MORITA ADJUNCTIONS AND MORITA DUALITY

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These notes are concerned with the close connection between duality theory, Morita equivalences, Morita duality, and Koszul duality. We explain the useful observation that a duality is always an adjunction, and how the notions of Morita equivalence, Morita adjunction, and Morita duality are related to each other. We will assume the reader is familiar with symmetric monoidal categories, functors, natural transformations and adjunctions. For applications to rings, some homological algebra is assumed. For applications to ring spectra, some stable homotopy theory is assumed.

1. Adjunctions and duality

An *adjunction* between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $L : \mathcal{C} \leftrightarrow \mathcal{D} : R$ with "unit" and "counit" natural transformations

$$\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow R \circ L \qquad \epsilon : L \circ R \longrightarrow \mathrm{id}_{\mathcal{D}}$$

such that the composites

$$L \cong L \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{L\circ\eta} L \circ R \circ L \xrightarrow{\epsilon\circ L} \mathrm{id}_{\mathcal{D}} \circ L \cong L$$
$$R \cong \mathrm{id}_{\mathcal{C}} \circ R \xrightarrow{\eta\circ R} R \circ L \circ R \xrightarrow{R\circ\epsilon} R \circ \mathrm{id}_{\mathcal{D}} \cong R$$

are the identity transformations of L and of R, respectively. The adjunction $(\mathcal{C}, \mathcal{D}, L, R, \eta, \epsilon)$ is an *adjoint equivalence of categories* if η and ϵ are natural isomorphisms. Of course, equivalences may be reversed, by taking R to be the left adjoint and L the right adjoint, and by taking the unit and counit to be ϵ^{-1} and η^{-1}, respectively.

Now, suppose that $\mathcal{C}$ is a symmetric monoidal category. Recall that (X, Y) is a *dual pair* in $\mathcal{C}$ if there are maps

$$c : I \longrightarrow Y \otimes X \qquad e : X \otimes Y \longrightarrow I$$

such that the composites

$$X \cong X \otimes I \xrightarrow{\mathrm{id}_X \otimes c} X \otimes Y \otimes X \xrightarrow{e \otimes \mathrm{id}_X} I \otimes X \cong X$$
$$Y \cong I \otimes Y \xrightarrow{c \otimes \mathrm{id}_Y} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}_Y \otimes e} Y \otimes I \cong Y$$

are the identity maps of X and Y, respectively. We say that the duality (X, Y, c, e) is *invertible* if c and e are isomorphisms.

With this presentation of the definitions, the next result is obvious.

Proposition 1.1. *A duality between X and Y is an adjunction between $X \otimes -$ and $Y \otimes -$, as functors from $\mathcal{C}$ to $\mathcal{C}$. The duality is invertible iff the adjunction is an equivalence.*

This observation is completely elementary but it gives the most powerful and elegant proofs of the standard theorems about dualizable objects [LMSM86]. For instance, if $\mathcal{C}$ is a closed symmetric monoidal category, then $X \otimes -$ has two right adjoints, $Y \otimes -$ and $F(X, -)$. Since these are canonically isomorphic, the assembly map

$$F(X, Z) \otimes W \longrightarrow F(X, Z \otimes W)$$

is isomorphic to the associativity isomorphism

$$(Y \otimes Z) \otimes W \cong Y \otimes (Z \otimes W)$$

and is therefore an isomorphism.

The duality also gives an adjunction between $- \otimes Y$ and $- \otimes X$. Since $\mathcal{C}$ is symmetric, these functors are isomorphic to $Y \otimes -$ and $X \otimes -$. We conclude that $X \otimes -$ is both a left and a right adjoint, and its two adjoints are isomorphic. Of course, this is not quite enough to imply that $X \otimes -$ is an equivalence.

