HOMOTOPY COLIMITS VIA THE BAR CONSTRUCTION

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The purpose of this expository note is to give explicit proofs for some well-known results on homotopy colimits. As described in Shulman's excellent paper [Shu06], we may think of homotopy colimits in two distinct ways: "globally" as left derived functors of the colimit, and "locally" by taking cofibrant replacement of each level of the diagram, and then applying a bar construction (the Bousfield-Kan formula). Our focus here will be on this second approach, and in particular the situations where it is unnecessary to make the objects cofibrant.

We will discuss the case of unbased spaces, which is treated in almost as much detail in the appendix of [DI04]. Then we will discuss based spaces, a general approach, and then the cases of prespectra and orthogonal G-spectra as defined in [MMSS01] and [MM02].

1. Hocolims of Unbased Spaces

As a concrete example, consider the pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

If $A \longrightarrow B$ is an h-cofibration (i.e. has the homotopy extension property) then we call the pushout $B \cup_A C$ a *homotopy pushout.* Another way of saying this is, the construction is *homotopical.* To be precise:

Proposition 1.1 (Gluing Lemma)**.** *If we have a weak equivalence of pushout diagrams*

$$\begin{tikzcd} C \arrow[d, "\sim"] & A \arrow[l] \arrow[r] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ C' & A' \arrow[l] \arrow[r] & B' \end{tikzcd}$$

where the maps $A \longrightarrow B$ and $A' \longrightarrow B'$ are h-cofibrations, then the map of pushouts

$$B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$$

is a weak equivalence.

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If one is familiar with model categories then it is not too difficult to prove this proposition, but under the additional assumption that A, A', B, B', C, and C' are all "q-cofibrant" (e.g. are CW-complexes). On the other hand, Strøm's model structure on spaces can be used to get the above statement but only when the vertical maps are homotopy equivalences. Clearly, model categories alone do not give us the above version of the gluing lemma. We have to do something more inherently topological.

Proof. We may replace each pushout with the corresponding double mapping cylinder:

$$B \cup_{A\times\{0\}} A \times I \cup_{A\times\{1\}} C \longrightarrow B \cup_A C$$

This collapse map is a homotopy equivalence. To see this, we define a homotopy inverse by using the homotopy extension property to deform usual inclusion

$$B \hookrightarrow B \cup_{A\times\{0\}} (A \times I)$$

to a map which sends $A \subset B$ to $A \times 1$. Piecing this together with the identity $C \longrightarrow C$ gives a map

$$B \cup_A C \longrightarrow B \cup_{A\times\{0\}} A \times I \cup_{A\times\{1\}} C$$

and it is straightforward to check this is a homotopy inverse.

Now we can assume that each of our pushouts is a double mapping cylinder. The induced map between these mapping cylinders is clearly an isomorphism on π_0, and on each component it is an isomorphism on π_1 by the van Kampen theorem for fundamental groupoids. Finally, the Mayer-Vietoris exact sequence applies to all double mapping cylinders and homology with all twisted coefficient systems, so the proof is finished by the lemma that follows. □

Lemma 1.2. *A map $X \longrightarrow Y$ of path-connected spaces is a weak homotopy equivalence iff it is an isomorphism on π_1, and on simplicial homology with all twisted coefficient systems.*

Proof. This is a standard result in some circles; the proof here is an adaptation of [Hat02], Prop 4.21. We first take $f : X \longrightarrow Y$ to be any weak equivalence of topological spaces. By definition it is an isomorphism on π_0 and π_1. Let $\mathcal{A}$ be any twisted coefficient system, given by a π_1-action on some abelian group A. Let $Mf = (X \times I) \cup_X Y$ be the mapping cylinder. By excision, it suffices to prove that the homology of the pair (Mf, X) with coefficients in $\mathcal{A}$ is zero. Any homology class α in Mf is represented by a finite collection of simplices $\Delta^n \longrightarrow Mf$ that have been lifted to the bundle $\mathcal{A}$. Let $K \longrightarrow Mf$ be the Δ-complex obtained by taking each of these n-simplices and gluing faces together precisely when those faces coincide in Mf. Let $L \subset K$ be the subcomplex of those $(n-1)$-simplices whose image is entirely in X. Then α is in the image of the map $(K, L) \longrightarrow (Mf, X)$ on relative homology with coefficients in $\mathcal{A}$. Since $\pi_*(Mf, X) = 0$, this map of pairs is homotopic to one that sends all of K into X. By a standard lemma homotopic maps induce the same map on homology, but once K goes into X the induced map must be zero. Therefore our homology class α must have been zero too. This proves $H_*(Mf, X; \mathcal{A}) = 0$, so by the excision long exact sequence, $H_*(X; \mathcal{A}) \longrightarrow H_*(Y; \mathcal{A})$ is an isomorphism.

