\xrightarrow{\sim} F^G(X, F(G/H_+, Y)) \\ &\cong F^G(X \wedge G/H_+, Y) \\ &\cong F^G(G/H_+, F(X,Y)) \\ &\cong F^H(X,Y) \end{aligned}$$

The first one commutes with coproducts in the $Y$ variable by assumption, and so the last one does as well. (Commutativity with the coproduct-commuting map is automatic because these are all functorial in $Y$.) □

(5) ⇒ (3) Consider the subcategory $\mathcal{C}$ of spectra $Y$ for which

$$F(X, \mathbb{S}) \wedge Y \longrightarrow F(X,Y)$$

is an equivalence. We always derive our mapping spectra so $\mathcal{C}$ is closed under weak equivalence.

Obviously $\mathcal{C}$ contains $\mathbb{S}$, but we also claim that it contains $\Sigma^\infty_+ G/H \simeq F(\Sigma^\infty_+ G/H, \mathbb{S})$. Setting $Y = F(\Sigma^\infty_+ G/H, \mathbb{S})$ gives

$$\begin{array}{ccc} F(X,\mathbb{S}) \wedge F(\Sigma^\infty_+ G/H, \mathbb{S}) & \longrightarrow & F(X, F(\Sigma^\infty_+ G/H, \mathbb{S})) \\ \downarrow \sim & & \downarrow \cong \\ F(X \wedge \Sigma^\infty_+ G/H, \mathbb{S}) & \xrightarrow{\cong} & F(X \wedge \Sigma^\infty_+ G/H, \mathbb{S})) \end{array}$$

where the left-vertical equivalence is by general duality theory (cf. LMS III.1.3).

Now that $\mathcal{C}$ contains all the suspension spectra $\Sigma^\infty_+ G/H$, by our proof of (1) ⇒ (2) we know that it contains all of the domains and codomains of the generating cofibrations. Furthermore $\mathcal{C}$ is closed under fibers/cofibers/retracts, and by assumption it is closed under all (possibly infinite) coproducts. Then just as in the case of modules, this is enough to show that every $G$-spectrum is in $\mathcal{C}$. In particular $X \in \mathcal{C}$. □

(3) ⇒ (1) The coevaluation map $\mathbb{S} \longrightarrow X \wedge DX$ must factor thru $X' \wedge DX$ for some finite cell complex $X'$. Then the diagram

$$\begin{array}{ccccc} & & X' \wedge DX \wedge X & \xrightarrow{1\wedge \mathrm{ev}} & X' \\ & \nearrow^{\mathrm{coev}' \wedge 1} & \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{coev}\wedge 1} & X \wedge DX \wedge X & \xrightarrow{1\wedge \mathrm{ev}} & X \end{array}$$

shows that $X$ is a retract of $X'$ in the homotopy category and therefore is semi-finite. □

## 5. Appendix: Finite cell complexes and filtered colimits

Our goal here is to be rigorous to the point of pedantic about how you prove that $\mathbb{S} \longrightarrow X \wedge DX$ factors in the homotopy category through $X' \wedge DX$ for some finite spectrum $X'$. The rough idea is "maps out of compact objects commute with filtered colimits," but if we interpret this rough idea too literally in a topological setting (as opposed to a simplicial setting) then it is false.

Let $\mathbf{C}$ be either the category of based topological spaces, orthogonal spectra, orthogonal $R$-module spectra, or orthogonal $G$-spectra. We will say that a space $A$ is compact if it has the usual open covering property, and in that case consider its free spectrum $F_nA$, its free $R$-module spectrum $R \wedge F_nA$, and its free $G$-spectrum $F_VA$ (where $A$ might have a $G$-action).

