FUNDAMENTAL THEOREMS FOR THH

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In this expository note we give illustrated proofs of three "fundamental theorems" for topological Hochschild homology (THH) of ring spectra and spectral categories.

- Morita invariance: THH is unchanged when we pass from a spectral category to its thick closure (Theorem 2.1).
- Homotopy invariance: weak equivalences of spectral functors give homotopic maps on THH (Theorem 5.3).
- Additivity: cofiber sequences of spectral functors split after applying THH (Theorem 6.3 and Theorem 6.9).

We also deduce some standard corollaries: THH preserves products (Proposition 2.5), the existence of the Dennis trace map (in the discussion after Theorem 6.9), and the THH version of the Eilenberg swindle (Corollary 6.4).

The localization theorem from [BM12] also deserves to be here, but the author has not yet spent enough time with it so it will have to wait until a future version.

Our perspective on these proofs was developed during the writing of [CLM$^+$a],[1] but the ideas are older than that. Many of them go back to the origins of THH with Waldhausen and Dennis [Wal79]. They were subsequently developed by Dundas and McCarthy [DM96], Blumberg and Mandell [BM12], and many others [BGT13, HS18, CP19]. These results also have conceptually similar but technically distinct proofs for THH of small stable ∞-categories, see for instance [BGT13, HS18].

1. Definitions

Throughout, the word "spectra" will refer to orthogonal spectra (see e.g. [MMSS01, Sch07]), but any well-behaved symmetric monoidal category of spectra will do.

A **spectral category** C is a (small) category enriched in spectra. We write the composition from left to right, because this is more natural when thinking about bar constructions. So in detail, a spectral category C consists of a set of objects ($\operatorname{ob} C$), a spectrum $C(a,b)$ for every pair of objects $a, b \in \operatorname{ob} C$, and composition maps

$$C(a,b) \wedge C(b,c) \to C(a,c)$$

that are associative and unital.

Example 1.1. • A ring spectrum A is the same thing as a spectral category with just one object.

[1] Many thanks to my co-authors for sharing their wisdom and helping to develop these awesome pictures.

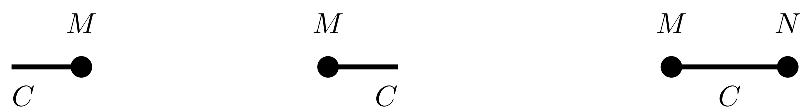
- If A is a ring spectrum, there is a spectral category ${}_A\mathrm{Mod}$ whose objects are left A-module spectra M. The mapping spectrum ${}_A\mathrm{Mod}(M, N)$ is the derived spectrum of A-linear maps from M to N, so it has the homotopy type of the A-linear mapping spectrum $F_A(M, N)$ when M and N are cofibrant and fibrant.

The **underlying category** C_0 is obtained by restricting to spectrum level 0 and forgetting the topology. The **homotopy category** $\pi_0 C$ is obtained by taking π_0 of all the mapping spectra $C(a, b)$, giving a category enriched in abelian groups.

A **left module** M over a spectral category C is an assignment of each object $a \in \mathrm{ob}\, C$ to a spectrum $M(a)$, and multiplication maps

$$C(a, b) \wedge M(b) \to M(a)$$

that are associative and unital. A right module is defined similarly. We represent left and right modules as a vertex with a single line coming out, as in Figure 1.2.



(A) Left module (B) Right module (C) Bar construction $B(M; C; N)$

FIGURE 1.2

If M and N are a left and a right module over C, respectively, then the **two-sided bar construction** $B(M; C; N)$ is the realization of the simplicial spectrum that at level n is

$$\bigvee_{c_0,\ldots,c_n \in \mathrm{ob}\, C} M(c_0) \wedge C(c_0, c_1) \wedge C(c_1, c_2) \wedge \ldots \wedge C(c_{n-1}, c_n) \wedge N(c_n).$$

The face maps compose the mapping spectra and the degeneracies insert units. We represent bar constructions by connecting the lines for the two modules together, see Figure 1.2.

If C and D are spectral categories, a (C, D)-**bimodule** is an assignment of each ordered pair $(c, d) \in \mathrm{ob}\, C \times \mathrm{ob}\, D$ to a spectrum $M(c, d)$, multiplication maps

$$C(c, c') \wedge M(c', d) \to M(c, d), \qquad M(c, d) \wedge D(d, d') \to M(c, d')$$

that are associative and unital, and the additional "associativity" condition that the two composites

$$C(a, b) \wedge M(b, d) \wedge D(d, e) \rightrightarrows M(a, e)$$

are equal. (In other words, the two actions commute with each other.)

If M is a (C, D)-bimodule and N is a (D, E)-bimodule then we can form the collection of two-sided bar constructions

$$B(M(c, -); D; N(-, e)), \qquad (c, e) \in \mathrm{ob}\, C \times \mathrm{ob}\, E.$$

Since the C and E actions on the outside commute with the D action, they make this collection of spectra into a (C, E)-bimodule that we denote $B(M; D; N)$. We depict these bimodules and bar constructions using vertices with two lines coming out, see Figure 1.3.

(A) Bimodules (B) A bar construction with bimodules

FIGURE 1.3

There is a canonical (C, C)-bimodule whose spectra are just the mapping spectra $C(a, b)$, and whose actions come from the composition of C. We call this bimodule C by abuse of notation. The following lemma is a standard result about bar constructions and can be proven by an explicit simplicial homotopy.

Lemma 1.4 (Bar lemma)**.** *For any (C, D)-bimodule M there are canonical equivalences of (C, D)-bimodules*

$$B(C; C; M) \xrightarrow{\sim} M, \qquad B(M; D; D) \xrightarrow{\sim} M$$

that simply compose all the mapping spectra in C or D into M.

In pictures, this means that a vertex labeled by the bimodule C can simply be deleted without changing the picture, see Figure 1.5.

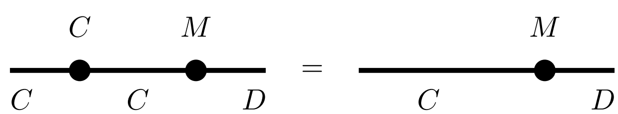


FIGURE 1.5. The bar lemma.

We can also tensor bimodules to themselves, if the same category acts on both sides. If M is a (C, C)-bimodule then the **cyclic bar construction**, or **topological Hochschild homology** $\mathrm{THH}(C; M)$, is the realization of the simplicial spectrum that at level n is

$$\bigvee_{c_0, \ldots, c_n \in \mathrm{ob}\, C} C(c_0, c_1) \wedge C(c_1, c_2) \wedge \ldots \wedge C(c_{n-1}, c_n) \wedge M(c_n, c_0).$$

We draw this as a circle with a vertex for M, as in Figure 1.6. Motivated by Figure 1.5, if the bimodule is C itself, we write this as $\mathrm{THH}(C) = \mathrm{THH}(C; C)$ and remove the vertex from the picture.

(A) $\mathrm{THH}(C; M)$ (B) $\mathrm{THH}(C)$

FIGURE 1.6

Throughout this note we assume that the mapping spectra of C are nice enough that these bar constructions agree with their left-derived functors. For instance, it is enough if each spectrum in C, M and N is cofibrant in the stable model structure on orthogonal spectra, or more generally in the flat model structure. Any spectral category, module, or bimodule can have its mapping spectra cofibrantly replaced so that this is true.

Remark 1.7. Spectral categories do not have nerves. The reason is that the terminal object in spectra $*$ is not equivalent to the unit object $\mathbb{S}$. In fact, smashing with the zero spectrum $*$ returns the zero spectrum $*$, and so

$$B(*; C; *) \simeq *.$$

This is why we focus instead on two-sided bar constructions $B(M; C; N)$ and the cyclic nerve $\mathrm{THH}(C; C)$.[2]

A functor of spectral categories $F\colon C \to D$ is a map on object sets and on mapping spectra

$$C(a, b) \to D(F(a), F(b)),$$

respecting composition and units. For one-object categories, this is the same thing as a ring homomorphism.

For a left D-module M we let ${}_FM$ denote the **pullback** along F, a C-module whose value at each $c \in \operatorname{ob} C$ is $M(F(c))$ and whose C-action is through the homomorphism F. We can similarly define pullbacks on the right M_F or on both sides ${}_FM_F$.

It is easy to see from the definition that F induces a map on THH

$$\mathrm{THH}(C) \to \mathrm{THH}(D).$$

More generally, F induces a map of cyclic bar constructions

$$\mathrm{THH}(C; {}_FM_F) \to \mathrm{THH}(D; M)$$

for any (D, D)-bimodule M, and a map of bar constructions

$$B(M_F; C; {}_FN) \to B(M; D; N)$$

for any D-modules M and N.

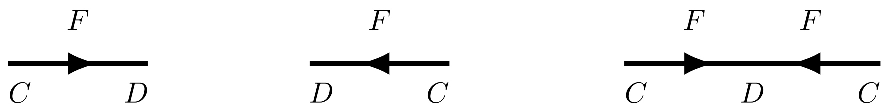
For any functor $F\colon C \to D$, if we pull back the canonical (D, D)-bimodule D, we call the resulting bimodule ${}_FD$ the **base-change bimodule** for F. It is a (C, D)-bimodule whose value at (c, d) is the spectrum $D(F(c), d)$. We similarly get a base-change bimodule on the other side D_F, and a double base-change bimodule ${}_FD_F$. We illustrate these bimodules in our diagrams by an arrow rather than a vertex, see Figure 1.8.