That takes care of one direction. Now assume that $X \longrightarrow Y$ is an isomorphism on π_1 and on homology with all twisted coefficient systems. We would like to make an argument involving the universal covers of X and Y, but they may not exist. So consider the CW replacements $\Gamma X \xrightarrow{\sim} X$ and $\Gamma Y \xrightarrow{\sim} Y$. The above argument guarantees that $\Gamma f : \Gamma X \longrightarrow \Gamma Y$ is also an isomorphism on twisted homology. ΓX has a universal cover $\widetilde{\Gamma X}$, whose homology with $\mathbb{Z}$ coefficients is naturally isomorphic to the homology of ΓX with coefficients given by $\mathbb{Z}[\pi_1(X)]$ as a module over itself. Therefore $\widetilde{\Gamma f} : \widetilde{\Gamma X} \longrightarrow \widetilde{\Gamma Y}$ is a map of simply connected CW complexes and an isomorphism on homology with $\mathbb{Z}$ coefficients. By the Hurewicz theorem, $\widetilde{\Gamma f}$ is a weak equivalence. But the higher homotopy groups of $\widetilde{\Gamma X}$ are mapped isomorphically to the higher homotopy groups of X, and similarly for Y, so the original map f is also an isomorphism on all homotopy groups. □

Now that we have treated homotopy pushouts, we will lead into more general homotopy colimits. In everything that follows, we work with compactly generated weak Hausdorff spaces. Recall that a *simplicial space* $X_\bullet$ is a diagram $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Top}$ from the opposite of the category of finite ordered sets into spaces. Concretely, it is a sequence of spaces $X_0, X_1, \ldots$ with face maps $d_0, \ldots, d_n : X_n \longrightarrow X_{n-1}$ and degeneracy maps $s_0, \ldots, s_n : X_n \longrightarrow X_{n+1}$ satisfying some relations. The *geometric realization* of $X_\bullet$ is the space

$$\left(\coprod_n X_n \times \Delta^n \right) / \sim \qquad \Delta^n = \left\{ t \in \mathbb{R}^{n+1} : \sum t_i = 1, t_i \geq 0 \quad \forall i \right\}$$

with equivalence relations

$$\begin{aligned} (x_n, t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) &= (d_i x_n, t_0, \ldots, t_{n-1}) \\ (s_i x_n, t_0, \ldots, t_{n+1}) &= (x_n, t_0, \ldots, t_i + t_{i+1}, \ldots, t_n) \end{aligned}$$

We may think of the expressions on the right as being simpler than the expressions on the left. "Simplifying" in this way allows us to write each point in this realization uniquely as a pair (x, t) in which all the barycentric coordinates of t are nonzero (otherwise apply the first relation) and the point $x \in X_n$ is nondegenerate (otherwise apply the second relation). We may define the *n-skeleton* of the realization to consist of all points whose standard form (x, t) has $x \in X_k$ with $k \leq n$. This is also the realization of the simplicial space $(\mathrm{Sk}^n X)_\bullet$, defined as

$$(\mathrm{Sk}^n X)_m = \{ x \in X_m : x = s_{i_1} s_{i_2} \ldots s_{i_j} y, y \in X_k, k \leq n \}$$

Clearly $|X_\bullet|$ is filtered by $\{|\mathrm{Sk}^n X_\bullet|\}_{n=0}^{\infty}$.

Now, any time a space X is covered by closed subspaces A and B there is a pushout square

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

In the case we see $|\mathrm{Sk}^n X_\bullet|$ is covered by the closed subspace $|\mathrm{Sk}^{n-1} X_\bullet|$ and the closed image of $X_n \times \Delta^n$. Manipulating this square a bit yields the pushout square

$$\begin{tikzcd} L_n X \times \Delta^n \cup_{L_n X \times \partial\Delta^n} X_n \times \partial\Delta^n \arrow[r] \arrow[d] & X_n \times \Delta^n \arrow[d] \\ |\mathrm{Sk}^{n-1} X_\bullet| \arrow[r] & |\mathrm{Sk}^n X_\bullet| \end{tikzcd} \tag{1}$$

where $L_n X = \bigcup_{i=0}^{n-1} s_i(X_{n-1}) \subset X_n$ is the nth latching object of X. (See for example [Dug08].) We are interested in when (1) is a homotopy pushout square, and when the bottom map is a closed inclusion. For both of these conditions it suffices to show that the top arrow is an h-cofibration. We use the following standard property of h-cofibrations:

Proposition 1.3 (Pushout-Product Axiom)**.** *If $A \longrightarrow X$ and $B \longrightarrow Y$ are h-cofibrations then*

$$(A \times Y) \cup_{(A \times B)} (B \times X) \longrightarrow (B \times Y)$$

is an h-cofibration. If one of the two input maps is a homotopy equivalence then the larger map is as well.

Proof. This appears in many places, for instance in ([May99], 6.4). The proof uses the equivalent characterization that (X, A) is an *NDR pair*. □

By the pushout-product lemma, the top arrow of (1)

$$L_n X \times \Delta^n \cup_{L_n X \times \partial\Delta^n} X_n \times \partial\Delta^n \longrightarrow X_n \times \Delta^n$$

is an h-cofibration when the two maps

$$\partial\Delta^n \longrightarrow \Delta^n$$
$$L_n X \longrightarrow X_n$$

are h-cofibrations. The first is always an h-cofibration, so we should focus our attention on the case when the second map is an h-cofibration as well:

Definition 1.4. The simplicial space $X_\bullet$ is *Reedy cofibrant* if the nth latching map $L_n X \longrightarrow X_n$ is an h-cofibration for all $n \geq 0$.