These are all examples that fit into a more general notion of "compact." We can say that a map $A \longrightarrow X$ in $\mathbf{C}$ is an *h-cofibration* if the inclusion

$$A \wedge I_+ \cup_{A \wedge 0_+} X \wedge 0_+ \longrightarrow X \wedge I_+$$

has a retract in $\mathbf{C}$. Every cofibration in $\mathbf{C}$ is an $h$-cofibration, and every $h$-cofibration is on each spectrum level a closed inclusion of based spaces. Then following Mandell-May-Schwede-Shipley we say that an object $X$ of $\mathbf{C}$ is *small* or *compact* if for any sequence of $h$-cofibrations $Y_n \longrightarrow Y_{n+1}$ the natural map of sets

$$\operatorname{colim} \mathbf{C}(X, Y_n) \longrightarrow \mathbf{C}(X, \operatorname{colim} Y_n)$$

is a bijection. In other words, every map from $X$ into the colimit factors through some finite stage $Y_n$. (In fact the above map is a homeomorphism but we won't use that.) While this general notion is nice, it's a little more annoying to use in the proofs, so we will stick to the more concrete notion of compact spaces and free spectra on such spaces.

Next we recall filtered diagrams. These are just diagrams indexed by a category $\mathbf{D}$ which is *filtered*. This means that for each finite category (finitely many arrows) $\mathbf{F}$ and functor $\mathbf{F} \xrightarrow{\varphi} \mathbf{D}$, there is a co-cone, which is a constant functor $\mathbf{F} \xrightarrow{c} \mathbf{D}$ and a natural transformation $\varphi \longrightarrow c$. More concretely, this is equivalent to the pair of conditions

- Every pair of objects $d, e \in \mathbf{D}$ is joined by a third object $d \longrightarrow f \longleftarrow e$.
- Every pair of parallel maps $d \rightrightarrows e$ is coequalized by some third map $e \longrightarrow f$.

From the co-cone definition it is easy to see that $B\mathbf{D} = |N\mathbf{D}|$ is a CW complex which is weakly contractible, and therefore contractible by Whitehead's theorem. Therefore, given a diagram $\mathbf{D} \longrightarrow \mathbf{Top}_*$ of based spaces, the natural map from the unbased hocolim to the based hocolim is a homotopy equivalence.

Clearly the sequential category

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots$$

is filtered, so one begins to wonder if the above definition of "compact" could be strengthened to allow all filtered colimits along $h$-cofibrations:

**Question.** Do compact spaces $A$ have the property that maps in from $A$ commute with all filtered colimits?

From what the author has seen in the literature, it seems that most authors go to great pains to avoid claiming this. For instance Strickland's notes on CGWH spaces distinguish filtered diagrams whose colimits commute with $\mathrm{Map}(A, -)$ as "strongly filtered." So it's not likely to be true.

This is really annoying, but there are two workarounds. The first workaround is to only allow our filtered colimit to be the poset of finite subcomplexes of some given cell complex. Then maps in from a compact object really do commute with the colimit. However to prove it everything must be proven one level of the cellular filtration at a time. The second workaround is to pass to homotopy colimits, but the proof of that relies on the first workaround, so we'll do them in order.

Our main result is a straightforward adaptation of LMS, A.4.2 and EKMM, III.2.3 to the world of orthogonal spectra:

**Proposition 5.1.** *Let $A \longrightarrow X$ be a relative cell complex in any of our four settings. Let $K$ be a compact based space, or a free spectrum built out out of a compact space or compact $G$-space. Then any map $K \longrightarrow X$ factors through a finite cell subcomplex $A \longrightarrow X_\alpha$. Furthermore, $X$ is the filtered colimit of its finite cell subcomplexes.*

We know well what a cell complex of spaces is, but we will recall how to generalize this to any cofibrantly-generated model category. A *(relative) cell complex* is a countable sequence of pushouts along coproducts of maps in $I$, the set of generating cofibrations:

$$A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow \mathrm{colim}_n X_n \cong X$$

$$\begin{array}{ccc} \bigvee_{b \in B_k} \mathrm{domain}_{f_b} & \xrightarrow{f_b \in I} & \bigvee_{b \in B_k} \mathrm{codomain}_{f_b} \\ \downarrow & \ulcorner & \downarrow \\ X_k & \longrightarrow & X_{k+1} \end{array}$$

Since everything will have to be proven one $X_k$ at a time, it makes the most sense to define the notion of *finite subcomplex* $X_\alpha \longrightarrow X$ inductively.