We illustrate more general pullbacks ${}_FM$ by putting an arrow next to M as in Figure 1.9. This is consistent because by the bar lemma

$$B({}_FD; D; M) \simeq {}_FM.$$

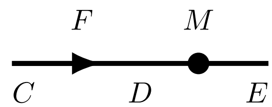
It also explains the terminology of "base-change bimodule" – multiplying by a base-change bimodule has the effect of pulling back M.

[2] More generally, enriched functors do not have colimits or homotopy colimits. They only have *weighted* colimits and homotopy colimits [Kel82, Shu06]. For spectral categories, the weighted homotopy colimit of a diagram M with weights given by N is precisely the two-sided bar construction $B(M; C; N)$.



(A) ${}_F D$ (B) D_F (C) ${}_F D_F \simeq B({}_F D; D; D_F)$

FIGURE 1.8



(A) ${}_F M \simeq B({}_F D; D; M)$

FIGURE 1.9

A **Dwyer-Kan embedding** is a functor $C \to D$ in which each of the maps $C(c, d) \to D(Fc, Fd)$ is a (stable) equivalence of spectra.

If C is a spectral category, then the collection of all left or right modules over C forms another spectral category with derived mapping spectra between them. There is a Dwyer-Kan embedding of C into left C-modules

$$C \to {}_C\mathrm{Mod}$$

sending $a \in \mathrm{ob}\, C$ to the "representable" left module $C(-, a)$. It is not a stretch to call this the **Yoneda embedding**. Similarly, C^{op} embeds into all right C-modules

$$C^{\mathrm{op}} \to \mathrm{Mod}_C$$

by sending a to the right module $C(a, -)$. One should think of these representable modules as the "free rank one modules" over the "ring" C.

An important special case is when C has one object, so it is a ring spectrum $A = C(a, a)$. Then this is the embedding of A into its category of left modules, sending the unique object to the free rank one *left* module A, and sending each element $a \in A$ to the endomorphism of A that multiplies on the *right* by a.

There are three distinct notions of "equivalence" of spectral categories, in increasing order of generality:

- $F \colon C \to D$ is a **pointwise equivalence** if it is a Dwyer-Kan embedding and is the identity on the object sets.
- $F \colon C \to D$ is a **Dwyer-Kan equivalence** if it is a Dwyer-Kan embedding and induces an equivalence of homotopy categories $\pi_0 C \xrightarrow{\sim} \pi_0 D$.
- $F \colon C \to D$ is a **Morita equivalence** if it is a Dwyer-Kan embedding and is surjective up to thick closure.

The last point requires further explanation. F is **surjective up to thick closure** if, after embedding into the category of modules over D, D is obtained from the image of C by taking retracts, cofibers, fibers, and extensions. To be concrete, it is enough to show that for each object $d \in \operatorname{ob} D$, the set $\{Fc : c \in \operatorname{ob} C\}$ can be enlarged to contain d by a finite sequence of steps of the following form.

- Take a retract in $\pi_0 D$ of an object already in the set and add it to the set.
- Take a pair of composable morphisms $d \to d' \to d''$ in the underlying category D_0 such that the induced maps of modules
$$D(d'', -) \to D(d', -) \to D(d, -), \qquad D(-, d) \to D(-, d') \to D(-, d'')$$
can be extended up to equivalence to cofiber sequences of modules, and two of the objects d, d', and d'' are already in the set; add the third object to the set.

Example 1.10.

- Cofibrant replacement of spectral categories is an example of pointwise equivalence. For any spectral category C, there is another spectral category C' on the same objects and a pointwise equivalence $C' \to C$. In other words, each map $C'(a, b) \to C(a, b)$ is an equivalence of spectra.
- Restricting C to a skeleton (a full subcategory with one object in each isomorphism class in $\pi_0 C$) is an example of a Dwyer-Kan equivalence.
- A module P over a ring spectrum A is **perfect** if it is obtainable from the free rank-one module by finite sums, cofibers, and retracts. Let ${}_A\mathrm{Perf} \subseteq {}_A\mathrm{Mod}$ denote the full subcategory on those A-modules that are perfect. Then the restriction of the Yoneda embedding to $A \to {}_A\mathrm{Perf}$ is an example of a Morita equivalence.

2. Morita invariance

Our first main theorem is that THH respects all three of the above notions of equivalence. Since we have the implications

$$\text{pointwise equivalence} \;\Rightarrow\; \text{Dwyer-Kan equivalence} \;\Rightarrow\; \text{Morita equivalence},$$

it suffices to prove that THH respects Morita equivalences.

Theorem 2.1 (e.g. [BM12, 6.4])**.** *If $F : C \to D$ is a Morita equivalence of spectral categories then the map*
$$\mathrm{THH}(C) \to \mathrm{THH}(D)$$
induced by F is an equivalence of spectra.

More generally, if F is a Morita equivalence then for any (D,D)-bimodule Q, the map
$$\mathrm{THH}(C; {}_FQ_F) \to \mathrm{THH}(D; Q)$$
induced by F is an equivalence, as is every map of two-sided bar constructions
$$B(M_F; C; {}_FN) \to B(M; D; N).$$

Remark 2.2. We are assuming that the mapping spectra are nice enough that THH preserves pointwise equivalences. So really, as soon as you arrange things so that THH preserves pointwise equivalences, then it also preserves Morita equivalences.

Before we explain the proof, let's list some consequences.

Corollary 2.3. *The embedding of A into ${}_A\mathrm{Perf}$ induces an equivalence on* THH*:*
$$\mathrm{THH}(A) \xrightarrow{\sim} \mathrm{THH}({}_A\mathrm{Perf}).$$

Two ring spectra are Morita equivalent if their categories of perfect modules are Dwyer-Kan equivalent.

Corollary 2.4. *If A and B are Morita equivalent then*
$$\mathrm{THH}(A) \simeq \mathrm{THH}(B).$$

In particular, $\mathrm{THH}(A) \simeq \mathrm{THH}(M_n(A))$, where $M_n(A) = F_A(A^{\vee n}, A^{\vee n}) \cong \prod^n \bigvee^n A$ is the ring of $n \times n$ matrices in the ring spectrum A.

Another surprising corollary involves the product of two spectral categories. Given two spectral categories C and D, define $C \times D$ to have object set $\mathrm{ob}\, C \times \mathrm{ob}\, D$, and morphisms from (c, d) to (c', d') given by the product $C(c, c') \times D(d, d')$.

We similarly define the smash product category $C \wedge D$ to have objects $\mathrm{ob}\, C \times \mathrm{ob}\, D$ and morphisms the smash product $C(c, c') \wedge D(d, d')$. It is not hard to check that THH respects smash products:
$$\mathrm{THH}(C \wedge D) \cong \mathrm{THH}(C) \wedge \mathrm{THH}(D).$$
It is much more surprising that THH also respects Cartesian products.

Proposition 2.5. *cf.* [DGM13, 1.4.4] *Assuming* $\mathrm{ob}\, C$ *and* $\mathrm{ob}\, D$ *are nonempty,*[3] *the projection functors* $C \times D \to C$, $C \times D \to D$ *induce an equivalence*
$$\mathrm{THH}(C \times D) \xrightarrow{\sim} \mathrm{THH}(C) \times \mathrm{THH}(D).$$

In particular, THH of a product ring decomposes into THH of the factors.

To prove Proposition 2.5 we introduce the wedge and zero objects for spectral categories. C has a zero object $*$ if
$$C(*, c) \cong C(c, *) \cong *,$$
in other words the spectrum of maps in or out of $*$ is the zero spectrum, for all objects $c \in \mathrm{ob}\, C$. Given a spectral category C we let C_+ denote the category obtained by adding an extra zero object. It is easy to see that the inclusion $C \to C_+$ induces an isomorphism of spectra
$$\mathrm{THH}(C) \cong \mathrm{THH}(C_+).$$
For two spectral categories C and D, we define the wedge $C_+ \vee D_+$ to have object set $\mathrm{ob}\, C \amalg \{*\} \amalg \mathrm{ob}\, D$, mapping spectra coming from C or D if both objects are in the same category, and zero otherwise. It is also easy to see that the inclusions $C, D \to C_+ \vee D_+$ induce an isomorphism
$$\mathrm{THH}(C) \vee \mathrm{THH}(D) \cong \mathrm{THH}(C_+) \vee \mathrm{THH}(D_+) \cong \mathrm{THH}(C_+ \vee D_+).$$
Finally, we check that the inclusions
$$C_+ \vee D_+ \to C_+ \times D_+ \leftarrow C \times D$$
are Morita equivalences. It is clear they are Dwyer-Kan embeddings. The first one is surjective up to thick closure because every object (c, d) in the product fits into a cofiber sequence with the objects $(c, *)$ and $(*, d)$ coming from the wedge. The second one is surjective up to thick closure because each $(c, *)$ or $(*, d)$ is a retract of some object (c, d).[4]

[3] Note that if a spectral category has empty object set then its THH is the zero spectrum $*$.

[4] This is where we use the assumption that $\mathrm{ob}\, C$ and $\mathrm{ob}\, D$ are nonempty.

