PROJECT DESCRIPTIONS FOR COLLABORATIVE WORKSHOP IN K-THEORY AND SCISSORS CONGRUENCE

1. Spherical Scissors Congruence

Inbar Klang and Cary Malkiewich

Scissors congruence K-theory is a form of algebraic K-theory that describes polytopes up to cut-and-paste relations. Calculating the lowest K-group (K_0) amounts to solving variants of Hilbert's 3rd problem, in other words describing polytopes up to scissors congruence. The higher K-groups capture higher additive cut-and-paste invariants, and have not been explored as thoroughly as K_0.

In this project we will study the higher scissors congruence K-groups of spherical geometry. A good deal is known about K_0 here: there is a rational splitting into "reduced" scissors congruence K-theory pieces, and a coalgebra structure coming from the Dehn invariant. Our group will work out the versions of these theorems that apply to the K-theory spectrum, and consequently to the higher K-groups. For the splitting, we expect to use a fair amount of homotopy orbit spectral sequences and homological algebra. For the coalgebra structure, we expect to first approach the problem using operads and symmetric spectra, and if this approach fails, to resort to more infinity-categorical techniques.

2. Equivariant trace methods

Teena Gerhardt and J.D. Quigley

Hochschild homology and cyclic homology are classical invariants of algebras. These theories have topological analogues, called topological Hochschild homology (THH) and topological cyclic homology (TC) that are important tools for studying algebraic K-theory. Indeed, there are trace maps from algebraic K-theory to THH and TC that have facilitated great advances in algebraic K-theory. In recent years, Real and equivariant analogues of algebraic K-theory have been developed, along with Real and equivariant analogues of TC and THH. Examples include the Real TC and THH of rings with anti-involution, and the twisted TC and THH of genuine C_n-rings.

There are many open computational questions about these new equivariant theories. This project will focus on understanding variants of topological cyclic homology and topological Hochschild homology with an equivariant input. In particular, we aim to use recent advances in trace methods and equivariant stable homotopy theory to study cyclotomic structures in these equivariant settings, and extend techniques for computing TC and THH to Real and twisted analogues.

3. Witt vectors, bicategorical traces and K-theory

Jonathan Campbell and Kate Ponto

This project seeks to extend and understand various definitions of, and constructions with, the Witt vectors in terms of bicategorical traces and K-theory. It has been known since the '70s that much of the combinatorics of the Witt vectors can be captured via the K-theory of endomorphisms. Recent work of Campbell–Ponto, Campbell–Lind–Malkiewich–Ponto–Zakharevich and Dotto–Krause–Nikolaus–Patchkoria motivated by trace methods indicate there is a much richer structure to be explore— the K-theory of endomorphisms is simply the K-theory of one very particular notion of bimodule, and perhaps the construction is most natural in this setting.

We expect that situating Witt vectors as manifestly bicategorical objects will illuminate more of their combinatorics. The goal is then to use the full duality and trace theory in bicategories a la Ponto to make the construction of non-commutative Witt vectors and their generalizations completely natural. For those seeking even deeper confusion, there is a nearly completely unexplored norm map on Witt vectors, a kind of multiplicative Verschiebung. We have no idea how this interacts with other constructions or with the bicategorical structure, but the answer almost certainly involves some fun with symmetric monoidal bicategories.

Example reading for flavor:

- Almkvist, J. Algebra 28 (1974), 375-388
- Campbell, arXiv:1910.10206 2019
- Campbell–Lind–Malkiewich–Ponto–Zakharevich, arXiv:2005.04334 2020
- Campbell–Ponto, AGT 19 (2019) 965-1017
- Dotto–Krause–Nikolaus–Patchkoria, arXiv:2002.01538 2020

4. Scissors congruence K-theory for manifolds

Mona Merling and Julia Semikina

There is an analogous definition to that of scissors congruence of polyhedra for smooth manifolds, which was first introduced under the name SK ("scheiden und kleben" = "cut and paste") in [KKNO73]: Given a closed smooth oriented manifold M, one can cut it along a codimension 1 submanifold Σ with trivial normal bundle and paste back the two pieces along an orientation preserving diffeomorphism $\Sigma \to \Sigma$ to obtain a new manifold. In [HMR$^+$20] we constructed a scissors congruence K-theory spectrum that recovers classical SK-groups for manifolds with boundary on π_0 and constructed a derived version of the Euler characteristic as a map to $K(\mathbb{Z})$. It turns out that mysteriously on π_1 the map that lifts the Euler characteristic, an SK invariant, detects Kervaire semicharacteristic, a finer, so-called SKK invariant. The SKK group is known to be π_1 of the cobordism category. The K-theory of manifolds has tantalizing connections to cobordism categories [HRS22, MRS23]. A lot of these connections are as of yet still not understood. In this project we propose to further study scissors congruence K-theory of manifolds. One possible direction is to better understand the connection with cobordism categories. Another potential direction is to study an equivariant version of cut and paste groups for manifolds, and construct these as the fixed points of a genuine G-spectrum.

References

[HMR+20] R. Hoekzema, M. Merling, C. Rovi, J. Semikina, and L. Wells. Cut and paste invariants of manifolds via algebraic K-theory. *arXiv:2001.00176*, 2020.

[HRS22] R. Hoekzema, C. Rovi, and J. Semikina. A K-theory spectrum for cobordism cut and paste groups. *arXiv:2210.00682*, 2022.

[KKNO73] U. Karras, M. Kreck, W. D. Neumann, and E. Ossa. *Cutting and pasting of manifolds; SK-groups*. Publish or Perish, Inc., Boston, Mass., 1973. Mathematics Lecture Series, No. 1.

[MRS23] M. Merling, G. Raptis, and J. Semikina. Parametrized cut-and-paste theory of manifolds and cobordisms. *In preparation*, 2023.

5. Double Steinberg coinvariants

Alexander Kupers and Robin Sroka

The homology of general linear groups and algebraic K-theory are, among others, related by important representations called "Steinberg modules" (and variants thereof). Recent work of Galatius, Kupers, and Randal–Williams on general linear groups over infinite fields shows that their low-degree homology with coefficients in the "double Steinberg module," i.e. the tensor product of the Steinberg module with itself, can be used to study homological stability questions. In particular, they compute the zeroth of these homology groups, the "double Steinberg coinvariants." The goal of this project is to generalise these results to other groups (e.g. special linear groups or symplectic groups) and other rings (e.g. local rings), and to examine mysterious similarities to double scissors congruences and polylogarithms.