We first define the identity map $A \longrightarrow A$ to be a finite subcomplex of $A \longrightarrow X$. Then we assume that we have defined when a map of spectra $X_\alpha \longrightarrow X_k$ is a finite subcomplex of $X_k$. Under that assumption, a map of spectra $X_\alpha \longrightarrow X_{k+1}$ is defined to be a *finite subcomplex of* $X_{k+1}$ if up to isomorphism it is a pushout of some finite subcomplex $X_\beta \longrightarrow X_k$ along some finite subset of the cells used to build $X_{k+1}$ from $X_k$. More precisely, we require that our maps $X_\alpha \longrightarrow X_{k+1}$ and $X_\beta \longrightarrow X_k$ fit into a commuting diagram of the form

$$\begin{array}{ccc}
\bigvee_{b\in S_k\subset B_k}\text{domain}_{f_b} & \xrightarrow{f_b\in I} & \bigvee_{b\in S_k\subset B_k}\text{codomain}_{f_b} \\
\downarrow & \ulcorner & \downarrow \\
X_\beta & \longrightarrow & X_\alpha \\
\downarrow & & \downarrow \\
X_k & \longrightarrow & X_{k+1}
\end{array}$$

in which the subset $S_k \subset B_k$ is finite, the top square is a pushout, and the compositions of the vertical maps give a subset of the attaching maps used to construct $X_{k+1}$ out of $X_k$. Intuitively, this is just the idea that $X_\alpha$ has been built out of finitely many of the cells used to build $X$. Finally, $X_\alpha \longrightarrow X$ is defined to be a finite subcomplex of $A \longrightarrow X$ if it factors through some finite stage $X_{k+1}$, and the factoring map $X_\alpha \longrightarrow X_{k+1}$ is a finite subcomplex as just defined.

Now we'll work on the proposition.

*Proof.* First we prove that when $K$ is a compact space or a free spectrum on a compact space, any map $K \longrightarrow X$ factors through a finite subcomplex. By compactness, it must factor through some finite level $X_{k+1}$ of the complex, so WLOG $X$ has only finitely many levels. Now we induct on the number of levels. When there are zero levels and the complex is $A = A$, there is nothing to prove. Inductively, we know that any compact object $L \longrightarrow X_k$ factors through some finite subcomplex, and we prove the same for $K \longrightarrow X_{k+1}$.

Now for spaces, the inductive step goes as follows. We only really need to know that each generating cofibration is a closed inclusion. Then the map $X_k \longrightarrow X_{k+1}$ is automatically a closed inclusion, and each point in the complement is in the image of the interior of a unique cell. If $K$ intersected infinitely many of these interiors, we could choose a point in each one, and this would give a subset of $X_{k+1}$ which is infinite, discrete (by the universal property of pushout), and in the image of $K$. Its preimage in $K$ would be an infinite discrete closed subspace, which is a contradiction. Therefore $K$ touches only finitely many interiors, and we take one such finite set of cells $S_k$ and restrict to that set. Again by the universal property of pushout, the pushout of $X_k$ and these cells is a closed subspace of $X_{k+1}$, so the map from $K$ into that subspace is automatically continuous.

Now we use the inductive hypothesis twice. Each cell in $S_k$ has an attaching map that lands in a finite subcomplex of $X_k$ by inductive hypothesis, and the union of these finite subcomplexes is still a finite subcomplex. Removing the preimage of the interiors of the cells from $K$ gives a smaller compact space $L$ which maps into $X_k$, and by inductive hypothesis this also lands in a finite subcomplex of $X_k$. Taking the union of all these things, we get a finite subcomplex of $X_{k+1}$ in which $K$ lands and the induction is complete.