Therefore by 2.1, the maps marked $\sim$ below are equivalences.

$$\begin{tikzcd}
\mathrm{THH}(C) \vee \mathrm{THH}(D) \arrow[r, leftrightarrow, "\cong"] & \mathrm{THH}(C_+ \vee D_+) \arrow[r, "\sim"] & \mathrm{THH}(C_+ \times D_+) \arrow[r] & \mathrm{THH}(C_+) \times \mathrm{THH}(D_+) \arrow[d, leftrightarrow, "\cong"] \\
 & & \mathrm{THH}(C \times D) \arrow[u, "\sim"] \arrow[r] & \mathrm{THH}(C) \times \mathrm{THH}(D)
\end{tikzcd}$$

The composite $\mathrm{THH}(C) \vee \mathrm{THH}(D) \to \mathrm{THH}(C) \times \mathrm{THH}(D)$ is just the inclusion of the wedge into the product, so it is an equivalence of spectra. Therefore the remaining maps are equivalences too, proving Proposition 2.5.

3. Proof of Morita invariance

The first step is to define the map of cyclic bar constructions depicted in Figure 3.1. Intuitively, we're applying F to the circle bit by bit, rather than doing it all at once.

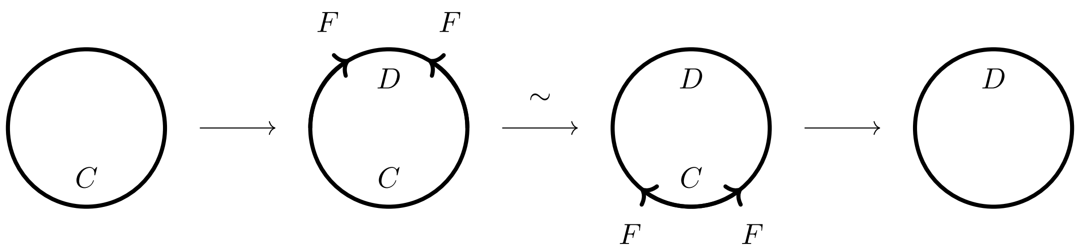


FIGURE 3.1

Focusing on the part of the circle where the action is, we get maps of bimodules depicted in Figure 3.2. Again, each one applies F to a segment in C to create a segment in D. To prove that the entire composite is an equivalence, is suffices to show that each of these maps is an equivalence of bimodules.

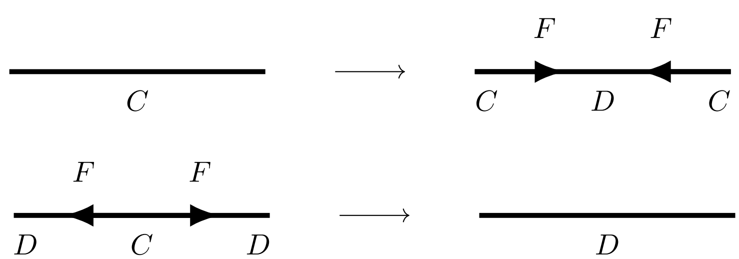


FIGURE 3.2

Lemma 3.3. *If F is a Morita equivalence, in other words a Dwyer-Kan embedding that is surjective up to thick closure, then the map of (C, C)-bimodules*

$$C \to {}_F D_F,$$

and the map of (D, D)-bimodules

$$B(D_F; C; {}_F D) \to B(D; D; D) \simeq D$$

induced by F are both equivalences of bimodules.

Proof. For the first one, if suffices to show that each map $C(c,c') \to D(Fc,Fc')$ is an equivalence, but this follows by definition because F is a Dwyer-Kan embedding.

For the second one, we need to show for each $d, d' \in \operatorname{ob} D$ that

$$B(D(d,-);C;D(-,d')) \longrightarrow D(d,d') \tag{3.4}$$

is an equivalence. If $d = Fc$ and $d' = Fc'$ are in the image of F, then we get the commuting square

$$\begin{tikzcd} B(C(c,-);C;C(-,c')) \arrow[r, "\sim"] \arrow[d, "\sim"'] & C(c,c') \arrow[d, "\sim"] \\ B(D(Fc,-);C;D(-,Fc')) \arrow[r] & D(Fc,Fc'). \end{tikzcd}$$

The vertical maps are equivalences since F is a Dwyer-Kan embedding and the top is an equivalence by the bar lemma. Hence the bottom is an equivalence too. This proves that (3.4) is an equivalence for *some* of the pairs of objects in D.

But recall that F is surjective up to thick closure. Therefore every other object in D can be obtained from these objects by retracts and cofiber sequences, and these induce retracts and cofiber sequences in both the source and target of (3.4). Therefore by a standard induction up cofiber sequences, (3.4) is an equivalence for *every* pair of objects in D. □

The second step of the proof is to write down the map on THH corresponding to the picture in Figure 3.1, and prove that it is homotopic to the desired map $\mathrm{THH}(C) \to \mathrm{THH}(D)$ coming from F. The following lemmas are the essential tools here.

Lemma 3.5 (Dennis-Waldhausen-Morita argument)**.** *For each (C,D)-bimodule M and a (D,C)-bimodule N, there is a natural "rotator" isomorphism*

$$\mathrm{THH}(C; B(M;D;N)) \cong \mathrm{THH}(D; B(N;C;M)).$$

Proof. Both sides are the realization of a bisimplicial spectrum in which level (m,n) is

$$\bigvee_{\substack{c_0,\ldots,c_m \in \operatorname{ob} C \\ d_0,\ldots,d_n \in \operatorname{ob} D}} C(c_0,c_1)\wedge\ldots\wedge C(c_{m-1},c_m)\wedge M(c_m,d_0)\wedge D(d_0,d_1)\wedge\ldots\wedge D(d_{n-1},d_n)\wedge N(d_n,c_0).$$

□

Lemma 3.6. *If M is a (D,D)-bimodule then the collapse maps $B(D;D;M) \to M$ and $B(M;D;D) \to M$ and the rotator isomorphism fit into a diagram*

$$\begin{tikzcd} \mathrm{THH}(D; B(D;D;M)) \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, "\sim"'] & & \mathrm{THH}(D; B(M;D;D)) \arrow[dl, "\sim"] \\ & \mathrm{THH}(D;M) & \end{tikzcd}$$

that commutes up to homotopy.

Proof. The diagram does not commute strictly because different simplicial directions are collapsed away. One can write down an explicit simplicial homotopy as in [LM19, 5.6] or [Lin13]. Alternatively, one can interpret all three of these constructions as left-derived functors of the operation that takes each bimodule M to the spectrum M/D defined as the coequalizer

$$\bigvee_{d,d' \in \mathrm{ob}\, D} D(d',d) \wedge M(d,d') \rightrightarrows \bigvee_{d,d' \in \mathrm{ob}\, D} M(d,d') \to M/D.$$

Each of three maps we defined between them lies over the identity of M/D, hence both routes in the diagram are maps of left-derived functors of M/D. But left-derived functors are unique in the homotopy category of functors over M/D, hence the two routes must agree in the homotopy category of functors. Therefore they agree up to homotopy for any one particular bimodule M. See e.g. [Shu06, Mal19] for more details. □

Remark 3.7. These lemmas are part of the verification that our circle diagrams can be used in rigorous proofs. If we draw a circle studded with several bimodules, then there are several ways to interpret it using THH and the two-sided bar construction, but all these representations are equivalent to each other. See for instance Figure 3.8. This can be captured formally by the idea that THH is a **shadow** or a **trace theory** on spectral categories and bimodules, see [Pon10, PS13]. And not only are all the interpretations of each picture equivalent, but any two equivalences obtained by expanding and collapsing bar constructions, or rotating the circle, are in fact homotopic [MP18].

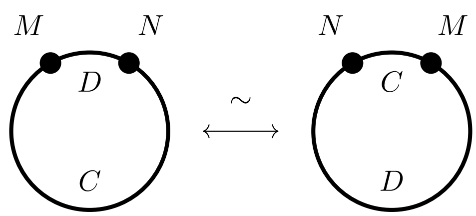


FIGURE 3.8

Now we can write down the maps on THH corresponding to our picture in Figure 3.1. They form the right-hand route in the following diagram. Again, the intuition is that they apply F to the circle in stages, turning each C into a D.

(3.9)
$$\begin{tikzcd}
\mathrm{THH}(C;C) \arrow[d, "\sim"'] & \\
\mathrm{THH}(C; {}_F D_F) \arrow[d] & \mathrm{THH}(C; B({}_F D; D; D_F)) \arrow[l, "\sim"'] \arrow[d, "\cong"] \\
\mathrm{THH}(D;D) & \mathrm{THH}(D; B(D_F; C; {}_F D)) \arrow[l, "\sim"']
\end{tikzcd}$$

The composite of the vertical maps on the left is the map on THH induced by F. The square commutes up to homotopy – mapping everything forward to D, this reduces to Lemma 3.6.

The maps marked $\sim$ are equivalences by Lemma 3.3 and the bar lemma (Lemma 1.4). Hence the remaining map is also an equivalence. Therefore the composite along the left-hand side, in other words the map induced by F, is an equivalence. This proves the Morita invariance of THH (Theorem 2.1).

For coefficients in a module Q, we instead see that the following square commutes up to homotopy.