We recall that for a given left adjoint L, the right adjoint R is defined by a universal property, and so it is unique up to a canonical isomorphism that preserves η and ϵ. This implies that duals are unique – any two duals of X must be isomorphic, in a way that agrees with the evaluation and coevaluation maps. We therefore say that X is *dualizable* if such a dual exists, and *invertible* if that unique duality is an invertible duality. In other words, we can think of dualizability or invertibility as a condition on X, rather than a set of data.

How do we make that condition more concrete? Suppose again that $\mathcal{C}$ is closed. Then the dual of X, if it exists, must be the right adjoint of $X \otimes -$ applied to the unit, which gives $F(X, I)$. Under this identification, the evaluation map is forced to be the usual evaluation map

$$X \otimes F(X, I) \longrightarrow I$$

which "plugs in" the X into the function object $F(X, I)$. (Because evaluation is adjoint to an isomorphism $Y \longrightarrow F(X, I)$, and the usual evaluation is adjoint to the identity of $F(X, I)$.)

If X is invertible this map must be an equivalence. Conversely, if this is an equivalence then it is trivial to verify that tensoring with X and with $F(X, I)$ give inverse equivalences $\mathcal{C} \longrightarrow \mathcal{C}$. Therefore:

Proposition 1.2. *In a closed symmetric monoidal category $\mathcal{C}$, the object X is invertible iff the evaluation map*

$$X \otimes F(X, I) \longrightarrow I$$

is an isomorphism.

Now for the weaker condition of X being dualizable. If X were dualizable, then the assembly map

$$F(X, I) \otimes X \longrightarrow F(X, X)$$

would be an isomorphism. Under this identification, coevaluation would be the map $I \longrightarrow F(X, X)$ adjoint to the identity of X. In other words, it "picks out the identity" inside the object of all maps from X to X. (Because that's the unit of the adjunction between $X \otimes -$ and $F(X, -)$ on the object I.)

On the other hand, if we *assume* that this assembly map is an isomorphism, then we can lift the identity $I \longrightarrow F(X, X)$ to a map $I \longrightarrow F(X, I) \otimes X$. That gives a candidate coevaluation map. It always forms a duality. The identities to check become

$$\begin{tikzcd}
F(I,X) \arrow[r, "\mathrm{id}_X \otimes c"] \arrow[dr, "\mathrm{id}_X \otimes \eta"'] & F(I,X) \otimes F(X,I) \otimes F(I,X) \arrow[r, "e \otimes \mathrm{id}_X"] \arrow[d, "\cong"] & F(I,X) \\
& F(I,X) \otimes F(X,X) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
F(X,I) \arrow[r, "c \otimes \mathrm{id}_Y"] \arrow[dr, "\eta \otimes \mathrm{id}_Y"'] & F(X,I) \otimes F(I,X) \otimes F(X,I) \arrow[r, "\mathrm{id}_Y \otimes e"] \arrow[d, "\cong"] & F(X,I) \\
& F(X,X) \otimes F(X,I) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

The left-hand triangles commute by definition of c, and the right-hand triangles commute because both routes simply compose the three mapping objects together. It follows that the two diagrams give the identity maps of X and of $F(X, I)$. (As is typical, this is easy to check in any concrete example of a closed symmetric monoidal category $\mathcal{C}$, but a bit of a pain to check in general.) In conclusion:

Proposition 1.3. *In a closed symmetric monoidal category $\mathcal{C}$, the object X is dualizable iff the assembly map*

$$F(X,I) \otimes X \longrightarrow F(X,X)$$

is an isomorphism.

The assembly map always exists, so this drives home the point that dualizability is just a condition on X.

2. Morita adjunctions

Now we will sketch the essentials of Morita theory, in a very suggestive way. Recall that Morita theory has two sides. First, given an additive category $\mathcal{C}$, when can we recognize it as a category of modules over a ring? Second, given two categories of modules, when are they equivalent? These questions are really two sides of the same question. The first is about the existence of a module category modeling $\mathcal{C}$, while the second is about non-trivial equivalences between module categories, which addresses how unique that model of $\mathcal{C}$ can be.