For spectra the argument is almost exactly the same, because a map out of a free spectrum on $K$ is nothing more than a map from the space $K$ into a particular spectrum level of $X$. On that spectrum level, each of the generating cofibrations is a closed inclusion, and every pushout square of spectra is now a pushout square of spaces. So by the same argument, the map out of $FK$ lands in $X_k$ with only finitely many cells attached, and removing the "interiors" of these cells gives a compact subspace $L$ whose free spectrum $FL$ lands in $X_k$. By inductive hypothesis both $FL$ and the attaching maps for our new cells factor through finite subcomplexes; we take their union and conclude that $FK$ factors through a finite subcomplex.

In particular, this proof shows that each cell $b \in B_k$ of $X$ has attaching map contained in some finite subcomplex of $X_k$, and so the cell is part of a finite subcomplex of $X_{k+1}$.

Now we'll prove that $X$ is the filtered colimit of its finite subcomplexes. Let $\mathbf{F}$ denote the partially ordered set of finite subcomplexes of $A \longrightarrow X$. Then $\mathbf{F}$ is filtered because any two finite subcomplexes $X_\alpha$ and $X_\beta$ are contained in their union, which may simply be built out of the union of the cells of $X_\alpha$ and $X_\beta$. There is obviously a map

$$\operatorname*{colim}_{\alpha \in F} X_\alpha \longrightarrow X$$

We show that it is a homeomorphism on each spectrum level. It's clearly a continuous bijection because every point of $X$ comes from some cell, every cell is in some $X_\alpha$, and two points only become the same in the colimit if they were in the same cell to begin with. On the other hand, the map is closed because a closed subset of the colimit gives a subset of $X$ which is closed when restricted to each cell, and so it is closed in $X$.

One could also prove this map is an isomorphism by showing both have the universal property that restricting to the cells

$$\mathbf{C}(X, Y) \longrightarrow \mathbf{C}(A, Y) \times \prod_k \prod_{b \in B_k} \text{codomain}_{f_b}$$

is injective and hits exactly those maps which inductively agree on $\prod_{b \in B_k} \text{domain}_{f_b}$ and so extend to a map out of $X_{k+1}$, for all $k$. This approach seems to require trickier book-keeping though. □

Returning to more general homotopy colimits, we get quite easily

**Lemma 5.2.** *Let* $[-,-]$ *denote homotopy classes of maps before taking fibrant replacement (in the spectrum case). Then if* $K$ *is a compact space or a free spectrum on a compact space, and* $\mathbf{A} \xrightarrow{X} \mathbf{Top}$ *is a filtered diagram, the map*

$$\operatorname*{colim}_{\alpha \in \mathbf{A}} [K, X_\alpha] \longrightarrow [K, \operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha]$$

*is a bijection of sets. In other words, every map from $K$ into a filtered hocolim is homotopic to a map that factors through a single level of the diagram.*

*Proof.* It suffices to show that map of sets is surjective, because then the surjectivity for $K \times I$ implies injectivity for $K$. The spectrum case reduces easily to the space case because hocolims of spectra are defined levelwise. The space case reduces to the case of unbased hocolims, so we assume the hocolim is unbased. To prove surjectivity, we take

$$K \longrightarrow \operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha$$

and project to

$$K \longrightarrow \operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha \longrightarrow \operatorname*{hocolim}_{\alpha \in \mathbf{A}} * = B\mathbf{A}$$