(3.10)
$$\begin{tikzcd}
\mathrm{THH}(C; {}_F Q_F) \arrow[d] & \mathrm{THH}(C; B({}_F Q; D; D_F)) \arrow[l, "\sim"'] \arrow[d, "\cong"] \\
\mathrm{THH}(D; Q) & \mathrm{THH}(D; B(D_F; C; {}_F Q)) \arrow[l, "\sim"']
\end{tikzcd}$$

The top map is an equivalence by the bar lemma. The bottom map $B(D_F; C; {}_F Q) \to Q$ is an equivalence by the same inductive argument as in Lemma 3.3. Alternatively, we can rearrange the map as follows and use Lemma 3.3 directly to conclude it is an equivalence.

$$B(D_F; C; {}_F Q) \xleftarrow{\sim} B(D_F; C; B({}_F D; D; Q)) \xleftrightarrow{\cong} B(B(D_F; C; {}_F D); D; Q) \xrightarrow{\sim} B(D; D; Q) \xrightarrow{\sim} Q.$$

At any rate, this proves that the map induced by F (the left-hand edge) is an equivalence. The corresponding picture is shown in Figure 3.11.

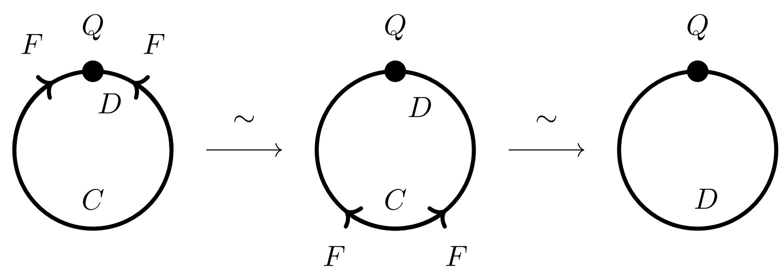


FIGURE 3.11

4. INTERPRETATION BY TRACES

Before we move on, it is useful to make the previous proof more conceptual. The idea is that the map on THH induced by F agrees with a certain non-commutative "trace map," and that the bimodule ${}_F D$ is "invertible." Then the theorem follows because the trace of an invertible object is always invertible.

To simplify notation, we use a circle dot to represent bar constructions, and double angle brackets to represent cyclic bar constructions:

$$M \odot_D N = B(M; D; N), \qquad \langle\!\langle M \rangle\!\rangle_D = \mathrm{THH}(D; M)$$

So for instance the Dennis-Waldhausen-Morita argument can be written as

$$\langle\!\langle M \odot_D N \rangle\!\rangle_C \cong \langle\!\langle N \odot_C M \rangle\!\rangle_D$$

or more simply as

$$\langle\!\langle M \odot N \rangle\!\rangle \cong \langle\!\langle N \odot M \rangle\!\rangle$$

if the spectral categories C and D are understood.

Let C and D be spectral categories. We say the (C,D)-bimodule M is **dualizable** over D (or dualizable on the right) if there is a (D,C)-bimodule M^*, a "coevaluation" map in the homotopy category of (C,C)-bimodules

$$\mathrm{coev}\colon\ C \to M \odot_D M^*$$

and an "evaluation" map in the homotopy category of (D,D)-bimodules

$$\mathrm{ev}\colon\ M^* \odot_C M \to D$$

such that the following composites give identity maps in the homotopy category of bimodules:

$$M \simeq C \odot_C M \xrightarrow{\mathrm{coev}\odot 1} M \odot_D M^* \odot_C M \xrightarrow{1\odot \mathrm{ev}} M \odot_D D \simeq M$$
$$M^* \simeq M^* \odot_C C \xrightarrow{1\odot \mathrm{coev}} M^* \odot_C M \odot_D M^* \xrightarrow{\mathrm{ev}\odot 1} D \odot_D M^* \simeq M^*$$

These are called the "triangle identities." See Figure 4.1. Of course these triangle identities could be written in the bar construction notation, it is just more cumbersome to read:

$$M \simeq B(C;C;M) \xrightarrow{\mathrm{coev}} B(B(M;D;M^*);C;M) \cong B(M;D;B(M^*;C;M)) \xrightarrow{\mathrm{ev}} B(M;D;D) \simeq M$$

$$M^* \simeq B(M^*;C;C) \xrightarrow{\mathrm{coev}} B(M^*;C;B(M;D;M^*)) \cong B(B(M^*;C;M);D;M^*) \xrightarrow{\mathrm{ev}} B(D;D;M^*) \simeq M^*$$

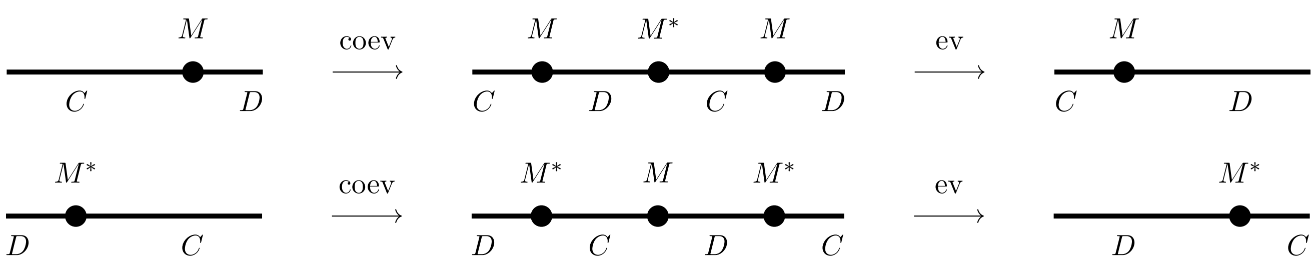


FIGURE 4.1

This is like duality in a symmetric monoidal category, except that the tensor product is not commutative, so we are not free to switch terms past each other. However, if we use a cyclic bar construction, then the Dennis-Waldhausen-Morita argument allows us to rotate a module on one end around to the other end (Figure 3.8).

It turns out, this is just enough structure to define a trace. The trace of (the identity map of) M is the composite described by the following symbols, or by the pictures in Figure 4.2.

$$\langle\!\langle C\rangle\!\rangle \xrightarrow{\mathrm{coev}} \langle\!\langle M \odot M^*\rangle\!\rangle \xleftrightarrow{\cong} \langle\!\langle M^* \odot M\rangle\!\rangle \xrightarrow{\mathrm{ev}} \langle\!\langle D\rangle\!\rangle$$

We give an equivalent picture in Figure 4.3. The author prefers this one because it gives the intuition that M and M^* are a "particle-antiparticle pair" that are created at the top, travel around the circle to the bottom, and then collide and are annihilated.

We say the (C,D)-bimodule M is **invertible** if it is dualizable on the right and its coevaluation and evaluation maps are both equivalences of bimodules (i.e. isomorphisms in the homotopy category). In particular, tensoring M with M^* gives back either C or D,

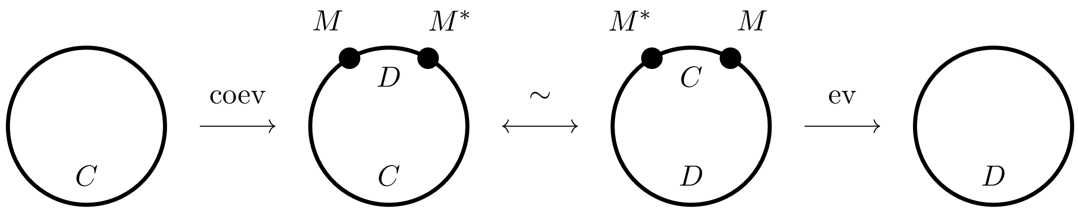


Figure 4.2

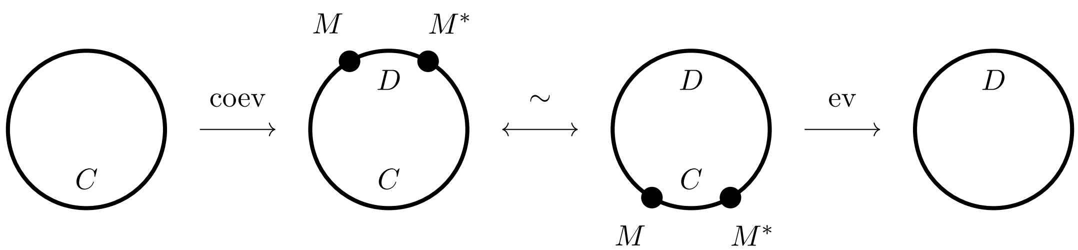


Figure 4.3

depending on which side the tensoring is performed. If M is invertible then its trace is an equivalence of spectra, because each of the three steps is an equivalence.

Lemma 4.4. [Pon10, Appendix][PS12, Lem. 7.6] *For any spectral functor $F\colon C \to D$, the base-change bimodule ${}_FD$ is dualizable on the right (over D). If in addition F is a Morita equivalence then the bimodule ${}_FD$ is invertible.*

Proof. The coevaluation and evaluation maps are essentially the maps induced by F that we considered in Lemma 3.3. For the first triangle identity it suffices to show the outside route of the following diagram gives the identity of ${}_FD$. The unlabeled maps collapse bar constructions and all the regions commute, proving the first triangle identity. The second one is similar.

$$\begin{tikzcd}
 & B(B({}_FD;D;D_F);C;{}_FD) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\sim"'] & B({}_FD;D;B(D_F;C;{}_FD)) \arrow[d, "B(1;1;\mathrm{ev})"] \\
B(C;C;{}_FD) \arrow[ur, dashed, "B(\mathrm{coev};1;1)"] \arrow[r, "B(F;1;1)"] \arrow[dr, "\sim"'] & B({}_FD_F;C;{}_FD) \arrow[d] & B({}_FD;D;D) \arrow[dl, "\sim"] \\
 & {}_FD &
\end{tikzcd}$$

By Lemma 3.3, if F is a Morita equivalence then the coevaluation and evaluation maps are equivalences, hence by definition ${}_FD$ is invertible. □

The discussion beneath (3.9) proves:

Lemma 4.5. *For any spectral functor $F\colon C \to D$, the trace of the identity of ${}_FD$ is homotopic to the map $\mathrm{THH}(C) \to \mathrm{THH}(D)$ induced by F.*

If F is a Morita equivalence, then ${}_F D$ is invertible, hence its trace is an equivalence of spectra (an isomorphism in the homotopy category). This is the conceptual proof of the Morita invariance of THH (Theorem 2.1).