We will focus on the uniqueness question, and summarize the main result as follows.

Theorem 2.1. *Given rings A and R, not necessarily commutative, any equivalence of categories $A - Mod \simeq R - Mod$ is given up to isomorphism by an $R - A$ bimodule M and an $A - R$ bimodule M^*. The equivalence is given by the adjoint functors $M \otimes_A -$ and $M^* \otimes_R -$, and so we have isomorphisms*

$$c : A \xrightarrow{\cong} M^* \otimes_R M, \qquad e : M \otimes_A M^* \xrightarrow{\cong} R$$

satisfying the triangle identities

$$M \cong M \otimes_A A \xrightarrow{\mathrm{id}_M \otimes c} M \otimes_A M^* \otimes_R M \xrightarrow{e \otimes \mathrm{id}_M} R \otimes_R M \cong M \tag{1}$$

$$M^* \cong A \otimes_A M^* \xrightarrow{c \otimes \mathrm{id}_M^*} M^* \otimes_R M \otimes_A M^* \xrightarrow{\mathrm{id}_M^* \otimes e} M^* \otimes_R R \cong M^*$$

Such an equivalence is called a *Morita equivalence* between A and R. By definition, e is a map of left R-modules, but since $M^* \cong M^* \otimes_R R$ and we have functoriality of the isomorphism in the R coordinate, e must be a map of right R-modules too. Similarly c must be a map of $A - A$ bimodules.

The easy converse to this theorem is, given A, R, M, M^*, c, and e, as above, they give an equivalence of module categories.

A Morita equivalence looks suspiciously like an invertible duality. This is not strictly true, because the tensor products are taken alternately over A and over R. However the idea can be made to work, by switching from symmetric monoidal categories to *bicategories*. These are like symmetric monoidal categories, except that each object is additionally marked with two colors for its "source" and "target." One is only allowed to form maps between objects with the same source and target, and in addition the tensor products can only be formed when the target color of one object lines up with the source color of the other object. There are several excellent references on the subject, but we encourage the interested reader to try to first make the idea precise.

Our next question is: if a Morita equivalence is like an invertible duality of bimodules, what concept corresponds to an ordinary duality?

Definition. A*Morita adjunction* consists of rings A and R, bimodules M and M^*, and bimodule maps c and e as above, so that the composites in (3) are the identity. We don't require c and e to be isomorphisms.

Again, because of the way we have defined things, it is obvious that a Morita adjunction is just an adjunction between $M \otimes_A -$ and $M^* \otimes_R -$. Because any two right adjoints are isomorphic, we get an isomorphism of functors

$$M^* \otimes_R - \cong \mathrm{Hom}_R(M, -)$$

and in particular

$$M^* \cong \mathrm{Hom}_R(M, R)$$

Furthermore we know that $\mathrm{Hom}_R(M, -)$ is right exact and commutes with coproducts, because it's isomorphic to the tensoring functor $M^* \otimes_R -$. By a quick inspection of $\mathrm{Hom}_R(M, -)$ applied to some surjective map of R-modules

$$\bigoplus R \longrightarrow M$$

we conclude that M must be perfect (i.e. finitely generated projective) as a left R-module.

The observation $M^* \cong \mathrm{Hom}_R(M, R)$ has a startling consequence. If we give just a ring R and a perfect left R-module M, there is only one possible Morita equivalence using this data. M^* must be $\mathrm{Hom}_R(M, R)$, and A must be

$$M^* \otimes_R M = \mathrm{Hom}_R(M, R) \otimes_R M \cong \mathrm{Hom}_R(M, M)$$

(The last map is an isomorphism because M is perfect.) So we take A to be the ring $\mathrm{Hom}_R(M, M)$, and M^* to be $\mathrm{Hom}_R(M, R)$. This fits into a Morita equivalence so long as the map

$$M \otimes_{\mathrm{Hom}_R(M,M)} M^* \longrightarrow R$$

is an equivalence. This happens when M is a "generator," which means that maps out of M detect all maps of R-modules. Equivalently, R is a direct summand of a direct sum of copies of M ([Bas62], Lemma 1).