Since $K$ is compact, by the above proposition it lands in a finite subcomplex of $B\mathbf{A}$, which is the image of $B\mathbf{F} \longrightarrow B\mathbf{A}$ for some finite category $\mathbf{F}$. Taking a cocone on $\mathbf{F}$ gives a bigger subcomplex of $\mathbf{A}$ which is literally a cone on $B\mathbf{F}$, and pushing $K$ along this cone deforms it into a single 0-cell $\alpha \in \mathbf{A}$. Upstairs in $\operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha$, this homotopy has a canonical lift, and at the end-point of the homotopy we now have $K$ landing in $X_\alpha$, which proves surjectivity. In the case of $G$-spectra both $K$ and the $X_\alpha$ have $G$-actions and all maps must be equivariant, but the indexing category $\mathbf{A}$ has no $G$-action, so no complications arise. □

Finally we need reassurance that filtered homotopy colimits preserve fibrant spectra.

**Proposition 5.3.** *In orthogonal $R$-module spectra or orthogonal $G$-spectra, a filtered homotopy colimit of fibrant objects is fibrant.*

*Proof.* Do $R$-modules first. Homotopy colimits are constructed levelwise by the Bousfield-Kan construction, assuming all $X_\alpha$ are cofibrant. In the homotopy colimit

$$\operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha$$

if all the spectra are fibrant then for each $n$ the adjoint structure maps

$$X_\alpha(n) \xrightarrow{\sim} \Omega X_\alpha(n+1)$$

are weak equivalences of based spaces. Now the adjoint structure map on the homotopy colimit can be factored as

$$\operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha(n) \xrightarrow{\sim} \operatorname*{hocolim}_{\alpha \in \mathbf{A}} \Omega X_\alpha(n+1) \xrightarrow{\sim} \Omega \operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha(n+1)$$

The first arrow is a weak equivalence because hocolims preserve weak equivalences of diagrams. The second arrow is an isomorphism on $\pi_0$ by applying the above lemma

$$\operatorname*{colim}_{\alpha \in \mathbf{A}} [K, X_\alpha] \longrightarrow [K, \operatorname*{hocolim}_{\alpha \in \mathbf{A}} X_\alpha]$$

with $K = S^1$, and it's an isomorphism on $\pi_n$ for $n \geq 0$ by applying the lemma with $K = S^1 \wedge (S^n_+)$. Therefore that second arrow is a weak equivalence, and we conclude that the adjoint structure map on the homotopy colimit is a weak equivalence.

Now do $G$-spectra. Homotopy colimits are still constructed levelwise by the Bousfield-Kan construction, assuming all $X_\alpha$ are cofibrant. In the homotopy colimit

$$\operatorname*{hocolim}_{\alpha\in\mathbf{A}} X_\alpha$$

if all the spectra are fibrant then for each $V$, $W$ the adjoint structure maps

$$X_\alpha(V) \xrightarrow{\sim} \Omega^W X_\alpha(V \oplus W)$$

are $G$-equivalences of based spaces (meaning a weak equivalence on the fixed points for all closed subgroups). Now the adjoint structure map on the homotopy colimit can be factored as

$$\operatorname*{hocolim}_{\alpha\in\mathbf{A}} X_\alpha(V) \xrightarrow{\sim} \operatorname*{hocolim}_{\alpha\in\mathbf{A}} \Omega^W X_\alpha(V \oplus W) \xrightarrow{\sim} \Omega^W \operatorname*{hocolim}_{\alpha\in\mathbf{A}} X_\alpha(V \oplus W)$$

The first arrow is a weak equivalence because hocolims preserve weak equivalences and commute with fixed points. The second arrow is an isomorphism on $\pi_0^H$ by applying the above lemma

$$\operatorname*{colim}_{\alpha\in\mathbf{A}} [K, X_\alpha] \longrightarrow [K, \operatorname*{hocolim}_{\alpha\in\mathbf{A}} X_\alpha]$$

with $K = S^W \wedge (G/H_+)$, and it's an isomorphism on $\pi_n^H$ for $n \geq 0$ by applying the lemma with $K = S^W \wedge (S^n_+) \wedge (G/H_+)$. Therefore that second arrow is a weak equivalence, and we conclude that the adjoint structure map on the homotopy colimit is a weak equivalence. □