For the case with coefficients, we have to generalize the trace to handle coefficients. Let P be a (C, C)-bimodule and let Q be a (D, D)-bimodule. Given a right dualizable (C, D)-bimodule M, we can take the trace of any map in the homotopy category

$$\varphi \colon P \odot_C M \to M \odot_D Q$$

by the rule

$$\langle\!\langle P \rangle\!\rangle \xrightarrow{\text{coev}} \langle\!\langle P \odot M \odot M^* \rangle\!\rangle \xleftrightarrow{\cong} \langle\!\langle M^* \odot P \odot M \rangle\!\rangle \xrightarrow{\phi} \langle\!\langle M^* \odot M \odot Q \rangle\!\rangle \xrightarrow{\text{ev}} \langle\!\langle Q \rangle\!\rangle.$$

This gives a map in the homotopy category $\mathrm{THH}(C; P) \to \mathrm{THH}(D; Q)$. See also Figure 4.6.

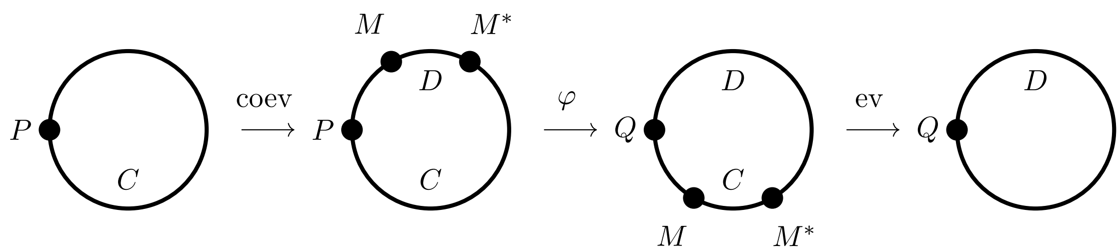


FIGURE 4.6

Continuing to let Q be a (D, D)-bimodule, let

- $M = {}_F D$,
- $P = {}_F Q_F = {}_F D \odot Q \odot D_F$,

and let ϕ be the map

$$({}_F D \odot Q \odot D_F) \odot ({}_F D) \xrightarrow{1 \odot 1 \odot \text{ev}} ({}_F D) \odot (Q). \tag{4.7}$$

Then the trace of ϕ is depicted in Figure 4.8.

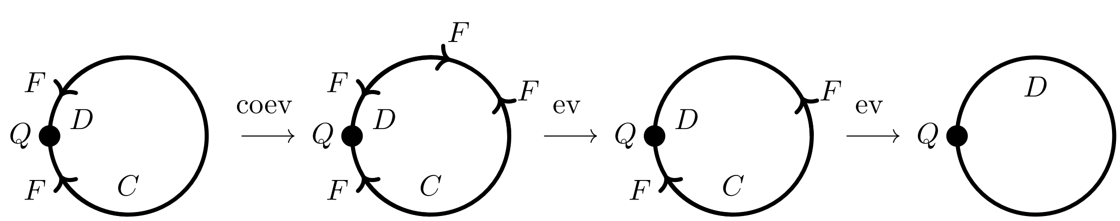


FIGURE 4.8

The first two maps cancel by the triangle identity, leaving us with the map we depicted earlier in Figure 3.11. Therefore our argument in the previous section (the homotopy commutativity of (3.10)) actually proves:

Lemma 4.9. *The trace of* (4.7) *as a twisted self-map of* ${}_FD$*, is homotopic to the map*

$$\mathrm{THH}(C; {}_FQ_F) \to \mathrm{THH}(D; Q)$$

induced by F.

If F is a Morita equivalence then (4.7) is an equivalence by Lemma 3.3, hence its trace is an equivalence. This is a more elaborate, but more conceptual, way of proving Morita invariance with coefficients.

One upshot of approaching the subject this way is that we can understand how various maps on categories of perfect modules translate across the equivalence $\mathrm{THH}(A) \simeq \mathrm{THH}({}_A\mathrm{Perf})$. Suppose A and B are ring spectra, and M is an (B, A)-bimodule that is dualizable over B. (This is equivalent to asking that M be perfect as a left B-module, after forgetting the A-action.) Then tensoring with M defines a functor

$$M \wedge_A -\colon {}_A\mathrm{Perf} \to {}_B\mathrm{Perf}.$$

We might wonder what the effect of this is on THH, after simplifying down to $\mathrm{THH}(A) \to \mathrm{THH}(B)$. It turns out it is exactly the trace of (the identity map of) M:

Proposition 4.10. *The following diagram commutes up to homotopy:*

$$\begin{tikzcd} \mathrm{THH}({}_A\mathrm{Perf}) \arrow[r, "M\wedge_A -"] & \mathrm{THH}({}_B\mathrm{Perf}) \\ \mathrm{THH}(A) \arrow[u, "\sim"'] \arrow[r, "\mathrm{tr}\,(\mathrm{id}_M)"] & \mathrm{THH}(B) \arrow[u, "\sim"'] \end{tikzcd}$$

This was proven directly in [LM19] and more conceptually in [CP19]. The conceptual proof is awesome: by Lemma 4.5, the map on the top and the Morita equivalences on the sides are all given by traces. It follows formally that their composite must also be a trace, hence the map along the bottom is the trace of something. It remains to walk through the definitions and to see that it is just the trace of id_M.

5. Homotopy invariance

Definition 5.1. A natural transformation η of two spectral functors $F, G\colon C \to D$ is an assignment of each object $a \in \mathrm{ob}\, C$ to a morphism $\eta(a)\colon Fa \to Ga$ in the underlying category D_0, satisfying the enriched naturality condition: for each pair $a, b \in \mathrm{ob}\, C$, the following square of spectra commutes.

$$\begin{tikzcd} C(a,b) \arrow[r, "F"] \arrow[d, "G"'] & D(Fa, Fb) \arrow[d, "\eta(b)\circ -"] \\ D(Ga, Gb) \arrow[r, "-\circ\eta(a)"'] & D(Fa, Gb) \end{tikzcd}$$

We say η is a natural isomorphism if each $\eta(a)$ is an isomorphism. The following is then a corollary of the Morita invariance theorem.

Corollary 5.2. *If F and G are naturally isomorphic then they induce homotopic maps on* THH.

Proof. Let $I = \{0 \leftrightarrow 1\}$ denote the category with two objects and just one isomorphism between them. Similarly let $C \times I$ denote the spectral category with objects the pairs $(a, i) \in \operatorname{ob} C \times \{0, 1\}$ and morphism spectra $(C \times I)((a, i), (b, j)) = C(a, b)$. We can check that a natural isomorphism between F and G gives a functor $C \times I \to D$ such that the inclusion of each copy of C gives F and G, respectively. Clearly both inclusions $C \to C \times I$ and the projection $C \times I \to C$ are Dwyer-Kan equivalences, so by by Theorem 2.1 they induce equivalences on THH, as shown below.

$$\begin{tikzcd}
\mathrm{THH}(C) \arrow[d, equal] \arrow[dr, "\sim"'] \arrow[drr, "F"] & & \\
\mathrm{THH}(C) \arrow[d, equal] & \mathrm{THH}(C \times I) \arrow[l, "\sim"'] \arrow[r] & \mathrm{THH}(D) \\
\mathrm{THH}(C) \arrow[ur, "\sim"] \arrow[urr, "G"'] & &
\end{tikzcd}$$

Therefore, in the homotopy category F and G both induce the same map as the zig-zag along the center row of the diagram. Therefore they are homotopic. $\square$

More generally, we say that the natural transformation η is a weak equivalence if each of the maps of spectra

$$D(d, Fa) \overset{\eta(a)\circ -}{\to} D(d, Fb), \qquad D(Ga, d) \overset{-\circ\eta(a)}{\to} D(Fa, d)$$

is an equivalence. In this case the maps induced by F and G are still homotopic:

Theorem 5.3 (Homotopy invariance)**.** *If F and G are naturally weakly equivalent then they induce homotopic maps on* THH.

Morally, this is also a consequence of the Morita invariance theorem. However, on a technical level, the above proof does not work here. It would require replacing $C \times I$ with $C \times [1]$, where $[1] = \{0 \to 1\}$ is the category with one arrow that is not an isomorphism. It is not hard to show that

$$\mathrm{THH}(C \times [1]) \simeq \mathrm{THH}(C) \vee \mathrm{THH}(C),$$

and so on THH this does not induce a homotopy between the two maps given by F and G. (In fact, it induces nothing more than the original two maps!) So $C \times [1]$ is *not* Dwyer-Kan or Morita equivalent to C, and the proof fails.[5]

Remark 5.4. The cyclic nerve does not send natural transformations to homotopies, unlike the ordinary nerve. In other words, both kinds of nerve send equivalences of categories to homotopy equivalences, but this generalizes in different directions (to adjunctions for the ordinary nerve, and to Morita equivalences for the cyclic nerve).