The Morita equivalence constructed out of R and M is unique. If we had some other A and M^* that worked, the above composite would send the identity element of A to the identity of M, because it's the unit of the adjunction. Therefore we must have an isomorphism of rings $A \cong \mathrm{Hom}_R(M, M)$, respecting the action on M.

Let's return to Morita adjunctions. We've seen that a Morita adjunction implies that M is a perfect R-module. Conversely, if M is perfect, then to get an adjunction we must take $M^* = \mathrm{Hom}_R(M, R)$. As soon as we specify how A acts on M, we can lift the action along the assembly map and get the coevaluation map

$$A \longrightarrow M^* \otimes_R M$$

This gives a duality, by essentially the argument we gave at the end of section 1.

It's important to notice that the ring A was totally unimportant for this to work. We only needed it to be some ring that commuted with the R-action. So the "universal" choice for A is $\mathrm{Hom}_R(M, M)$, but any old ring mapping into $\mathrm{Hom}_R(M, M)$ will give a Morita adjunction. To summarize:

Proposition 2.2.

- *Morita equivalences into $R - Mod$ correspond to perfect left R-modules M that are generators.*
- *Morita adjunctions with left adjoint going into $R - Mod$ correspond to choices of $R - A$ bimodules M which are perfect over R.*
- *Each Morita adjunction is an equivalence iff the induced map $A \longrightarrow \mathrm{Hom}_R(M, M)$ is an isomorphism, and M is a generator.*
- *Every ring Morita equivalent to R is of the form $\mathrm{Hom}_R(M, M)$ for some perfect generator R-module M.*

Proposition 2.3. *The following are equivalent for a left R-module M:*

(1) *$M \otimes_{\mathbb{Z}} -$ is a left Morita adjoint.*
(2) *$M \otimes_A -$ is a left Morita adjoint for some ring A.*
(3) *M is perfect (finitely generated and projective).*
(4) *The assembly map $\mathrm{Hom}_R(M, R) \otimes_R M \longrightarrow \mathrm{Hom}_R(M, M)$ is an isomorphism of abelian groups.*

We conclude that while Morita equivalences are not so common, Morita adjunctions are very common. By the Eilenberg-Watts theorem, any sufficiently nice functor of module categories (right exact, coproduct-preserving) must come about by tensoring with a bimodule. Such a functor then gives a Morita adjoint as long as the resulting bimodule is perfect over R.

There is also a Morita theory for derived categories, which we will not cover here. In essence, we continue to work with rings but we let the A-R bimodules and maps between them be objects and morphisms in the derived category $\mathcal{D}(A \otimes R^{\mathrm{op}})$. See [Sch04].

3. Morita adjunctions for ring spectra

Schwede and Shipley have performed a beautiful generalization of classical Morita theory to the setting of ring spectra and module spectra. Instead of searching for equivalences of categories, we look for Quillen equivalences between module categories. We recall the main "uniqueness" result from [SS03].

Theorem 3.1. *If A and R are ring spectra whose categories of module spectra are equivalent as spectral model categories, then there is an $R-A$ bimodule M such that the derived smash product $M \wedge_A^L -$ gives an equivalence from the homotopy category A-modules to the homotopy category of R-modules.*

Now let us take their framework and flesh out the structure a bit more. Let's work in orthogonal spectra. Recall the following standard result.[1]

Proposition 3.2. *The following are equivalent for a cofibrant left R-module M:*

1. *M is perfect (a homotopy retract of a finite cell spectrum).*
2. *$F_R(M, -)$ commutes with all wedge sums up to equivalence.*
3. *The derived assembly map $F_R(M, R) \wedge_R N \longrightarrow F_R(M, N)$ is a weak equivalence for all R-modules N.*
4. *The derived assembly map $F_R(M, R) \wedge_R M \longrightarrow F_R(M, M)$ is a weak equivalence.*

Now let's add more structure to these Morita equivalences. Assume that the underlying spectra of A and R are cofibrant. (In particular, we could take them to be a cofibrant rings.) Then if M is a cofibrant bimodule, the functor $M \wedge_A -$ is a Quillen left adjoint from A-modules to R-modules, with right adjoint $F_R(M, -)$.

