

The gluing lemma is left-properness

Cary Malkiewich

Let \mathbf{C} be a category that contains all pushouts, equipped with a subcategory of cofibrations and a subcategory of weak equivalences. Assume these properties:

- all isomorphisms are weak equivalences
- weak equivalences satisfy the 2 out of 3 property
- each map of \mathbf{C} factors into a cofibration followed by a weak equivalence (it doesn't need to be functorial)
- pushouts of cofibrations are cofibrations

Then we can show that the following two hypotheses are equivalent:

Hypothesis 0.1 (Gluing Lemma). *Pushouts along cofibrations are homotopical. In other words, if we have a weak equivalence of pushout diagrams*

$$\begin{array}{ccccc} C & \longleftarrow & A & \longrightarrow & B \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ C' & \longleftarrow & A' & \longrightarrow & B' \end{array}$$

where the maps $A \rightarrow B$ and $A' \rightarrow B'$ are cofibrations, then the map of pushouts

$$B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$$

is a weak equivalence.

Hypothesis 0.2 (Left Properness). *A pushout of a weak equivalence along a cofibration is a weak equivalence. In other words, if we have a pushout diagrams*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow \sim & & \downarrow \\ C & \longrightarrow & B \cup_A C \end{array}$$

where $A \rightarrow B$ is a cofibration and $A \rightarrow C$ is a weak equivalence, then $B \rightarrow B \cup_A C$ is a weak equivalence.

Gluing Lemma \Rightarrow *Left Properness*. Consider the map of pushout diagrams

$$\begin{array}{ccccc} A & \xlongequal{\quad} & A & \longrightarrow & B \\ \downarrow \sim & & \parallel & & \parallel \\ C & \xleftarrow{\sim} & A & \longrightarrow & B \end{array}$$

Isomorphisms are weak equivalences, so this is a weak equivalence of pushout diagrams. The induced map of pushouts is $B \rightarrow B \cup_A C$, and by the gluing lemma this is a weak equivalence. \square

Left Properness \Rightarrow *Gluing Lemma*. Apply the factorization to $A \rightarrow C$ to get

$$\begin{array}{ccccc} A & \longrightarrow & \tilde{A} & \xrightarrow{\sim} & C \\ \downarrow \sim & & & & \downarrow \sim \\ A' & \longrightarrow & & & C' \end{array}$$

such that $A \rightarrow \tilde{A}$ is a cofibration. Now add in a pushout:

$$\begin{array}{ccccc} A & \longrightarrow & \tilde{A} & \xrightarrow{\sim} & C \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ A' & \longrightarrow & P & \xrightarrow{\sim} & C' \end{array}$$

The map $A' \rightarrow P$ is now a cofibration by the pushout property. The vertical $\tilde{A} \rightarrow P$ is a weak equivalence by left properness, and then $P \rightarrow C'$ is a weak equivalence by 2 out of 3. This shows that we may break up our pushouts into two stages, each with an extra assumption. So we just need to prove the gluing lemma under the extra assumption that $A \rightarrow C$ and $A' \rightarrow C'$ are cofibrations, then prove it again under the extra assumption that $A \rightarrow C$ and $A' \rightarrow C'$ weak equivalences.

If both are cofibrations then by

$$\begin{array}{ccc} A' & \longrightarrow & B \cup_A A' \\ \downarrow & & \downarrow \\ C' & \longrightarrow & B \cup_A C' \end{array}$$

the map $B \cup_A A' \rightarrow B \cup_A C'$ is a cofibration. In the composite

$$B \rightarrow B \cup_A A' \rightarrow B'$$

the first map is a weak equivalence because it is the pushout of the weak equivalence $A \rightarrow A'$ by $A \rightarrow B$. The composite is a weak equivalence by assumption. By 2 out of 3, we conclude that $B \cup_A A' \rightarrow B'$ is a weak equivalence. Now in the pushout square

$$\begin{array}{ccc} B \cup_A A' & \longrightarrow & B \cup_A C' \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B' \cup_{A'} C' \end{array}$$

we have seen that the top leg is a cofibration and the left leg is a weak equivalence, so $B \cup_A C' \rightarrow B' \cup_{A'} C'$ is a weak equivalence. Finally in the pushout square

$$\begin{array}{ccc} C & \longrightarrow & B \cup_A C \\ \downarrow & & \downarrow \\ C' & \longrightarrow & B \cup_A C' \end{array}$$

the left leg is assumed to be a weak equivalence and the top map is a cofibration as the pushout of $A \rightarrow B$, so the map $B \cup_A C \rightarrow B \cup_A C'$ is a weak equivalence. Composing our two weak equivalences

$$B \cup_A C \rightarrow B \cup_A C' \rightarrow B' \cup_{A'} C'$$

gives the desired weak equivalence.

For the second half, assume $A \rightarrow C$ and $A' \rightarrow C'$ are both weak equivalences. Then we show the following maps are weak equivalences in this order:

- $B \rightarrow B \cup_A A'$ by left properness ($A \rightarrow B$ is a cofibration)
- $B \cup_A A' \rightarrow B'$ by 2 out of 3
- $B' \rightarrow B' \cup_{A'} C'$ by left properness ($A' \rightarrow B'$ is a cofibration)
- $B \cup_A A' \rightarrow B' \cup_{A'} C'$ by composition
- $B \cup_A A' \rightarrow B \cup_A C'$ by left properness ($A' \rightarrow B \cup_A A'$ is a cofibration)
- $B \cup_A C' \rightarrow B' \cup_{A'} C'$ by 2 out of 3
- $B \cup_A C \rightarrow B \cup_A C'$ by left properness ($C \rightarrow B \cup_A C$ is a cofibration)
- $B \cup_A C \rightarrow B' \cup_{A'} C'$ by composition

and that finishes the proof. □