Spectra and stable homotopy theory
(draft version, first 6 chapters)

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This textbook provides an elementary, illustrated introduction to stable homotopy theory. We cover much of the same material from Adams' classical reference [Ada74, III] but from a more modern, and hopefully simpler, point of view. We also include foundational material on many of the developments in the subject since Adams' time, including the "higher algebra" of ring spectra and module spectra.

We intend to make the core ideas of the subject accessible to mathematicians and graduate students in a wide variety of fields, not just those with an especially strong background in algebraic topology and category theory. We only assume some background with the fundamental group and covering spaces, homology and cohomology, homotopy groups, and a basic familiarity with categories and functors. We do *not* assume that the reader is familiar with model categories, ∞-categories, or even simplicial sets. We develop these concepts as they arise.

In particular, our treatment differs from existing ones such as [BR20, Sch16] in that we develop the core theory without model categories or ∞-categories, using only elementary point-set topology and algebraic topology in the first four chapters. We then give a treatment of model categories in Chapter 5, and briefly introduce ∞-categories in a later chapter (not included in this draft), as we need the concepts for the more advanced theory.

Contents

0.1 Introduction

Algebraic topology is the study of topological spaces by algebraic invariants. What is an algebraic invariant? You've seen them before – an algebraic invariant is a rule that assigns each space to a group (or some other algebraic object), so that homeomorphic spaces (or homotopy equivalent spaces) go to isomorphic groups. If you are reading this book, then you have probably already seen:

- the fundamental group $\pi_1(X)$,
- the homotopy groups $\pi_n(X)$,
- the homology groups $H_n(X;G)$, and
- the cohomology groups $H^n(X;G)$.

This list is common and familiar, but it is far from complete. There are many more algebraic invariants of spaces out there. For example:

Example 0.1.1 (Bordism)**.** For a topological space X, define the abelian group $\mathfrak{N}_k(X)$ by taking equivalence classes of k-dimensional closed smooth manifolds M, and reference maps $f: M \to X$. Two such manifolds

$$f_0: M_0 \to X, \quad f_1: M_1 \to X$$

are equivalent if there is a cobordism between them, a compact $(k+1)$-dimensional smooth manifold W whose boundary is the disjoint union $M_0 \amalg M_1$, and a map $f: W \to X$ that restricts to $f_0 \amalg f_1$ on the boundary.

The groups $\mathfrak{N}_*(X)$ are an example of an **extraordinary homology theory** – they satisfy all of the Eilenberg-Steenrod axioms for homology, save the dimension axiom, the one that says that the homology of a point is concentrated in degree zero. In fact, $\mathfrak{N}_2(*) = \mathbb{Z}/2$, generated by the closed manifold $\mathbb{RP}^2$.

Example 0.1.2 (Topological K-theory)**.** For a (paracompact) topological space X, define the abelian group $KU^0(X)$ by taking isomorphism classes of complex vector bundles over X, and formally completing the monoid of such into an abelian group. Equivalently, we define

$$KU^0(X) = [X, \mathbb{Z} \times BU]$$

where $BU = \operatorname{colim}_{n\to\infty} BU(n)$ is the classifying space for stable complex vector bundles, and $[-,-]$ denotes homotopy classes of maps. We similarly define

$$KU^{2n}(X) = [X, \mathbb{Z} \times BU], \qquad KU^{2n+1}(X) = [X, U].$$

The groups $KU^*(X)$ are an example of an **extraordinary cohomology theory**, a contravariant theory that satisfies the Eilenberg-Steenrod axioms except the dimension axiom. In fact, $KU^2(*) = \mathbb{Z}$.

Now, for ordinary homology and cohomology, we prove the basic properties by working in the category of chain complexes. For instance, to show that homotopic maps $f, g: X \rightrightarrows Y$ induce the same map on homology, we first show that they induce a chain homotopy between the maps $f_\#, g_\#: C_*(X) \rightrightarrows C_*(Y)$ on the associated chain complexes. This study of chain complexes is the discipline of **homological algebra**.

It turns out that homological algebra is not sufficient for extraordinary homology and cohomology. In general, we cannot give a chain complex whose homology is $E_*(X)$, for a given extraordinary homology theory E. Instead, we have to use a new kind of homological algebra, where the objects are **spectra** instead of chain complexes.