(We observe that $M \wedge_A -$ sends cofibrations to cofibrations iff $M \wedge_A (A \wedge F_k(i))$ is a cofibration when i is the inclusion of a sphere into a disc. Since M is built up from cells of the form $A \wedge R \wedge F_k(S^n)$, which send the above to $F_k(S^n) \wedge R \wedge A \wedge F_k(i)$, we need the assumption that A is cofibrant. On the other hand, to preserve weak equivalences, we do the same argument with i the inclusion of a disc into disc cross interval, and we also check it with i as before but box product with a certain mapping cylinder of orthogonal spectra which is already a weak equivalence. Essentially we then need that smashing with $R \wedge A$ preserves weak equivalences of cofibrant orthogonal spectra. This is true without any additional assumptions on R. Of course, we need to assume R is cofibrant to get $M^* \wedge_R -$ to be left Quillen too.)

If we let $M^* \longrightarrow F_R(M, R)$ be a cofibrant replacement of $A-R$ bimodules, then $M^* \wedge R-$ admits a natural transformation into $F_R(M, -)$ by

$$M^* \wedge_R N \longrightarrow F_R(M, R) \wedge_R F_R(R, N) \longrightarrow F_R(M, N)$$

By the above proposition, on fibrant R-modules N this is an equivalence. (Cofibrancy of M^* makes the $\wedge_R$ derived.) If f denotes a monoidal fibrant replacement in orthogonal spectra, cA denotes a cofibrant replacement of A as an $A-A$ bimodule, we can lift the A-action map $A \longrightarrow F_R(M, M)$ up to homotopy to a map

$$c : cA \longrightarrow f(M^* \wedge_R M)$$

[1] I've given an exposition of the proof in the notes "Finite modules."

Together with $e : M \wedge_A M^* \longrightarrow R$, we get triangle identities

$$M \wedge_A cA \xrightarrow{\mathrm{id}_M \wedge c} M \wedge_A f(M^* \wedge_R M) \xrightarrow{e \wedge \mathrm{id}_M} fM \tag{2}$$

$$cA \wedge_A M^* \xrightarrow{c \wedge \mathrm{id}^*_M} f(M^* \wedge_R M) \wedge_A M^* \xrightarrow{\mathrm{id}^*_M \wedge e} f(M^*)$$

whose composites agree up to homotopy with the identifications

$$M \wedge_A cA \xrightarrow{\sim} M \xrightarrow{\sim} fM \tag{3}$$

$$cA \wedge_A M^* \xrightarrow{\sim} M^* \xrightarrow{\sim} f(M^*)$$

Definition. A*spectral Morita adjunction* consists of orthogonal ring spectra A and R whose underlying spectra are cofibrant, cofibrant bimodules M and M^*, and bimodule maps c and e as above, satisfying the two triangle identities in the sense outlined above.

These statements are only messy because we are choosing not to work in the homotopy category. In the homotopy category, the definition becomes exactly the same as the earlier one for rings.

As for ordinary rings, being a Morita adjoint means that $F_R(M, -)$ is a left adjoint on the homotopy category, so it commutes with arbitrary sums. From this we can deduce that M must be perfect over R.