So assume that we have a map $X_\bullet \longrightarrow Y_\bullet$ of Reedy cofibrant simplicial spaces, and each level $X_n \longrightarrow Y_n$ is a weak equvialence. We will show that the map on latching objects $L_n X \longrightarrow L_n Y$ is a weak equivalence. First observe that $s_i(X_{n-1}) \cong X_{n-1}$ is preserved up to weak equivalence for each value of i. From the observation that for $1 \leq k \leq n-1$ we have the pushout square

$$\begin{tikzcd} s_k\left(\bigcup_{i=0}^{k-1} s_i(X_{n-2})\right) \arrow[r] \arrow[d] & \bigcup_{i=0}^{k-1} s_i(X_{n-1}) \arrow[d] \\ s_k(X_{n-1}) \arrow[r] & \bigcup_{i=0}^{k} s_i(X_{n-1}) \end{tikzcd} \tag{2}$$

we may induct the following statements on n, and inside that induction on n, induct on k:

- $\bigcup_{i=0}^{k-1} s_i(X_{n-1}) \longrightarrow \bigcup_{i=0}^{k} s_i(X_{n-1})$ is a cofibration and similarly for Y
- $\bigcup_{i=0}^{k} s_i(X_{n-1}) \longrightarrow \bigcup_{i=0}^{k} s_i(Y_{n-1})$ is a weak equivalence

The induction on the first statement goes by composing some maps that we know to be cofibrations because of the inductive hypothesis

$$\bigcup_{i=0}^{k-1} s_i(X_{n-2}) \longrightarrow \bigcup_{i=0}^{k} s_i(X_{n-2}) \longrightarrow \bigcup_{i=0}^{k+1} s_i(X_{n-2}) \longrightarrow \ldots \longrightarrow \bigcup_{i=0}^{n-2} s_i(X_{n-2}) = L_{n-2}X \longrightarrow X_{n-1}$$

and then applying s_k to this composition to obtain a homeomorphic map

$$s_k\left(\bigcup_{i=0}^{k-1} s_i(X_{n-2})\right) \longrightarrow s_k(X_{n-1})$$

which is necessarily a cofibration. This is the left vertical in the square (2), so the right vertical map is a cofibration and the induction is complete. The induction on the second statement is the straightforward application of the gluing lemma to the above pushout square.

We conclude that the map on latching objects $L_nX \longrightarrow L_nY$ is a weak equivalence. Then by the gluing lemma again, we have weak equivalences

$$L_nX \times \Delta^n \cup_{L_nX \times \partial\Delta^n} X_n \times \partial\Delta^n \longrightarrow L_nY \times \Delta^n \cup_{L_nY \times \partial\Delta^n} Y_n \times \partial\Delta^n$$
$$X_n \times \Delta^n \longrightarrow Y_n \times \Delta^n$$

One final application of the gluing lemma to the square (1) finishes the inductive argument that the induced map of skeleta

$$|\mathrm{Sk}^n X_\bullet| \longrightarrow |\mathrm{Sk}^n Y_\bullet|$$

is a weak equivalence. We want to take the limit as $n \to \infty$, but for this we need one more result:

Proposition 1.5. *Given a sequential diagram of weak Hausdorff unbased spaces*

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \ldots$$

in which every map is an h-cofibration, the natural map

$$\operatorname*{colim}_i \pi_*(A_i) \xrightarrow{\cong} \pi_* \left(\operatorname*{colim}_i A_i \right)$$

is an isomorphism.

Proof. It is standard that h-cofibrations of weak Hausdorff spaces are always closed inclusions. For this proof we will only need that the maps of our colimit system are inclusions, and that finite unions of points in A_n are closed.

We will show that for a compact Hausdorff space K, each map $K \longrightarrow \operatorname{colim} A_i$ factors through some A_n. Then taking $K = S^n$, we see that the above map of groups is surjective. Taking $K = S^n \times I$ proves that the map is injective.

If $K \longrightarrow \operatorname{colim} A_i$ does not factor through some finite level, then we may construct a sequence $\{k_n\}_{n=0}^{\infty} \in K$ such that the image of k_n lies outside A_n. K is compact, so without loss of generality this sequence converges to some $k \in K$. The image of k is in some A_n. Take the union of all images of k_i with $i > n$; this union is closed because its restriction to each K_i is a finite union of points, which is always closed in a weak Hausdorff space. Its complement is an open subset of $\operatorname{colim} A_i$ that contains the limit k without containing any of the k_i for $i > n$, contradiction. $\square$

Now we can let $n \to \infty$ and obtain

Theorem 1.6. *If $X_\bullet \longrightarrow Y_\bullet$ is a map of Reedy cofibrant simplicial spaces, and each level of the map $X_n \longrightarrow Y_n$ is a weak homotopy equivalence, then the induced map $|X| \longrightarrow |Y|$ is a weak equivalence as well.*

Remark. This foundational theorem appears in many places but not always with the same assumptions; for instance it appears in 1972 as ([May72], Thm. 11.13) but with simple connectivity assumptions and a slightly stricter notion of Reedy cofibrant. The above version has been known for quite some time, as also noted in [DI04].

Now we will discuss the kind of simplicial space we are really interested in. Given a small category $\mathbf{C}$ and a diagram of spaces $X : \mathbf{C} \longrightarrow \mathbf{Top}$, we define its *uncorrected homotopy colimit* to be

$$\operatorname{uhocolim} X = B(X, \mathbf{C}, *) = \left| \coprod_{c_0 \xrightarrow[\varphi_1]{} \ldots \xrightarrow[\varphi_n]{} c_n} X(c_0) \right|$$

In other words, it is the realization of the simplicial space whose level n is a disjoint union with one copy of the topological space $X(c_0)$ for each n-tuple of composable arrows $c_0 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_n} c_n$. The face map d_i composes φ_i with φ_{i+1}, deleting c_i in the process, while acting as the identity on $X(c_0)$. The exception to this rule is the 0th face d_0, which not only deletes c_0 and φ_1 from our n-tuple of arrows, but also applies φ_1 to

$X(c_0)$ to land in $X(c_1)$. The degeneracies are easier: s_i inserts an identity arrow at the object c_i.

