FIXED POINTS AND COLIMITS

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This brief expository note records some facts about when fixed points and colimits of various kinds commute.

1. SPACE-LEVEL CONSTRUCTIONS

Let G be a topological group. Consider a diagram of unbased G-spaces X_α indexed by a category A. The colimit $\operatorname*{colim}_A X_\alpha$ then inherits a G-action. There is a natural map

$$\operatorname*{colim}_A X_\alpha^G \xrightarrow{\Phi} \left(\operatorname*{colim}_A X_\alpha \right)^G$$

and a similar map on the homotopy colimits, constructed from the Bousfield-Kan formula:

$$\operatorname*{hocolim}_A X_\alpha^G \xrightarrow{h\Phi} \left(\operatorname*{hocolim}_A X_\alpha \right)^G$$

We study when these maps are homeomorphisms.

It cannot be true in general: consider the circle S^1 with the $\mathbb{Z}/2$-action given by horizontal flip. Then the coequalizer of the identity map and the flip map is a closed interval I with trivial $\mathbb{Z}/2$-action. So the fixed points of the coequalizer is I. But the coequalizer of the fixed points is $\{0, 1\}$. Therefore colimits and fixed points do not commute in general, not even up to homotopy equivalence. However, in this example the homotopy coequalizer is the Klein bottle, with fixed points given by two circles, and this coincides with the homotopy coequalizer of the fixed points.

In what follows we use CGWH spaces.

Definition 1.1. We say G is *topologically finitely generated* if there is a finite collection of elements $g_1, g_2, \ldots, g_n$ such that the smallest closed subgroup containing all the g_i is G itself. We will sometimes abbreviate this to *good*.

All compact Lie groups are good, and these are the examples I have in mind. But there may be more.

Proposition 1.2.   (1) *If $\{X_\alpha\}_{\alpha \in A}$ has no nontrivial morphisms, then $\Phi = h\Phi$ is a homeomorphism. So fixed points commute with coproducts.*

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(2) *If G is good, then $h\Phi$ is a homeomorphism for any diagram of spaces. So fixed points of good groups commute with Bousfield-Kan homotopy colimits.*

(3) *If $\{X_\alpha\}_{\alpha \in A}$ is a filtered diagram of closed inclusions, then Φ is a closed inclusion. If in addition G is good then Φ is a homeomorphism. So fixed points of good groups commute with filtered colimits.*

(4) *If $\{X_\alpha\}_{\alpha \in A}$ is a pushout along a closed inclusion, then Φ is a homeomorphism. So fixed points commute with homotopy pushouts.*

Remark. All of our homotopy colimits are constructed in the *unbased* (or *unreduced*) way. However these results are still true if we take *based* (or *reduced*) homotopy colimits instead. These based homotopy colimits are given by a reduced form of the Bousfield-Kan construction, so long as all the spaces are well-based. A more general construction simply takes the unreduced homotopy cofiber of

$$\operatorname*{hocolim}_A * \longrightarrow \operatorname*{hocolim}_A X_\alpha$$

to get the reduced homotopy colimit. So using (4), we can see that the unbased form of (2) implies that fixed points commute with this construction of based hocolims, up to homeomorphism. This is a based form of (2).

Proof. (1) is straightforward. (2) follows from (3) and (4), since we can filter the Bousfield-Kan formula by skeleta, and each inclusion in the filtration is a pushout along a closed inclusion:

$$L_n X \times \Delta^n \cup_{L_n X \times \partial \Delta^n} X_n \times \partial \Delta^n \longrightarrow X_n \times \Delta^n$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$|\mathrm{Sk}^{n-1} X_\bullet| \longrightarrow |\mathrm{Sk}^n X_\bullet|$$

Specifically, the top horizontal is a pushout-product of two h-cofibrations, therefore it's an h-cofibration, therefore it's a closed inclusion.

For (3), we forget topology and consider Φ as a map of sets. Then Φ is injective, since any two points in $\operatorname*{colim}_A X_\alpha^G$ which go to the same point in $\operatorname*{colim}_A X_\alpha$ have representatives in X_α^G, X_β^G which map forward to the same point in X_γ^G, so our two representatives become the same in the colimit. This is one place where working with CGWH spaces is a bit of a pain: the colimit functor does not respect the forgetful map to sets in general, but it does if the morphisms of the filtered diagram are closed inclusions (Strickland Lemma 3.3). We remark in passing the annoying fact that filtered colimits along closed inclusions do not seem to be homotopy colimits in general.