[5]The existence of such a homotopy would also imply $\mathrm{THH}({}_{\mathbb{S}}\mathrm{Perf}) \simeq *$, since there is a natural transformation from zero to the identity on the spectral category ${}_{\mathbb{S}}\mathrm{Perf}$. But this contradicts the fact that $\mathrm{THH}({}_{\mathbb{S}}\mathrm{Perf}) \simeq \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S} \not\simeq *$.

One could say the above argument fails because it doesn't use weak equivalences in any way. To fix this, instead of replacing C by $C \times [1]$, replace D by a category that looks like $\mathrm{Hom}([1], D)$. This fails to be Dwyer-Kan equivalent to D, as it must, because of the counterexamples above. But, if we restrict to those arrows in D that are weak equivalences, we get a smaller category $w_1 D$ that is Dwyer-Kan equivalent to D. Then we can dualize the above proof to show Theorem 5.3.

To define this Hom category, we take the objects to be arrows $a \to b$ in D_0. The spectrum of maps from the arrow $(a \to b)$ to the arrow $(c \to d)$ is given by the homotopy pullback

$$D(a,c) \times^h_{D(a,d)} D(b,d) = D(a,c) \times_{D(a,d)} D(a,d)^I \times_{D(a,d)} D(b,d).$$

Intuitively, a point in this mapping spectrum is a map from a to c, a map from b to d, and a homotopy making the square commute:

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d] & c \arrow[d] \\ b \arrow[r, dashed] & d \end{tikzcd}$$

We then compose mapping spectra by composing the maps along the top and bottom of the square, and concatenating the paths.

There is a small technical problem here. This does not define a spectral category, because the composition of these paths is not associative. This is the same issue that occurs with the loop space ΩX. As in that case, the composition of these paths is only associative up to homotopy.

One way to fix this is to force the path to be length zero, in other words use the strict pullback

$$D(a,c) \times_{D(a,d)} D(b,d).$$

This defines a spectral category that we call the **strict end**, $\mathrm{Hom}([1], D)$.

We could also fix this by a variant of the Moore path construction, letting the length of the path vary and requiring that the path be constant when it is length zero. This makes the composition strictly associative, so that we get an honest-to-god spectral category, but the mapping spectra are equivalent to the *homotopy* pullback described above, so we know their homotopy type. We call this construction the **Moore end**, $F([1], C)$.

More generally, we can define the strict end $\mathrm{Hom}(I, C)$ and Moore end $F(I, C)$ for any small category I and spectral category C. In both cases the objects are the I-diagrams in C_0. In the Moore end, the mapping spectra are given by a homotopy end of two diagrams, so a point in such a spectrum is a map at each object of I, a homotopy for each arrow of I, a 2-simplex of maps for each composable pair of arrows of I, and so on. The strict end is the same except that all the homotopies have length zero, so everything strictly commutes.

We can black-box the construction of $F(I, C)$ and just use the following facts about it:

- It is both a functor of C and a contravariant functor of I, i.e. a functor on the product category $\mathbf{Cat}^{\mathrm{op}} \times \mathbf{SpCat} \to \mathbf{SpCat}$.

- It receives a natural map from the strict end which is the identity on objects (so the object set is the collection of I-diagrams in C_0).
- When $I = [0]$ is the one-point category we get $C = \mathrm{Hom}([0], C) \xrightarrow{\sim} F([0], C)$.
- More generally, when $I = [k] = \{0 \to \cdots \to k\}$ the mapping spectrum between two diagrams $\phi, \gamma \colon [k] \to C_0$ is naturally equivalent to the homotopy limit of the zig-zag

$$\begin{tikzcd}[column sep=small] C(\phi(0),\gamma(0)) \arrow[dr] & & C(\phi(1),\gamma(1)) \arrow[dl] \arrow[dr] & \cdots & C(\phi(k),\gamma(k)) \arrow[dl] \\ & C(\phi(0),\gamma(1)) & & \cdots & \end{tikzcd}$$

- Even more generally, when I is a product of categories of the form $[k]$, the mapping spectra of $F(I, C)$ are are the homotopy limit of the corresponding grid of zig-zags.

Proof of Theorem 5.3. Given two functors $F, G \colon C \to D$ and a weak equivalence $\eta \colon F \Rightarrow G$, they define a map

$$C \to \mathrm{Hom}([1], D) \to F([1], D).$$

This lands in the full subcategory $w_1 D$ on those arrows in D that are weak equivalences. We check using the homotopy pullback axiom for the Moore end that both of the projections $w_1 D \rightrightarrows D$ and the inclusion of identity morphisms $D \to w_1 D$ are all Dwyer-Kan equivalences. Therefore we get the commuting diagram

$$\begin{tikzcd} & & \mathrm{THH}(D) \arrow[d, equal] \\ \mathrm{THH}(C) \arrow[urr, "F"] \arrow[r] \arrow[drr, "G"'] & \mathrm{THH}(w_1 D) \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & \mathrm{THH}(D) \arrow[l, "\sim"'] \arrow[d, equal] \\ & & \mathrm{THH}(D) \end{tikzcd}$$

and deduce that the maps induced on THH by F and G are homotopic. □

6. Additivity

Now we are ready for the additivity theorem. Roughly, additivity says that any time three functors $F_1, F_2, F_3 \colon C \to D$ form a cofiber sequence $F_1 \to F_2 \to F_3$, the map $\mathrm{THH}(C) \to \mathrm{THH}(D)$ induced by F_2 is homotopic to the sum of the maps induced by F_1 and F_3.

To make this precise, we need a notion of cofiber sequence for functors of spectral categories. The following framework is a convenient one.

Recall that a **Waldhausen category** is a category C_0 with cofibrations and weak equivalences such that

(1) every isomorphism is both a cofibration and a weak equivalence,
(2) there is a zero object $*$ and every object is cofibrant,
(3) C_0 has all pushouts along cofibrations (homotopy pushouts),
(4) the pushout of a cofibration is a cofibration, and

(5) a weak equivalence of homotopy pushout diagrams induces a weak equivalence of pushouts.

In particular, it is a category that has a notion of cofiber sequence. A cofiber sequence is a pushout square of the following form (horizontal maps are cofibrations).

$$\begin{tikzcd} a \arrow[r, tail] \arrow[d] & b \arrow[d] \\ * \arrow[r, tail] & c \end{tikzcd} \tag{6.1}$$

An **exact functor** $C_0 \to D_0$ is a functor preserving the zero object, the cofibrations and weak equivalences, and the pushouts along cofibrations. In particular it preserves these cofiber sequences.

Following [CLM+a], we define a **spectral Waldhausen category** to be a spectral category C and a Waldhausen category C_0 on the same objects. It may be that C_0 is the underlying category of C, but this is not necessary; it is enough to have a map from C_0 to the underlying category of C that is the identity on objects. We require three compatibility conditions:

(1) The zero object of C_0 is also a zero object for C.
(2) Every weak equivalence $c \to c'$ in C_0 induces stable equivalences

$$C(c', d) \xrightarrow{\sim} C(c, d), \qquad C(d, c) \xrightarrow{\sim} C(d, c').$$

(3) For every pushout square in C_0 along a cofibration

$$\begin{tikzcd} a \arrow[r, tail] \arrow[d] & b \arrow[d] \\ c \arrow[r, tail] & d \end{tikzcd}$$

and object e, the resulting two squares of spectra

$$\begin{tikzcd} C(a,e) & C(b,e) \arrow[l] \\ C(c,e) \arrow[u] & C(d,e) \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} C(e,a) \arrow[r] \arrow[d] & C(e,b) \arrow[d] \\ C(e,c) \arrow[r] & C(e,d) \end{tikzcd}$$

are homotopy pushout squares.

Example 6.2. For any ring spectrum A, the category ${}_A\mathrm{Mod}$ of left A-modules and the subcategory ${}_A\mathrm{Perf}$ of perfect A-modules are both spectral Waldhausen categories. More generally, for any spectral category C, ${}_C\mathrm{Mod}$ and ${}_C\mathrm{Perf}$ are spectral Waldhausen categories.

Note that $\mathrm{THH}(C) \simeq \mathrm{THH}({}_C\mathrm{Perf})$ by Morita invariance. In other words, every spectral category can be changed into a spectral Waldhausen category, without changing its THH. Therefore, when we formulate the additivity theorem, there is no loss of generality if we restrict attention to spectral Waldhausen categories.

A map of spectral Waldhausen categories $(C, C_0) \to (D, D_0)$ consists of an exact functor $F_0 \colon C_0 \to D_0$ and a spectral functor $F \colon C \to D$ that agree along the inclusion of C_0 and D_0 into the underlying categories of C and D.

A natural transformation between two such maps F and G assigns each object $a \in \operatorname{ob} C$ to a morphism $F(a) \to G(a)$ in D_0, such that these morphisms give a natural transformation both of ordinary functors $F_0 \Rightarrow G_0$ and of spectral functors $F \Rightarrow G$ (Definition 5.1). A cofiber sequence of functors is a commuting square of natural transformations

$$\begin{tikzcd} F_1 \arrow[r, tail] \arrow[d] & F_2 \arrow[d] \\ * \arrow[r, tail] & F_3 \end{tikzcd}$$

that is a pushout along a cofibration, when evaluated on each object $a \in \operatorname{ob} C$.