Exercise 1.7. If $\mathbf{C} = \{\bullet \longleftarrow \bullet \longrightarrow \bullet\}$ is the category that models pushout diagrams, prove that this uncorrected homotopy colimit construction is naturally *homeomorphic* to the double mapping cylinder construction.

Proposition 1.8. *$Y_\bullet = B_\bullet(X, \mathbf{C}, *)$ is always a Reedy cofibrant simplicial space.*

Proof. We calculate $L_n Y$ to be a disjoint union of $X(c_0)$ over a subset of all possible n-tuples of arrows, namely those n-tuples which contain at least one identity map. Clearly the inclusion of one of the pieces of a disjoint union is a cofibration. □

Putting this all together, we get

Theorem 1.9. *A map of diagrams $X \longrightarrow X'$ inducing weak equivalences $X(c) \longrightarrow X'(c)$ induces a weak equivalence of uncorrected homotopy colimits $B(X, \mathbf{C}, *) \longrightarrow B(X', \mathbf{C}, *)$.*

We will not go into model categories in much detail here, but we will summarize how these uncorrected hocolims fit in with the rest of the theory. The category of diagrams from $\mathbf{C}$ to spaces can be given the *projective* or *levelwise* model structure. Under this choice of model structure, the colimit functor into spaces is left Quillen. We compute its left derived functor on a diagram X by replacing X with a cofibrant diagram, and taking the colimit. Let QX denote the diagram we obtain by taking a functorial cofibrant replacement of the *levels* $X(c)$. Then QX is not usually a cofibrant diagram. However we may use a bar construction to build $B(QX, \mathbf{C}, \mathbf{C})$, and check that this is a cofibrant diagram by verifying the appropriate left lifting property. The colimit of this fattened diagram is B(QX,**C**,*). Therefore left derived functor of colim is naturally equivalent to $B(QX, \mathbf{C}, *)$:

$$\mathbf{L}\mathrm{colim}\, X \simeq B(QX, \mathbf{C}, *)$$

Our work in this section demonstrates that taking QX was unnecessary:

$$B(QX, \mathbf{C}, *) \xrightarrow{\sim} B(X, \mathbf{C}, *)$$

So we may alternatively compute $\mathbf{L}\mathrm{colim}\, X$ with the uncorrected Bousfield-Kan formula.

In short, homotopy colimits of unbased spaces need not be corrected.

2. Hocolims of Based Spaces

Recall that a map of based spaces $X \longrightarrow Y$ is a *based h-cofibration* if it satisfies the HEP, but with all maps basepoint-preserving. This is weaker than being an unbased h-cofibration. We say that X is *well-based* or has a *nondegenerate basepoint* if the inclusion of the basepoint $* \longrightarrow X$ is an unbased h-cofibration.

Clearly, we can always forget the basepoints, and conclude that unbased double mapping cylinders and pushouts along unbased h-cofibration preserve weak equivalences. This might seem unsatisfying; shouldn't there be a variant where we use based h-cofibrations? In other words, don't *reduced* double mapping cylinders preserve weak equivalences? The answer is no. That would imply that wedge sums of based spaces preserve weak equivalences. Now, let $X \subset \mathbb{R}^2$ be the infinite shrinking wedge of circles, with CW replacement $\Gamma X \xrightarrow{\sim} X$. Then the map of wedge sums

$$\Gamma X \vee \Gamma X \longrightarrow X \vee X$$

is not surjective on π_1, so a wedge of weak equivalences is not in general a weak equivalence. This is why nondegenerate basepoints are essential when studying based spaces.

However, a based h-cofibration between well-based spaces is an unbased h-cofibration [May99]. So we get the following essentially for free:

Proposition 2.1. *If we have a weak equivalence of pushout diagrams of well-based spaces*

$$\begin{tikzcd} C \arrow[d, "\sim"] & A \arrow[l] \arrow[r] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ C' & A' \arrow[l] \arrow[r] & B' \end{tikzcd}$$

where the maps $A \longrightarrow B$ and $A' \longrightarrow B'$ are based h-cofibrations, then the map of pushouts

$$B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$$

is a weak equivalence.

Now we also get a similar colimit result, but here we don't need nondegenerate basepoints:

Proposition 2.2. *Given a sequential diagram of weak Hausdorff based spaces*

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \dots$$

in which every map is a based h-cofibration, the natural map

$$\operatorname*{colim}_i \pi_*(A_i) \xrightarrow{\cong} \pi_*\left(\operatorname*{colim}_i A_i\right)$$

is an isomorphism.