Given cofibrant R and cofibrant perfect M, to get a Morita equivalence or adjunction, M^* must be some cofibrant replacement of $F_R(M, R)$. To get a spectral Morita equivalence we must have A a cofibrant replacement of $F_R(M, M)$, and then M^* must be made cofibrant as an $A - R$ bimodule. This will successfully give an equivalence so long as M generates the homotopy category of R-modules.

To get an adjunction we may again take A to be any old ring acting on R, and then we may lift the map $A \longrightarrow F_R(M, M)$ to $cA \longrightarrow f(M^* \wedge_R M)$ and get a Morita adjunction as above. In summary, the propositions from the last section remains true with little modification:

Proposition 3.3.

- *Morita equivalences into $R - Mod$ correspond to perfect left R-modules M that are generators.*
- *Morita adjunctions with left adjoint going into $R - Mod$ correspond to choices of $R - A$ bimodules M which are perfect over R.*
- *Each Morita adjunction is an equivalence iff the induced map $A \longrightarrow \mathrm{Hom}_R(M, M)$ is an isomorphism, and M is a generator.*
- *Every ring Morita equivalent to R is of the form $\mathrm{Hom}_R(M, M)$ for some perfect generator R-module M.*

Proposition 3.4. *The following are equivalent for a cofibrant left R-module M:*

1. *$M \wedge_{\mathbb{S}} -$ is a left Morita adjoint.*
2. *$M \wedge_A -$ is a left Morita adjoint for some ring spectrum A.*

(3) *M is perfect (a homotopy retract of a finite cell spectrum).*
(4) *The derived assembly map $F_R(M,R) \wedge_R M \longrightarrow F_R(M,M)$ is a weak equivalence of spectra.*

4. Morita duality

Now we will start to say things that are not so well-known, and should be known better. There is a notion of "Morita duality" between two rings, and surprisingly, it is *more general* than a Morita equivalence. And it is neither more nor less general than a Morita adjunction. We will resolve the resulting confusion by giving four conditions on an A-B bimodule P, and showing that various subsets of these conditions imply Morita equivalence, Morita adjunction, and Morita duality. All of our statements can be interpreted as applying to rings and bimodules on the nose, or ring spectra and the homotopy category of bimodules.

Definition. Let A and B be rings (or ring spectra) and let P be an A-B bimodule (spectrum). Consider the following four conditions on P

(D_A) P is dualizable over A, in other words the natural map

$$\mathrm{Hom}_A(P,A) \otimes_A P \to \mathrm{Hom}_A(P,P)$$

is an isomorphism.

(D_B) P is dualizable over B, in other words the natural map

$$P \otimes_B \mathrm{Hom}_B(P,B) \to \mathrm{Hom}_B(P,P)$$

is an isomorphism.

(C_A) P satisfies the centralizer condition over A, in other words the natural map

$$A \to \mathrm{Hom}_B(P,P)$$

is an isomorphism.