Returning to topology, we prove Φ is a closed inclusion. It's easily continuous. Given a closed subset $C \subset \operatorname*{colim}_A X_\alpha^G$, its preimage in each X_α^G is closed, but $X_\alpha^G \subset X_\alpha$ is a closed inclusion by the fact that the G-action is continuous. So each preimage of $C \subset \operatorname*{colim}_A X_\alpha$ is a closed subspace of each X_α, so by definition C is closed inside

$\operatorname*{colim}_{A} X_\alpha$. It's contained inside the subspace $\left(\operatorname*{colim}_{A} X_\alpha\right)^G$ and is therefore relatively closed in that subspace. Therefore Φ is a closed map, which completes the proof that it is a closed inclusion.

Finally we show that image of Φ is exactly the fixed points $\left(\operatorname*{colim}_{A} X_\alpha\right)^G$ when G is good. Any point in the colimit that is fixed comes from a point $x \in X_\alpha$ for which the stabilizer is some closed subgroup $H_0 \leq G$. Taking a generator g not in H_0, $gx \neq x$, but they give the same point in the colimit, so x and gx become equal in some X_β, where the image of x is now stabilized by a strictly larger closed subgroup $H_1 \leq G$. Repeating for each of the generators, we eventually get an image of x in some X_γ which is fixed by all of G. This gives a preimage in $\operatorname*{colim}_{A} X_\alpha^G$ of our chosen point in $\left(\operatorname*{colim}_{A} X_\alpha\right)^G$, finishing the proof that Φ is surjective and therefore a homeomorphism.

The proof of (4) is similar. Given a pushout diagram of G-spaces

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \\
C & &
\end{array}
$$

in which $A \longrightarrow B$ is a closed inclusion, the underlying set of the pushout is $C \amalg (B - A)$, with the obvious G-action which preserves this decomposition. Then it's clear that

$$ C^G \amalg (B^G - A^G) \longrightarrow (C \amalg (B - A))^G $$

is a bijection. Continuity is again immediate, and it's a closed map because a closed subset of the domain gives a closed subset of C^G and B^G with the same restriction to A^G, but again fixed points are closed so this gives a closed subset of C and B with the same restriction to A, which is a closed subset of the pushout. $\qquad\square$

2. SPECTRUM-LEVEL CONSTRUCTIONS

For spectra, we restrict to G finite or compact Lie, which are always *good* as defined above. Now consider a general diagram X_α of orthogonal G-spectra. Here the colimit is defined levelwise, and the homotopy colimit is defined with the Bousfield-Kan formula levelwise. One might expect that we need to make the individual spectra X_α cofibrant as well, but this is unnecessary; see the notes "Homotopy colimits via bar constructions" on this webpage for a detailed take on this almost-standard fact.

Recall that when X is an orthogonal G-spectrum and $H \leq G$ is a closed subgroup, there are three notions of fixed points. The *categorical fixed points* are defined levelwise as

$$ (X^H)_n := (X_n)^H $$

These define a Quillen right adjoint so they are derived when X is fibrant. Since colimits of G-spectra are defined levelwise, it is immediate that $(-)^H$ commutes with all coproducts, homotopy pushout squares, and sequential colimits along closed inclusions. Therefore they commute with homotopy colimits on the nose. This isn't so useful on its own, because $(-)^H$ does not preserve stable equivalences.

The *genuine fixed points* $(fX)^H$ are obtained by deriving the categorical ones by taking fibrant replacement first. Fibrant replacement preserves coproducts, homotopy pushout squares, and sequential colimits along closed inclusions up to stable equivalence. Therefore

Proposition 2.1. *Genuine fixed points of G-spectra with all homotopy colimits up to equivalence.*

Finally, the *geometric fixed points* of X are defined as a coequalizer

$$\bigvee_{V,W} F_{W^H} S^0 \wedge \mathbf{J}_G^H(V,W) \wedge X(V)^H \rightrightarrows \bigvee_{V} F_{V^H} S^0 \wedge X(V)^H \longrightarrow \Phi^H X$$

Coproducts, pushouts along a closed inclusion, and sequential colimits along closed inclusions all get applied to the $X(V)$ in the above formula. These three operations all commute with fixed points, smashing with a space, and taking coequalizers, therefore they all commute with $\Phi^H X$ on the nose:

Proposition 2.2. *If we model homotopy colimits of G-spectra with the Bousfield-Kan formula, then Φ^H commutes with them on the nose.*

This statement about the geometric fixed points is not so widely known, but appears for instance in Blumberg and Mandell's paper on the model* category of cyclotomic spectra.

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