Proof. Same proof as before, we just have to explain why based h-cofibrations have to be inclusions. If $A \longrightarrow X$ is a based h-cofibration then the reduced cylinder $X \wedge I_+$ retracts onto the reduced mapping cylinder $(A \wedge I_+) \cup_{A \wedge \{1\}_+} X$. Therefore the inclusion of compactly generated weak Hausdorff spaces.

$$(A \wedge I_+) \cup_{A \wedge \{1\}_+} X \hookrightarrow X \wedge I_+$$

has a retract and so it is a closed inclusion. The sequence of injective maps

$$A \times \{0\} \hookrightarrow X \times \{0\} \hookrightarrow X \wedge I_+$$

composes to give the same map as the composition of the closed inclusions

$$A \times \{0\} \hookrightarrow (A \wedge I_+) \cup_{A \wedge \{1\}_+} X \hookrightarrow X \wedge I_+$$

and so the composite is a closed inclusion, therefore the first map $A \hookrightarrow X$ must be a closed inclusion. These kinds of arguments are found in many places, for instance Lemma 1.6.2 of [MS06]. $\square$

Moving on, a *simplicial based space* $X_\bullet$ is a diagram $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Top}_*$ from the opposite of the category of finite ordered sets into based spaces. One may check that this is the same as a *based simplicial space*, i.e. a simplicial space $X_\bullet$ with a choice of map from the final object $*_\bullet$ of simplicial spaces back into $X_\bullet$. The *geometric realization* of $X_\bullet$ is the same space as before

$$\left(\coprod_n X_n \times \Delta^n \right) / \sim$$

though because the basepoints

$$\left(\coprod_n * \times \Delta^n \right) / \sim$$

all get glued to one point, we may rewrite it as

$$\left(\bigvee_n X_n \wedge \Delta^n_+ \right) / \sim$$

(We check that the result is homeomorphic by comparing universal properties.) In both cases the equivalence relation is as above:

$$(x_n, t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) = (d_i x_n, t_0, \ldots, t_{n-1})$$
$$(s_i x_n, t_0, \ldots, t_{n+1}) = (x_n, t_0, \ldots, t_i + t_{i+1}, \ldots, t_n)$$

Because the definition respects the forgetful functor into unbased spaces, we get the same results as above:

Proposition 2.3. *If $X_\bullet \longrightarrow Y_\bullet$ is a map of based simplicial spaces which are Reedy cofibrant as unbased simplicial spaces, and each $X_n \longrightarrow Y_n$ is a weak homotopy equivalence, then the induced map $|X| \longrightarrow |Y|$ is a weak equivalence as well.*

Given a diagram $X : \mathbf{C} \longrightarrow \mathbf{Top}_*$ of based spaces, we define its *uncorrected (reduced) homotopy colimit* to be the realization of a simplicial based space

$$\widetilde{\operatorname{uhocolim}}\, X = \widetilde{B}(X, \mathbf{C}, *) = \left| \bigvee_{c_0 \xrightarrow[\varphi_1]{} \dots \xrightarrow[\varphi_n]{} c_n} X(c_0) \right|$$

Alternatively, one may use the basepoint to form a map of uncorrected unbased homotopy colimits

$$B(*, \mathbf{C}, *) \hookrightarrow B(X, \mathbf{C}, *)$$

and then take the quotient. When the objects $X(c)$ are all cofibrant, this reduced hocolim will turn out to be the left derived functor of colim, from diagrams of based spaces to based spaces. The unreduced hocolim is in general not equivalent to the reduced one, in particular when $B(*, \mathbf{C}, *) = B\mathbf{C}$ is not contractible.

Warning 2.4. It is standard practice to drop the tilde and the word "reduced" here; in fact, we will do this when we move to spectra. The reader is encouraged to remember that the based hocolim is usually not equivalent to the unbased hocolim.

Now our above arguments run into a small snag: the inclusion of a smaller wedge sum into a bigger wedge sum is not always an unbased h-cofibration. We therefore get only this weaker result:

Proposition 2.5. *If each $X(c)$ is well-based then $Y_\bullet = \widetilde{B}_\bullet(X, \mathbf{C}, *)$ is a based simplicial space which is Reedy cofibrant as an unbased simplicial space.*

Theorem 2.6. *A map of diagrams of based spaces $X \longrightarrow X'$ inducing weak equivalences $X(c) \longrightarrow X'(c)$ induces a weak equivalence of uncorrected reduced homotopy colimits $\widetilde{B}(X, \mathbf{C}, *) \longrightarrow \widetilde{B}(X', \mathbf{C}, *)$ as long as every $X(c)$ and $X'(c)$ is well-based.*

As before, we can conclude that the left derived functor of colim on X is equivalent to the uncorrected reduced hocolim $\widetilde{B}(X, \mathbf{C}, *)$ for every diagram X with the property that every $X(c)$ is well-based. Put another way, hocolims of based spaces *do* need to be corrected, but it is unnecessary to make all the spaces of the diagram cofibrant. Making them well-based is good enough. Finally, we have

Proposition 2.7. *If each $X(c)$ is well-based then the obvious maps*

$$B\mathbf{C} \longrightarrow B(X, \mathbf{C}, *) \longrightarrow \widetilde{B}(X, \mathbf{C}, *)$$

form a homotopy cofiber sequence of unbased spaces.

Proof. Since each $X(c)$ is well-based, growing a whisker to get $wX(c)$ does not change it up to weak equivalence. Examining the constructions directly gives

$$C(B\mathbf{C}) \cup_{B\mathbf{C}} B(X, \mathbf{C}, *) \cong \widetilde{B}(wX, \mathbf{C}, *)$$

and the above proposition tells us that collapsing the whiskers

$$\widetilde{B}(wX, \mathbf{C}, *) \xrightarrow{\sim} \widetilde{B}(X, \mathbf{C}, *)$$

gives a weak equivalence. $\square$

3. A General Approach

At this point we would like to get a bit more abstract, and condense the previous two sections into one. This will help us streamline our thinking, and prove the corresponding results for spectra more quickly.