This new kind of homological algebra is called **stable homotopy theory**. It's also called higher algebra, spectral algebra, or brave new algebra.

This is a richer theory than classical homological algebra, and it has a far greater reach. The simplest way to illustrate this is to consider how many homomorphisms there are from $\mathbb{Z}/2$ to $\mathbb{Z}/2$. In ordinary algebra, there are two of them, the identity and the zero map. In homological algebra, there are more: the group

$$\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

gives an extra, "degree-shifting homomorphism" from $\mathbb{Z}/2$ to $\mathbb{Z}/2$.

In stable homotopy, there are many, many more. The homomorphisms from $\mathbb{Z}/2$ to $\mathbb{Z}/2$ in this setting can be organized into the Steenrod algebra $\mathscr{A}$, an infinite-dimensional noncommutative graded $\mathbb{Z}/2$-algebra. Each element of this algebra corresponds to a natural transformation on cohomology $H^*(X;\mathbb{Z}/2)$ with $\mathbb{Z}/2$ coefficients.

The tradeoff for this richness is that stable homotopy theory is more difficult to learn. It is based on topological spaces or homotopy types, instead of chain complexes. This makes the theory more subtle, and harder to master.

The goal of this book is to give a self-contained introduction to stable homotopy theory. It is the author's hope that the treatment here will make the subject more accessible to mathematicians in fields where higher algebra plays an important role. These include:

- geometric topology (surgery theory),
- symplectic geometry (Floer homotopy theory),
- algebraic geometry (motivic homotopy theory),
- number theory (algebraic K-theory and higher algebra), and of course
- algebraic topology and homotopy theory.

To accomplish this, we present spectra in a "low-tech" way, giving much of the foundational material before introducing simplicial methods and model categories. We also accompany the treatment with pictures and schematics to get the main ideas across to the busy, working mathematician.

0.2 Overview

In this section we briefly indicate what is in the book, and where to find it.

This book is about spectra. If you had to use a single phrase to say what spectra are, you might say one of the following. Spectra are:

- topological spaces in which suspension Σ has been inverted,
- CW complexes with negative-dimensional cells,

- abelian groups up to homotopy, or perhaps
- chain complexes with coefficients in the space of maps $S^n \to S^n$.

There are actually two categories of spectra:

Sp	Ho **Sp**
The point-set category of spectra,	The homotopy category of spectra,
aka the model category of spectra, and	aka the stable homotopy category.

We pass from **Sp** to Ho **Sp** by localizing, taking a certain class of maps called the "stable equivalences" and turning them into isomorphisms. This has the effect of passing to homotopy classes of maps, just like when we form the classical homotopy category of spaces (i.e. CW complexes with homotopy classes of maps between them) or the derived category of a ring R (projective chain complexes of R-modules and chain homotopy classes of maps between them).

There are actually different versions of the point-set category **Sp**, that all give the same homotopy category Ho **Sp**. We focus on three of them in this book: sequential spectra (also called prespectra), symmetric spectra, and orthogonal spectra. These objects are not the same up to isomorphism – they are only the same up to equivalence, and therefore they have the same homotopy category.

The category of spectra has a smash product operation $X \wedge Y$. In the analogy between spectra and abelian groups, this plays the role of the tensor product $X \otimes Y$. Using this new tensor product, we can define rings and modules in the world of spectra. The study of these objects constitutes the subject of spectral algebra, or higher algebra.

The most fundamental ring spectrum is the sphere spectrum $\mathbb{S}$. Every spectrum is a module over $\mathbb{S}$, in the same way that every abelian group is a module over $\mathbb{Z}$. There is also a ring spectrum $H\mathbb{Z}$, with the property that module spectra over $H\mathbb{Z}$ are essentially the same thing as unbounded chain complexes of abelian groups.

Along the ring homomorphism $\mathbb{S} \to H\mathbb{Z}$, every chain complex can be turned into a spectrum by "restriction of scalars," and every spectrum can be turned into a chain complex by "extension of scalars," or "taking chains with $\mathbb{Z}$ coefficients." This is the essence of the relationship between homological algebra and stable homotopy theory. Because the sphere spectrum $\mathbb{S}$ is to the left of $H\mathbb{Z}$, stable homotopy theory has a "deeper base," and sees more information than homological algebra does.

Chapter 1 recalls basic facts from algebraic topology that we need to know before starting into spectra. Most of this should be review, but the results on basic homotopy colimits are worth looking over.