The fact that hocolims preserve equivalences now easily gives

**Corollary 5.4.** *If $\{X_\alpha\}$ is a filtered diagram then taking fibrant approximations of each object $X_\alpha \longrightarrow fX_\alpha$ gives a map of homotopy colimits*

$$\operatorname*{hocolim}_{\alpha\in\mathbf{A}} X_\alpha \longrightarrow \operatorname*{hocolim}_{\alpha\in\mathbf{A}} fX_\alpha$$

*which is also a fibrant approximation.*

Combining the last three results, we obtain a derived version of our earlier lemma:

**Proposition 5.5.** *Let $[-,-]$ denote maps in the homotopy category. Then if $K$ is a compact space or a free spectrum on a compact space, and $\mathbf{A} \xrightarrow{X} \mathbf{Top}$ is a filtered diagram, the map*

$$\operatorname*{colim}_{\alpha\in\mathbf{A}} [K, X_\alpha] \longrightarrow [K, \operatorname*{hocolim}_{\alpha\in\mathbf{A}} X_\alpha]$$

*is a bijection of sets. In other words, every derived map from $K$ into a filtered hocolim is equivalent in the homotopy category to a derived map from $K$ into a single level of the diagram.*

Let $X$ be a cell complex; then we have seen that $X$ is a filtered colimit of its finite subcomplexes and every space or free spectrum on a compact space mapping into $X$ factors through some level. Therefore the homotopy groups of this colimit system behave identically to the homotopy groups of the homotopy colimit, and we conclude

**Corollary 5.6.** *If $A \longrightarrow X$ is a relative cell complex then the homotopy colimit of the finite subcomplexes of $X$ is weakly equivalent to the ordinary colimit:*

$$\operatorname*{hocolim}_{\alpha \in \mathbf{F}} X_\alpha \xrightarrow{\sim} \operatorname*{colim}_{\alpha \in \mathbf{F}} X_\alpha$$

The last two results justify our claim that maps from $\mathbb{S}$ into $X \wedge DX$ factor through $X' \wedge DX$ for some finite $X'$: just express the cofibrant approximation $cX$ as the filtered colimit of its finite subcomplexes $X_\alpha$, and smash the system with a cofibrant approximation $c(DX)$. Then

$$\begin{aligned} \text{hocolim } (X_\alpha \wedge c(DX)) &\xrightarrow{\cong} (\text{hocolim } X_\alpha) \wedge c(DX) \\ &\xrightarrow{\sim} (\text{colim } X_\alpha) \wedge c(DX) \\ &\xrightarrow{\cong} cX \wedge c(DX) \\ &= X \wedge^L DX \end{aligned}$$

and taking functorial fibrant approximations gives maps

$$f\,(X_\alpha \wedge c(DX)) \longrightarrow f(X \wedge^L DX)$$

which assemble into

$$\begin{array}{ccc} \text{hocolim } (X_\alpha \wedge c(DX)) & \xrightarrow{\sim} & X \wedge^L DX \\ \downarrow \sim & & \downarrow \sim \\ \text{hocolim } f\,(X_\alpha \wedge c(DX)) & \longrightarrow & f(X \wedge^L DX) \end{array}$$

The bottom map is an equivalence by 2-out-of-3 and both objects are fibrant, so by Ken Brown's lemma the map gives an isomorphism on the set of honest homotopy classes of maps out of $\mathbb{S}$. Therefore maps from $\mathbb{S}$ into the bottom-right corner lift up to homotopy to maps into the bottom-left corner, where they then factor through some $f\,(X_\alpha \wedge c(DX))$. This finishes the pedantic proof of the claim, and it works for $R$-module spectra (the smash product is over $R$) and $G$-spectra equally well.

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