To be precise, let us consider a pointed, cofibrantly generated model category $\mathbf{D}$, which is tensored over unbased spaces. This may sound like a lot of assumptions, but in essence we are only going to use the fact that $\mathbf{D}$ has a notion of "weak equivalence" and "cell" which allows us to build homotopy colimits as bar constructions and prove their homotopy invariance in the usual way, plus a well-behaved notion of "half-smash product" of an object $d \in \mathbf{D}$ with the n-simplex Δ^n or its boundary $\partial\Delta^n$. The reader is encouraged to think of taking an unbased space K, adding a disjoint basepoint K_+, and then taking the smash product; in fact we will write the tensoring with K as $- \wedge K_+$ to reinforce this intuition.

Smashing with $\Delta^1_+ = I_+$ gives a space that can be used to define homotopies. We define an *h-cofibration* in $\mathbf{D}$ to be a map $A \longrightarrow X$ satisfying the most obvious analogue of the homotopy extension property. Namely, any map $X \longrightarrow Y$ with a homotopy $A \wedge I_+ \longrightarrow Y$ of the restriction to A can be extended to a homotopy $X \wedge I_+ \longrightarrow Y$ of the whole map on X. By the usual argument, this is equivalent to insisting that the inclusion

$$(A \wedge I_+) \cup_{A \wedge \{0\}_+} (X \wedge \{0\}_+) \longrightarrow X \wedge I_+$$

has a retraction. (For based spaces, this was the notion of based h-cofibration.)

Now we want to assume that there is a full subcategory $\mathbf{D}' \subset \mathbf{D}$ of objects on which pushouts and colimits are homotopical. For concreteness, one may take $\mathbf{D}$ to be based spaces and $\mathbf{D}'$ to be well-based spaces. To be precise, we assume $\mathbf{D}'$ satisfies the following three hypotheses.

Hypothesis 3.1 (Gluing Lemma). *$\mathbf{D}'$ is closed under arbitrary coproducts and pushouts in which one leg is an h-cofibration. Furthermore, if we have a map of pushout diagrams in $\mathbf{D}'$*

$$\begin{tikzcd} C \arrow[d, "\sim"] & A \arrow[l] \arrow[r] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ C' & A' \arrow[l] \arrow[r] & B' \end{tikzcd}$$

where the vertical arrows are weak equivalences and the maps $A \longrightarrow B$ and $A' \longrightarrow B'$ are h-cofibrations, then the map of pushouts $B \cup_A C \longrightarrow B' \cup_{A'} C'$ is a weak equivalence.

Hypothesis 3.2 (Colimit Lemma). *If we have a map of sequential colimit diagrams in* $\mathbf{D}'$

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \dots \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \dots \end{array}$$

where the vertical arrows are weak equivalences and horizontal maps are h-cofibrations, the map of colimits

$$\operatorname*{colim}_i A_i \longrightarrow \operatorname*{colim}_i B_i$$

is a weak equivalence.

Hypothesis 3.3 (Pushout-Product Lemma). *If* $A \longrightarrow X$ *is an h-cofibration in* $\mathbf{D}'$ *then*

$$(A \wedge \Delta^n_+) \cup_{(A \wedge \partial \Delta^n_+)} (X \wedge \partial \Delta^n_+) \longrightarrow (X \wedge \Delta^n_+)$$

is an h-cofibration in $\mathbf{D}'$. *Here the left-hand side is defined as a pushout in the obvious way. In addition, the operations* $- \wedge \partial \Delta^n_+$ *and* $- \wedge \Delta^n_+$ *preserve weak equivalences between objects in* $\mathbf{D}'$.

The expert will notice that the first part of this pushout-product lemma would follow from a more general statement that there is a "classical" model structure on $\mathbf{D}$ which is *topological*, meaning that the pairing with unbased spaces acts on cofibrations the way you might expect. The existence of such a structure on unbased and based spaces was proven by Strøm, but the problem of building these structures for very general categories $\mathbf{D}$ has been the subject of much interesting recent research; cf. [BR12], [Col06], [MS06].

Define simplicial objects and geometric realization of $\mathbf{D}$ in the usual categorical way. We won't go into detail on this since it is clear what to do in each example we consider. We get the same pushout square as before

$$\begin{array}{ccc} L_n X \times \Delta^n \cup_{L_n X \times \partial \Delta^n} X_n \times \partial \Delta^n & \longrightarrow & X_n \times \Delta^n \\ \downarrow & & \downarrow \\ |\mathrm{Sk}^{n-1} X_\bullet| & \longrightarrow & |\mathrm{Sk}^n X_\bullet| \end{array}$$

from the categorical definition of realization and diagram-chasing, but again we won't go into detail because in the case of spectra below we may simply do the space-level argument on each spectrum level.

A simplicial object $X_\bullet$ of $\mathbf{D}$ is *Reedy cofibrant* if the latching maps $L_n X \longrightarrow X_n$ are all h-cofibrations. Tracing through our argument in the case of unbased spaces, we get everything we need for the the following proposition. (We also use that cofibrations are closed under pushouts, but that is always true of h-cofibrations.)