Chapter 2 develops the category of sequential spectra **Sp**. We prove the basic properties directly, and show how spectra represent extraordinary homology and cohomology theories.

Chapter 3 introduces homotopy categories and derived functors, allowing us to pass the results in Chapter 2 from the category of spectra **Sp** to the stable homotopy category Ho **Sp**.

Chapter 4 introduces the smash product $\wedge$ as a black box, and explains how this makes both spectra **Sp** and the homotopy category Ho **Sp** into symmetric monoidal categories. There is also some discussion of rings and modules in a general symmetric monoidal category, along with duality and traces.

Chapter 5 develops the theory of model categories, and proves that **Sp** is a model category. This is an important technical framework that is needed for the more sophisticated mathematics that occurs in the second half of this book.

Chapter 6 defines symmetric spectra $\mathbf{Sp}^{\Sigma}$, orthogonal spectra $\mathbf{Sp}^{O}$, and their smash product. The black-boxed properties from Chapter 4 are proven explicitly here. This relies heavily on the technology of model categories from the previous chapter.

?? introduces simplicial sets, simplicial spaces, and more generally simplicial objects in any category. We then define the bar construction. The bar construction allows us to show that $H\mathbb{Z}$ is a ring spectrum, and plays a crucial role in the next couple of chapters.

?? uses the bar construction to define homotopy colimits and limits of spectra in general, and proves that they are the left- and right-derived functors of the ordinary colimit and limit, respectively.

?? introduces operads and grouplike E_∞ spaces, which we think of as "abelian groups up to coherent homotopy." Such spaces are equivalent to connective spectra, using the bar construction. This makes the idea precise that spectra are "essentially the same thing" as abelian groups. In particular, this is how every bounded-below chain complex creates a spectrum.

?? develops the higher algebra of ring spectra and module spectra. A fundamental result is the Morita theorem that says every "stable" homotopy theory is equivalent to modules over some ring spectrum. In particular, unbounded chain complexes are equivalent to modules over the Eilenberg-Maclane spectrum $H\mathbb{Z}$.

?? discusses how to construct spectral sequences, with a particular focus on the Serre, homotopy orbits, and Adams spectral sequences. We don't go especially far into computational techniques in this book, especially those involving the Adams spectral sequence and its generalizations – see e.g. [Rav86] for more in that direction.

0.3 Why study spectra?

What are spectra good for?

We've already discussed how spectra are needed to work with extraordinary homology theories. We want to indicate why such theories are important, beyond the ones with obvious geometric content like bordism and K-theory. Even in the abstract, extraordinary homology theories are a powerful tool in homotopy theory. Why is that?

The problem with ordinary homology $H_*(X)$ is that it has too few groups. It only has a single group H_0 when X is a point. This is a boon when we are trying to distinguish spaces up to homotopy equivalence, but it is a curse when we are trying to count homotopy classes of based maps, $[X, Y]_*$.

For instance, the set of homotopy classes $[S^3, S^2]_*$ is nonzero. But ordinary homology can't see that. Every map $S^3 \to S^2$ must be zero on homology. Unpacking the reason why, it is precisely because the homology of a sphere is concentrated in a single degree. If we pass to *extraordinary* homology, then a map $S^3 \to S^2$ induces a map

$$E_3(S^3) \longrightarrow E_3(S^2)$$

which could be (and in many cases is) nonzero.

Of course, there are limits to what even extraordinary homology can see. Since suspension is an isomorphism on homology, we can't see the difference between maps $[X, Y]_*$ and the limit of these maps under the suspension operation:

$$\{X, Y\} = \operatorname{colim}\Big([X, Y]_* \longrightarrow [\Sigma X, \Sigma Y]_* \longrightarrow [\Sigma^2 X, \Sigma^2 Y]_* \longrightarrow \cdots \Big).$$

These are the **stable maps** from X to Y. In the stable homotopy category $\operatorname{Ho}\mathbf{Sp}$, these are exactly the maps that appear from X to Y. All of the "unstable" information from the finite stages of this system has been thrown out.

It is far easier to compute these stable maps than the unstable ones $[X, Y]_*$, in part because we have long exact sequences that parallel the ones from homological algebra. Philosophically, these stable maps are a "linear approximation" of the unstable ones. Or, to put it differently:

Spectra are to spaces what linear algebra is to algebra.