Theorem 6.3 (Additivity, first formulation)**.** *If $F_1 \to F_2 \to F_3$ is a cofiber sequence of functors of spectral Waldhausen categories $C \to D$, then on* THH*, the map induced by F_2 is homotopic to the sum of the maps induced by F_1 and F_3.*

We prove this in the next section. We first list some corollaries and a second formulation.

Corollary 6.4 (Eilenberg Swindle)**.** *If the spectral Waldhausen category C has countable direct sums then* $\mathrm{THH}(C) \simeq *$.

To be specific, we assume that the underlying Waldhausen category has countably infinite coproducts $\bigvee_{i=1}^{\infty} a_i$, and that these are also coproducts for the spectral enrichment, in the sense that the inclusions $a_i \to \bigvee_{i=1}^{\infty} a_i$ induce an isomorphism[6]

$$C\left(\bigvee_{i=1}^{\infty} a_i, b\right) \xrightarrow{\cong} \prod_{i=1}^{\infty} C(a_i, b).$$

In particular, the category ${}_C\mathrm{Mod}$ of left C-modules always has such countably infinite coproducts.

Proof. We define a spectral functor $F\colon C \to C$ taking a to $\bigvee_{i=1}^{\infty} a$. On mapping spectra we pick the map $C(a,b) \to C(\vee_{i=1}^{\infty} a, \vee_{i=1}^{\infty} b)$ such that the composition

$$C(a,b) \longrightarrow C\left(\bigvee_{i=1}^{\infty} a, \bigvee_{i=1}^{\infty} b\right) \xrightarrow{\cong} \prod_{i \geq 1} C\left(a, \bigvee_{i=1}^{\infty} b\right).$$

is the product of the maps that compose with each of the inclusions $b \to \bigvee_{i=1}^{\infty} b$. A straightforward diagram-chase verifies that this is a spectral functor. Furthermore the maps

$$a \longrightarrow \bigvee_{i=1}^{\infty} a \longrightarrow \bigvee_{i=1}^{\infty} a$$

that include the first summand and that shift the summands one slot to the right, both define spectral natural transformations.

This defines a cofiber sequence on the underlying Waldhausen category, hence by definition we have a cofiber sequence $\mathrm{id} \to F \to F$ of functors from C to itself. Therefore by additivity (Theorem 6.3), the identity map on $\mathrm{THH}(C)$ is homotopic to the difference $\mathrm{THH}(F) - \mathrm{THH}(F)$, which is zero. $\square$

Corollary 6.5. *For any spectral category C,* $\mathrm{THH}({}_C\mathrm{Mod}) \simeq *$.

[6] By standard enriched category theory, the ordinary coproduct will automatically be an enriched coproduct if C is not only enriched in spectra but also cotensored, see [Kel82, §3.8], [Shu06, §11].

This is why we restrict to perfect modules when taking THH of a category of modules.

To prove Theorem 6.3 we will show it follows from a different formulation of additivity. Recall the Moore end construction $F(I, C)$ of the previous section. It makes sense for any spectral Waldhausen category as well, by taking the objects to be diagrams $I \to C_0$ in the Waldhausen category C_0. The underlying Waldhausen category of $F(I, C)$ has the pointwise cofibrations and weak equivalences, meaning the maps that are cofibrations or weak equivalences, respectively, at each object $i \in I$.

Lemma 6.6. [CLM+b, 5.4] *This respects Waldhausen structures, defining a functor*

$$\mathbf{Cat}^{\mathrm{op}} \times \mathbf{SpWaldCat} \to \mathbf{SpWaldCat}.$$

Taking $I = [1] \times [1]$ to be the commuting square category, we define the category of cofiber sequences

$$S_2C \subseteq F([1] \times [1], C)$$

to be the full subcategory on those commuting squares of the form (6.1).[7] So the objects are cofiber sequences $a \to b \to c$ in C_0, and the mapping spectrum from $a \to b \to c$ to $a' \to b' \to c'$ consists of points in various mapping spectra and homotopies between them, as illustrated in Figure 6.7.

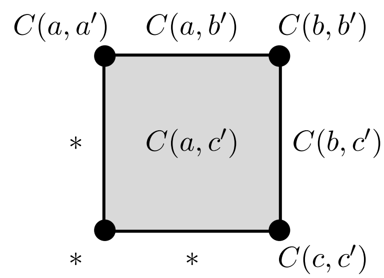


FIGURE 6.7

Using Lemma 6.6, we get four maps of spectral Waldhausen categories

$$\begin{array}{rcl@{\qquad}rcl} s\colon C & \to & S_2C & i\colon C & \to & S_2C \\ a & \mapsto & a \to a \to * & c & \mapsto & * \to c \to c \\ d\colon S_2C & \to & C & \pi\colon S_2C & \to & C \\ a \to b \to c & \mapsto & a & a \to b \to c & \mapsto & c \end{array} \tag{6.8}$$

such that $d \circ s$ and $\pi \circ i$ are the identity functor of C.

Theorem 6.9 (Additivity, second formulation)**.** *The functors s and i induce an equivalence*

$$\mathrm{THH}(C) \vee \mathrm{THH}(C) \xrightarrow{\sim} \mathrm{THH}(S_2C).$$

[7] It's not necessary to know this, but we also restrict the cofibrations so that they agree with the usual cofibrations on S_2C. This doesn't change the fact that it's a spectral Waldhausen category, since fewer cofibrations means fewer conditions to check.

Equivalently, d and π induce an equivalence to the product,

$$\mathrm{THH}(S_2 C) \xrightarrow{\sim} \mathrm{THH}(C) \times \mathrm{THH}(C).$$

Morally, the two formulations of additivity we have described are equivalent to each other. On a technical level, we will see this second version implies the first one.

This second formulation is the kind we use to give an explicit construction of the Dennis trace. We define $S_n C$ similarly to $S_2 C$ and use an inductive argument to get

$$\bigvee^{n} \mathrm{THH}(C) \simeq \mathrm{THH}(S_n C).$$

As n varies the categories $S_n C$ form a simplicial object in spectral Waldhausen categories. By additivity, the resulting simplicial spectrum $\mathrm{THH}(S_\bullet C)$ is equivalent to $S^1_\bullet \wedge \mathrm{THH}(C)$, hence on realizations we get

$$|\mathrm{THH}(S_\bullet C)| \simeq \Sigma \mathrm{THH}(C).$$

Therefore "THH satisfies a property that K-theory satisfies universally," so it receives a trace from K-theory. In more detail, we define $w_m S_n C$ by taking flags of weak equivalences of objects in $S_n C$, then we include the objects of $w_m S_n C$ into the 0-skeleton of $\mathrm{THH}(w_m S_n C)$ to get

$$\mathrm{ob}\, w_m S_n C_0 \to \Omega^\infty \mathrm{THH}(w_m S_n C).$$

After realizing, we get the map of spaces

$$|w_\bullet S_\bullet C_0| \to \Omega^\infty \Sigma \mathrm{THH}(C)$$

and looping once more gives a map from the K-theory space to THH:

$$K(C_0) = \Omega |w_\bullet S_\bullet C_0| \to \Omega^\infty \mathrm{THH}(C)$$

By repeating this with iterates of the $S_\bullet$ construction we can make this a map of spectra $K(C_0) \to \mathrm{THH}(C)$. See e.g. [BM11, DGM13, CLM+a] for more details.

7. Proof of additivity

We first describe how to recover the first formulation from the second. Any cofiber sequence $F_1 \to F_2 \to F_3$ of functors $C \to D$ defines a functor $C \to S_2 D$ (essentially because the strict end maps into the Moore end), and then we have the commuting diagram in the homotopy category

$$\begin{tikzcd}
\mathrm{THH}(C) \arrow[r] \arrow[rr, bend left, "F_2"] \arrow[dr, "F_1 \vee F_3"'] & \mathrm{THH}(S_2 D) \arrow[r] \arrow[d, leftrightarrow, "\sim"] & \mathrm{THH}(D). \\
& \mathrm{THH}(D) \vee \mathrm{THH}(D) \arrow[ur, "\mathrm{id} \vee \mathrm{id}"'] &
\end{tikzcd}$$

Therefore the map on THH induced by F_2 is homotopic to the sum of the maps induced by F_1 and F_3.