Proposition 3.4. *If* $X_\bullet \longrightarrow Y_\bullet$ *is a map of Reedy cofibrant simplicial objects of* $\mathbf{D}$, *each level* X_n *and* Y_n *is in our chosen subcategory* $\mathbf{D}'$, *and each* $X_n \longrightarrow Y_n$ *is a weak equivalence, then the induced map* $|X| \longrightarrow |Y|$ *is a weak equivalence as well.*

Finally we define an uncorrected homotopy colimit of a diagram $X : \mathbf{C} \longrightarrow \mathbf{D}$ as the geometric realization of the following simplicial object of $\mathbf{D}$:

$$\operatorname{uhocolim} X = B(X, \mathbf{C}, *) = \left| \coprod_{c_0 \xrightarrow[\varphi_1]{} \dots \xrightarrow[\varphi_n]{} c_n} X(c_0) \right|$$

It is straightforward to see that every object of $\mathbf{D}$ is h-cofibrant, and every inclusion of a smaller coproduct into a larger one is an h-cofibration, so

Proposition 3.5. *The simplicial object $Y_\bullet = B_\bullet(X, \mathbf{C}, *)$ is always Reedy cofibrant.*

Notice that this does not contradict our discussion of based spaces. It says that every based simplicial space is Reedy cofibrant, but not necessarily Reedy cofibrant as an unbased simplicial space. Putting this all together, we get

Theorem 3.6. *A map of diagrams $X \longrightarrow X'$ from $\mathbf{C}$ to $\mathbf{D}$ inducing weak equivalences $X(c) \longrightarrow X'(c)$ induces a weak equivalence of uncorrected homotopy colimits $B(X, \mathbf{C}, *) \longrightarrow B(X', \mathbf{C}, *)$, so long as every $X(c)$ and $X'(c)$ is in our chosen subcategory $\mathbf{D}'$.*

Model category arguments tell us that the left derived functor of colim is given by taking cofibrant replacement of the objects $QX(c)$ and then applying the bar construction. By the above, this is equivalent to the uncorrected hocolim so long as every object $X(c)$ is in $\mathbf{D}'$. This abstract framework summarizes a large part of the argument from the previous two sections; the only work left is to actually verify the gluing lemma, colimit lemma, and pushout-product axiom.

4. Hocolims of Prespectra

Now we turn to prespectra, as described in [MMSS01]. One might expect this to be a direct extension of the above results for based spaces, but miraculously, in the stable setting the nondegenerate basepoint hypotheses actually disappear.

If $A \longrightarrow X$ is an h-cofibration of prespectra (as in the last section), then the levels $A_n \longrightarrow X_n$ are based h-cofibrations, so one might not expect pushouts along such maps to preserve equivalences of spectra. But they do:

Proposition 4.1. *The gluing lemma holds for the entire category of prespectra.*

Proof. This is sketched in [MMSS01], Thm. 7.4(iv) so we will only comment on what goes into the proof. Clearly we cannot prove this by working upwards from pushouts of based spaces. Instead, we use the fact that the homotopy cofiber of a map of prespectra is equivalent to a shift of the homotopy fiber, even when those prespectra are degenerately based. From this, we may prove that a homotopy pushout square of spectra is also a homotopy pullback square. More importantly, such a square gives a Mayer-Vietoris sequence on the stable homotopy groups.

Now given a weak equivalence of pushout diagrams as in the statement of the gluing lemma, we apply the Mayer-Vietoris sequence and use the five-lemma to conclude that the map of pushouts is a stable equivalence. □

At this point it's worth discussing what happens to our previous counterexample. Let X be the infinite shrinking wedge of circles, and ΓX either its CW replacement or X with a whisker grown onto it. Then

$$\Sigma^\infty \Gamma X \longrightarrow \Sigma^\infty X$$

is not a π_*-isomorphism of prespectra. We may take wedges, pushouts, and hocolims with $\Sigma^\infty X$, and these will have the same homotopy groups whether we take a cofibrant replacement of $\Sigma^\infty X$ or not beforehand, but of course the result will be different if we use $\Sigma^\infty \Gamma X$ instead. Put another way, cofibrant replacement does not commute with Σ^∞.

Proposition 4.2. *The colimit lemma holds for the entire category of prespectra.*

Proof. This is more general than [MMSS01], Thm. 7.4(v), but it should not be surprising because we know the analogous result for based spaces is true. As in that case, we prove the stronger statement

$$\operatorname*{colim}_i \pi_*(A_i) \xrightarrow{\cong} \pi_* \left(\operatorname*{colim}_i A_i \right)$$

We consider the colimit system indexed by both the indices i and the levels k of the prespectra.

$$\begin{aligned}
\operatorname*{colim}_i \pi_n(A_i) &\cong \operatorname*{colim}_i \operatorname*{colim}_k \pi_{n+k}((A_i)_k) \\
&\cong \operatorname*{colim}_k \operatorname*{colim}_i \pi_{n+k}((A_i)_k) \\
&\cong \operatorname*{colim}_k \pi_{n+k}\left(\operatorname*{colim}_i (A_i)_k\right) \\
&\cong \operatorname*{colim}_k \pi_{n+k}\left(\left(\operatorname*{colim}_i A_i\right)_k\right) \\
&\cong \pi_n\left(\operatorname*{colim}_i A_i\right)
\end{aligned}$$

Taking a particular element in $\pi_{n+k}((A_i)_k)$, we check that its image in $\pi_n\left(\operatorname*{colim}_i A_i\right)$ under these isomorphisms is the same as that of the natural map defined earlier, so this isomorphism is exactly the one that we expect. □