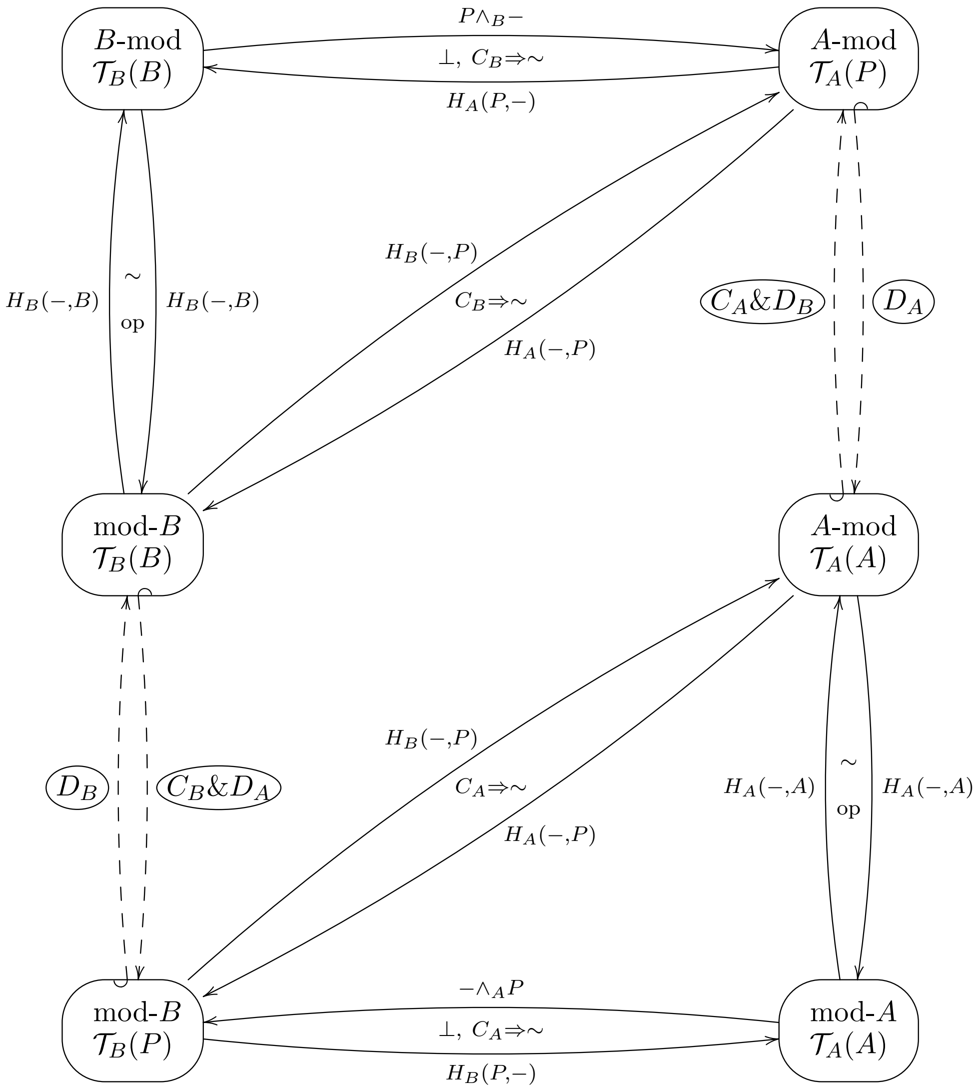
(C_B) P satisfies the centralizer condition over B, in other words the natural map

$$B \to \mathrm{Hom}_A(P,P)$$

is an isomorphism.

Definition. A *Morita duality* between A and B is an A-B bimodule P satisfying the two centralizer conditions C_A and C_B.

This implies that $\mathrm{Hom}_A(-,P)$ and $\mathrm{Hom}_B(-,P)$ give contravariant equivalences between certain subcategories of left A-modules and right B-modules. In fact, there are two such subcategories we can pick. We can either go between the thick subcategory of A-modules on A and the thick subcategory of B-modules on P, or between the thick subcategory of A-modules on P and the thick subcategory of B-modules on B. In general, neither of these cases is contained in the other. This is summarized in the diagram below. It is fairly elementary to prove that the dotted maps are inclusions of subcategories under the stated conditions.



As the diagram indicates, however, when we assume in addition the conditions D_A and D_B, we end up with an equivalence between A-modules finitely generated over A and B-modules finitely generated over B, which then extends to all modules. In other words, P gives a Morita equivalence if it satisfies all four of the above conditions.

We can also summarize this in the table below. Each term on the left is listed with the minimal set of conditions on the right that it must satisfy, and conversely if these conditions are satisfied then we get the term on the left.

Morita equivalence	D_A	D_B	C_A	C_B
Morita adjunction	D_A			
Morita duality			C_A	C_B

Morita duality is also closely related to Koszul duality. In derived bimodules, or bimodule spectra, Koszul duality is just Morita duality in which P is the ground ring k and the A-module structure on P arises from a ring map $A \to k$, in other words an augmentation.

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