To prove the second formulation, we start by observing that every cofiber sequence $(x \to y \to z)$ sits in a "cofiber sequence of cofiber sequences"

$$\begin{tikzcd} x \arrow[r] \arrow[d, equal] & x \arrow[r] \arrow[d] & * \arrow[d] \\ x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & z \arrow[d, equal] \\ * \arrow[r] & z \arrow[r] & z \end{tikzcd}.$$

Note that the outside cofiber sequences are obtained from $(x \to y \to z)$ by applying the functors $s \circ d$ and $i \circ \pi$ from (6.8). Since S_2C is a spectral Waldhausen category this means for each cofiber sequence $(a \to b \to c)$ we get a cofiber sequence of spectra

$$S_2C(a \to b \to c, x = x \to *) \longrightarrow S_2C(a \to b \to c, x \to y \to z) \longrightarrow S_2C(a \to b \to c, * \to z = z).$$

We would like to say that this gives a cofiber sequence of S_2C-bimodules

$$S_2C_{s\circ d} \longrightarrow S_2C_{\mathrm{id}} \longrightarrow S_2C_{i\circ\pi},$$

using the pullback notation M_F from previous sections. However this is not quite correct, because the maps do not commute with the action of S_2C on the right. To illustrate this, observe that given two homotopy-coherent maps of cofiber sequences

$$\begin{tikzcd} a \arrow[r] \arrow[d] & x \arrow[d, equal] \\ b \arrow[r] \arrow[d] & x \arrow[d] \\ c \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r] \arrow[d] & d \arrow[d] \\ y \arrow[r] \arrow[d] & e \arrow[d] \\ z \arrow[r] & f \end{tikzcd}$$

the following two composites are not identical, only homotopic:

$$\begin{tikzcd} a \arrow[r] \arrow[d] & x \arrow[r] \arrow[d, equal] & x \arrow[r] \arrow[d] & d \arrow[d] \\ b \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & e \arrow[d] \\ c \arrow[r] & * \arrow[r] & z \arrow[r] & f, \end{tikzcd}$$

or

$$\begin{tikzcd} a \arrow[r] \arrow[d] & x \arrow[r] \arrow[d, equal] & d \arrow[r] \arrow[d, equal] & d \arrow[d] \\ b \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & d \arrow[r] \arrow[d] & e \arrow[d] \\ c \arrow[r] & * \arrow[r] & * \arrow[r] & f. \end{tikzcd}$$

To fix this, we thicken up the bimodule $S_2C_{s\circ d}$ by taking the mapping spectrum from $(a \to b \to c)$ to $(x \to y \to z)$ to be the spectrum of homotopy-coherent maps between the two diagrams

$$\begin{tikzcd}[row sep=small, column sep=small]
 & a \arrow[dd] & & & x \arrow[dd] \\
a \arrow[ur, equal] \arrow[dd] & & \longrightarrow & x \arrow[ur, equal] \arrow[dd, equal] & \\
 & b \arrow[dd] & & & y \arrow[dd] \\
b \arrow[ur, equal] \arrow[dd] & & \longrightarrow & x \arrow[ur] \arrow[dd] & \\
 & c & & & z \\
c \arrow[ur, equal] & & \longrightarrow & * \arrow[ur] &
\end{tikzcd}$$

This is equivalent to the spectrum of maps from $(a \to b \to c)$ to $(x = x \to *)$, by restricting to the front face, and maps forward to the spectrum of maps from $(a \to b \to c)$ to $(x \to y \to z)$, by restricting to the back face. All these operations give maps of spectral categories, hence they give maps of S_2C-bimodules. So in summary we get a zig-zag of maps of S_2C-bimodules

$$S_2C_{s \circ d} \xleftarrow{\sim} \bullet \longrightarrow S_2C_{\mathrm{id}},$$

equivalently a map in the homotopy category of bimodules.

Representing these maps in the homotopy category as dashed arrows, we therefore get a cofiber sequence of bimodules

$$S_2C_{s \circ d} \dashrightarrow S_2C_{\mathrm{id}} \dashrightarrow S_2C_{i \circ \pi}.$$

Hence we can take $\mathrm{THH}(S_2C)$ with coefficients in these bimodules, giving a cofiber sequence of spectra

$$\mathrm{THH}(S_2C; S_2C_{s \circ d}) \dashrightarrow \mathrm{THH}(S_2C; S_2C_{\mathrm{id}}) \dashrightarrow \mathrm{THH}(S_2C; S_2C_{i \circ \pi}).$$

Our next goal is to show that the outside terms simplify to $\mathrm{THH}(C)$, so that this cofiber sequence takes the form

$$\mathrm{THH}(C) \dashrightarrow \mathrm{THH}(S_2C) \dashrightarrow \mathrm{THH}(C). \tag{7.1}$$

To do this we expand out $\mathrm{THH}(S_2C; S_2C_{s \circ d})$ in pictorial form, rearrange the picture, and use the fact that $d \circ s$ is the identity functor of C to cancel it out. See Figure 7.2. The same argument applies with $i \circ \pi$ in the place of $s \circ d$, because $\pi \circ i$ is also the identity of C.

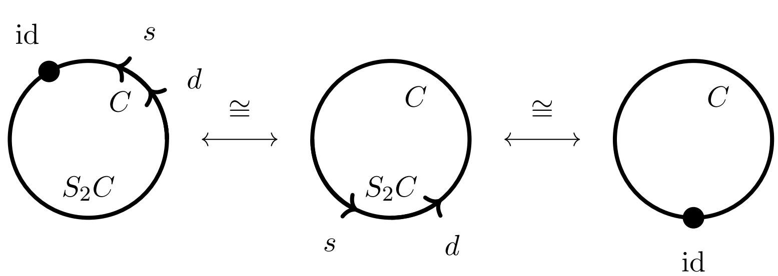


FIGURE 7.2

The final goal is to show that the maps we get in this new cofiber sequence agree (in the homotopy category) with the maps induced by s and π. To do this, we first observe that the functors s and d form an adjunction on the underlying categories of C and S_2C. A map of cofiber sequences

$$\begin{tikzcd} x \arrow[r] \arrow[d, equal] & a \arrow[d] \\ x \arrow[r] \arrow[d] & b \arrow[d] \\ * \arrow[r] & c \end{tikzcd}$$

corresponds precisely to a map $x \to a$. Then we observe that the same is true for the spectral enrichments, up to equivalence. In other words, the spectrum of maps of cofiber sequences depicted above is equivalent (though not isomorphic) to $C(x, a)$:

$$S_2C((x = x \to *), (a \to b \to c)) \xrightarrow{\sim} C(x, a).$$

Rewriting this as an equivalence

$$S_2C(s(x), (a \to b \to c)) \xrightarrow{\sim} C(x, d(a \to b \to c))$$

we check that it respects the left action of C and the right action of S_2C, hence is an equivalence of (C, S_2C)-bimodules

$$ {}_sS_2C \xrightarrow{\sim} C_d \tag{7.3}$$

Replacing all the instances of C_d with ${}_sS_2C$, the equivalence and then map

$$\mathrm{THH}(C) \simeq \mathrm{THH}(S_2C; S_2C_{s \circ d}) \to \mathrm{THH}(S_2C)$$

becomes a composite illustrated in Figure 7.4.

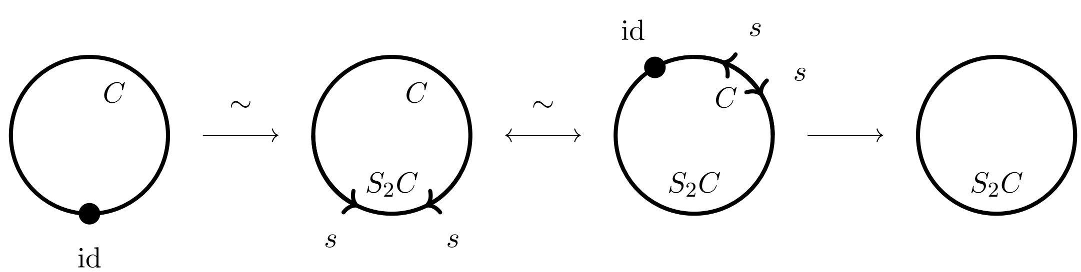


FIGURE 7.4

This looks suspiciously like the trace of the identity of the base-change bimodule ${}_sS_2C$ as in Figure 4.2.[8] We omit the diagram chasing that confirms this is the case, but it can be found in [CLM$^+$a], and it is done completely using the black-boxed properties we gave for the Moore end in Section 5.

Lemma 7.5. [CLM$^+$a, Lem 5.14] *Along the equivalence* (7.3)*, the maps in the homotopy category of bimodules constructed thus far*

$$C_{\mathrm{id}} \xrightarrow{\cong} C_{d \circ s}, \qquad S_2C_{s \circ d} \dashrightarrow S_2C_{\mathrm{id}}$$

[8]Recall that by Lemma 4.4 this bimodule is dualizable, and that since $s \colon C \to S_2C$ is a Dwyer-Kan embedding, the coevaluation map is an equivalence but the evaluation map is not necessarily an equivalence.

agree with the coevaluation and evaluation maps for the base-change bimodule from Lemma 4.4,

$$C \xrightarrow{\sim} {}_sS_2C \odot_{S_2C} S_2C_s, \qquad S_2C_s \odot_C {}_sS_2C \longrightarrow S_2C.$$

In summary, in our cofiber sequence from (7.1)

$$\mathrm{THH}(C) \dashrightarrow \mathrm{THH}(S_2C) \dashrightarrow \mathrm{THH}(C),$$

the first map is just the trace of the identity of the base-change bimodule ${}_sS_2C$. By Lemma 4.5, this trace agrees with the map on THH induced by the spectral functor s. A similar argument identifies the second map as induced by π, so we conclude that

$$\mathrm{THH}(C) \xrightarrow{s} \mathrm{THH}(S_2C) \xrightarrow{\pi} \mathrm{THH}(C)$$

is a cofiber sequence. The second map is split by i, so s and i induce an equivalence

$$\mathrm{THH}(C) \vee \mathrm{THH}(C) \xrightarrow[\sim]{s\vee i} \mathrm{THH}(S_2C).$$

This concludes the proof of additivity.

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