Proposition 4.3. *The pushout-product lemma holds for the entire category of prespectra.*

Proof. It's enough to just use the formal pairing result of Schwaňzl and Vogt, quoted in ([MS06], Thm. 4.3.2(i)). This requires knowing that h-cofibrations of unbased spaces are "strong," but that comes from Strøm's result quoted in ([MS06], Thm 4.4.4(ii)). The fact that $- \wedge K$ preserves weak equivalences is [MMSS01], 7.4(i). □

For a *simplicial prespectrum* $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \textbf{Prespectra}$, the geometric realization $|X_\bullet|$ and nth latching object $L_n X$ are computed levelwise. The simplicial prespectrum $X_\bullet$ is Reedy cofibrant if the latching maps $L_n X \longrightarrow X_n$, $n \geq 0$, are all h-cofibrations of prespectra as defined above. The generalities from the previous section give us

Proposition 4.4. *If $X_\bullet \longrightarrow Y_\bullet$ is a map of simplicial prespectra which are Reedy cofibrant, and each $X_n \longrightarrow Y_n$ is a stable homotopy equivalence, then the induced map $|X| \longrightarrow |Y|$ is a stable equivalence as well.*

Given a diagram $X : \mathbf{C} \longrightarrow \textbf{Prespectra}$, we define its uncorrected homotopy colimit to be the realization of a simplicial prespectrum

$$\operatorname{uhocolim} X = B(X, \mathbf{C}, *) = \left| \bigvee_{c_0 \xrightarrow[\varphi_1]{} \ldots \xrightarrow[\varphi_n]{} c_n} X(c_0) \right|$$

This is more like a reduced hocolim than an unreduced one, but we skip the tildes because it isn't really possible to make an unreduced hocolim of spectra (unless one wants a parametrized prespectrum over $B\mathbf{C}$). Our general framework then gives

Proposition 4.5. *The simplicial prespectrum $Y_\bullet = B_\bullet(X, \mathbf{C}, *)$ is always Reedy cofibrant.*

Theorem 4.6. *A map of diagrams of prespectra $X \longrightarrow X'$ inducing stable equivalences $X(c) \longrightarrow X'(c)$ induces a stable equivalence of uncorrected homotopy colimits $B(X, \mathbf{C}, *) \longrightarrow B(X', \mathbf{C}, *)$.*

So the left derived functor of colim is equivalent to the uncorrected hocolim $B(-, \mathbf{C}, *)$. Amazingly, hocolims of prespectra never need to be corrected. As a word of warning, though, the suspension spectrum functor Σ^∞ still does not preserve weak equivalences if the inputs are not well-based.

5. Hocolims of Orthogonal G-Spectra

Let G be a finite group, or a compact Lie group. Orthogonal G-spectra are defined in [MM02]; they are stable objects that have an action by the group G, a notion of weak equivalence that keeps track of fixed points, and the wonderfully useful property that suspending by a G-representation is invertible up to equivalence.

Proposition 5.1. *The gluing lemma holds for all orthogonal G-spectra.*

Proof. This is [MM02], 3.5(iv) and is proven by an equivariant analogue of the same result for prespectra. □

Proposition 5.2. *The colimit lemma holds for all orthogonal G-spectra.*

Proof. Again this is more general than [MM02], Thm. 3.5(v), but is not surprising. As before, for each closed subgroup $H \leq G$ we prove the stronger statement

$$\operatorname*{colim}_i \pi_n^H(A_i) \xrightarrow{\cong} \pi_n^H\left(\operatorname*{colim}_i A_i\right)$$

We prove this by examining the groups

$$\begin{cases} \pi_n\left((\Omega^V A_i(V))^H\right) & \text{if} \quad n \geq 0 \\ \pi_0\left((\Omega^V A_i(V \oplus \mathbb{R}^n))^H\right) & \text{if} \quad n < 0 \end{cases}$$

These form a colimit system indexed by both i and V, as V ranges over an indexing set of G-representations in a complete G-universe. The proof is otherwise identical to the case of prespectra. □

Proposition 5.3. *The pushout-product lemma holds for all orthogonal G-spectra.*

Proof. As in the nonequivariant case, it's enough to just use the formal pairing result of Schwaïzl and Vogt, quoted in ([MS06], Thm. 4.3.2(i)). This requires knowing that h-cofibrations of unbased spaces are "strong," but that comes from Strøm's result quoted in ([MS06], Thm 4.4.4(ii)). It is nontrivial to prove that $- \wedge K$ actually preserves G-equivalences when K is a CW-complex, but this is [MM02], III.3.11. □

Our general framework then gives

Theorem 5.4. *A map of diagrams of orthogonal G-spectra $X \longrightarrow X'$ inducing G-equivalences $X(c) \longrightarrow X'(c)$ induces a G-equivalence of uncorrected homotopy colimits $B(X, \mathbf{C}, *) \longrightarrow B(X', \mathbf{C}, *)$.*

Therefore genuine G-equivalences of spectra are preserved by uncorrected homotopy colimits, so the left derived functor of colimit is given by applying the Bousfield-Kan formula $B(X, \mathbf{C}, *)$ directly to X. In the short companion to these notes "Fixed points and colimits" we use this to show that the genuine fixed points functor commutes with hocolims of G-spectra up to equivalence, and geometric fixed points commute with hocolims on the nose, provided that we use the uncorrected bar construction to build our hocolims. This is the last standard fact that we have aimed to prove, so our work here is complete.

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