PUSHOUTS IN THE HOMOTOPY CATEGORY DO NOT EXIST

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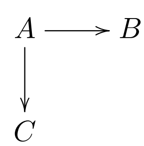
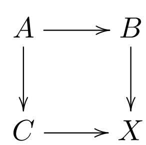
The author has heard from several sources that, in the homotopy category of spaces, most pushouts do not exist. The purpose of this note is to give the details of one particularly nice explicit counterexample; i.e. a pushout diagram in the homotopy category that does not have a pushout.

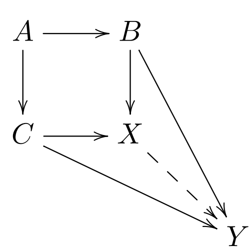
Let **CW** be the category of spaces that are homeomorphic to unbased CW complexes, with continuous maps between them. Let **H** be its homotopy category, which has the same objects, but morphisms are homotopy classes of maps. Since **H** is a category, one may take any diagram in **H** of the form



The *pushout* of this diagram, if it exists, is an object X of **H** that fits into a commuting square



and is initial among all such objects. That is, for any Y in **H** fitting into a similar square, there is a unique choice of dotted arrow below making the diagram commute:



Before we proceed, we should point out that there are other equivalent models for **H**. For instance, we could take all spaces and localize at the class of weak homotopy equivalences. But these models are equivalent as categories, so pushouts exist in our model of **H** iff they exist in all other models.

Now consider the diagram of unbased spaces

$$\begin{array}{ccc} * & \longrightarrow & S^2 \\ \downarrow & & \\ S^2 & & \end{array}$$

In the category of CW complexes **CW**, or in topological spaces **Top**, this diagram has pushout $S^2 \vee S^2$. However, if we regard this as a diagram in the homotopy category **H**, we will prove that it has no pushout.

Suppose X is the pushout. Then X is equipped with two homotopy classes of maps $S^2 \longrightarrow X$ so that this square commutes up to homotopy:

$$\begin{array}{ccc} * & \longrightarrow & S^2 \\ \downarrow & & \downarrow \\ S^2 & \longrightarrow & X \end{array}$$

This picks out two maps from S^2 into X which land in the same path component. By choosing a path connecting them, we may form a homotopy class of maps

$$f : S^2 \vee S^2 \simeq S^2 \cup_0 I \cup_1 S^2 \longrightarrow X$$

Going the other way, $S^2 \vee S^2$ receives an obvious collection of maps from our pushout diagram, giving a map $g : X \longrightarrow S^2 \vee S^2$. We first show that f and g are inverse homotopy equivalences.

The composition $g \circ f : S^2 \vee S^2 \longrightarrow S^2 \vee S^2$ is some unbased map. Since $S^2 \vee S^2$ is simply-connected, the forgetful functor from based maps to unbased maps is an isomorphism on homotopy classes of maps. To prove this we observe that there is a fibration sequence

$$\Omega(S^2 \vee S^2) \longrightarrow \mathrm{Map}_*(S^2 \vee S^2, S^2 \vee S^2) \longrightarrow \mathrm{Map}(S^2 \vee S^2, S^2 \vee S^2) \longrightarrow S^2 \vee S^2$$

and both end terms are path-connected. Therefore we deform $g \circ f$ to a based map and compute its based homotopy class. It is determined by the images of the two spheres in

$$\pi_2(S^2 \vee S^2) \xrightarrow{\cong} H_2(S^2 \vee S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

and this is measured by degree, which is an invariant of unbased classes of maps. From our definitions it is easy to see the degrees are $(1, 0)$ and $(0, 1)$ and so the composite

$$g \circ f : S^2 \vee S^2 \longrightarrow X \longrightarrow S^2 \vee S^2$$

is homotopic to the identity.

The other direction is easier: the composite

$$f \circ g : X \longrightarrow S^2 \vee S^2 \longrightarrow X$$

lies under the same maps from each S^2 into X, so by the uniqueness part of the universal property of X, this composite must be the identity.

Therefore f is a homotopy equivalence, so the obvious pushout $S^2 \vee S^2$ must be the pushout in the homotopy category. Now that we know it explicitly, we will derive a contradiction by showing it fails to have the required universal property.

Consider the space $S^2 \vee S^1 \vee S^2$. The obvious inclusion from $S^2 \vee S^2$ puts this space under our pushout diagram, so it must receive a unique homotopy class of maps from $S^2 \vee S^2$. Since $S^2 \vee S^2$ is simply-connected, the homotopy classes of unbased maps $S^2 \vee S^2 \longrightarrow S^2 \vee S^1 \vee S^2$ are in one-to-one correspondence with the homotopy classes of unbased maps from $S^2 \vee S^2$ into the universal cover

$$\bigvee_{n \in \mathbb{Z}} (S^2 \vee S^2)_{t^n}$$

up to action by the generator t of $\mathbb{Z} \cong \pi_1(S^2 \vee S^1 \vee S^2)$. Since the universal cover is simply-connected these correspond to based maps up to $\mathbb{Z}$-action. Again by the Hurewicz theorem, these correspond to bounded sequences of pairs of integers

$$(\ldots, (0,0), (0,0), (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n), (0,0), (0,0), \ldots)$$

up to a $\mathbb{Z}$-action that shifts both sequences. To agree with the given map in from the first copy of S^2, we need the sequence of first coordinates a_i to be the trivial sequence

$$(\ldots, 0, 0, 1, 0, 0, \ldots)$$

up to a shift. Similarly for the second copy, we need the second coordinates b_i to form the above sequence up to a shift, but not necessarily the same shift. So the two sequences of pairs

$$(\ldots, (0,0), (0,0), (1,1), (0,0), (0,0), \ldots)$$

$$(\ldots, (0,0), (0,0), (1,0), (0,1), (0,0), (0,0), \ldots)$$

give different homotopy classes of unbased maps $S^2 \vee S^2 \longrightarrow S^2 \vee S^1 \vee S^2$ that are homotopic when restricted to either copy of S^2. Geometrically, the first map is the obvious inclusion, and the second one winds the second copy of S^2 around S^1 once before capping off by covering the second copy of S^2.

In summary, the diagram

$$\begin{array}{ccc} * & \longrightarrow & S^2 \\ \downarrow & & \\ S^2 & & \end{array}$$

has no pushout in the homotopy category of unbased spaces. By almost exactly the same argument, the diagram

$$\begin{array}{ccc} S^0 & \longrightarrow & S^2_+ \\ \downarrow & & \\ S^2_+ & & \end{array}$$

has no pushout in the homotopy category of based spaces.

What made this counterexample run is the same idea that leads to *homotopy pushouts*. We are trying to build a space X with a square of continuous maps

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array}$$

that commutes up to homotopy. If this is all the data we have, then X cannot hope to satisfy the uniqueness part of the universal property of the pushout. We are only assured that when you restrict of your maps $B, C \longrightarrow Y$ to two maps $A \longrightarrow Y$, there *exists* a homotopy between them, but there is no canonical choice for which homotopy to pick. Unfortunately, different choices for this homotopy often give different connecting maps $X \longrightarrow Y$ which are not even homotopic (as we saw above). The way to remedy this is to change the desired universal property, by making the choice of homotopy a part of the data. This leads to a new universal property, which is satisfied by the double mapping cylinder

$$B \cup_{A \times 0} A \times I \cup_{A \times 1} C$$

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