However, the computations are still difficult. Computing the **stable homotopy groups of spheres**

$$\pi_k(\mathbb{S}) = \{S^k, S^0\}$$

is easier than computing the unstable groups $\pi_{k+n}(S^n)$, but is still an open problem.

Another motivation comes from differential topology. The **Pontyagin-Thom isomorphism** tells us that the homotopy groups of certain spectra classify manifolds up to cobordism. As a special case, the stable maps $\pi_d(\mathbb{S}) = \{S^d, S^0\}$ described just above are isomorphic to the set of closed framed d-dimensional manifolds, up to cobordisms that respect the framing.

This means that any calculation of stable homotopy groups of spheres gives as an immediate corollary a statement about framed manifolds. Our algebraic results have geometric consequences. More generally, variants of the Pontryagin-Thom isomorphism result play a central role in surgery theory, and therefore in the classification of high-dimensional manifolds.

Finally, and most importantly, spectra are interesting in their own right. They are a world where topology and algebra merge, having some of the best features of chain complexes and of spaces. The algebra of spectra is richer than classical algebra. Taking homology with "exotic" coefficients in a spectrum is more powerful than being limited to "ordinary" coefficients in an abelian group. This richness has striking and dramatic consequences in many fields of mathematics, far beyond pure homotopy theory and algebraic topology.

0.4 History

The history of spectra is somewhat backwards – the homotopy category Ho **Sp** was invented before the point-set category **Sp**.

The homotopy category was first defined by Lima in [Lim58]; see also [SW53, SW55] for early works anticipating the concept, and [Ada74] for the state of the art in the 1970s. The original motivations for the homotopy category of spectra included

- using stable maps $\{X, Y\} = \operatorname*{colim}_{n\to\infty} [\Sigma^n X, \Sigma^n Y]_*$ as a "linear approximation" to homotopy classes of maps, in order to compute new homotopy groups of spheres,
- being able to "dualize" a topological space X in the way one dualizes a vector space $V^* = \mathrm{Hom}_k(V, k)$,
- classifying smooth manifolds up to cobordism, and
- working with extraordinary homology and cohomology theories.

On the other hand, the point-set category **Sp** took longer to develop. One might say the basic foundations only came into place in the 1990s with the papers [HSS00, EKMM97, MMSS01, MM02]. Before then, it was common to work with the stable homotopy category, and the models for it were complicated and clunky. Peter May once remarked that the situation was like knowing what the derived category of a ring is, without knowing what an R-module is.

The problem wasn't how to define spectra, it was how to define the *smash product*. Like the tensor product of vector spaces, the smash product is an operation that combines two or more spectra together into a single spectrum, and it is supposed to be associative and commutative up to isomorphism. In other words, it makes spectra into a symmetric monoidal category. And the problem is that no one knew how to do this on the point-set level (it was even believed to be impossible) until the groundbreaking works of Hovey, Shipley, and Smith [HSS00], and Elmendorf, Kriz, Mandell, and May [EKMM97].

In the next few years these foundations were fleshed out by the work of many people, see e.g. [HPS97, SS00, MMSS01, SS03], giving a fully-developed theory that could be gleaned from research papers. In the past decade, this theory has been worked over and distilled into more expository works, including the present one.

The theory of spectra has also experienced numerous significant developments and applications since then. It would be impossible to give a complete survey of these developments, but to name a few of them:

- The smash product of spectra allowed for an era of "trace methods" in algebraic K-theory, greatly increasing our understanding and allowing for new calculations that connect to both differential topology and to number theory.
- Goodwillie and others further developed the theory of functor calculus, explaining how spectra emerge in the process of forming "linear approximations," "quadratic approximations," and so on, to functors on spaces.
- The theory of ∞-categories emerged as a new framework for understanding homotopy theoretic constructions and extending their reach to new settings. Lurie characterized the ∞-category of spectra by a simple universal property.
- Chromatic homotopy theory saw spectacular further development, including the creation of the ring spectrum of topological modular forms, allowing for further computations and insight into the homotopy groups of spheres, arguably the most fundamental problem in algebraic topology.
- Motivic homotopy theory and stable motivic homotopy theory were created, building new connections between the theory of spectra and algebraic geometry, important even to those who are only concerned with one of the two subjects.
- Floer homotopy theory created an increasingly important connection between stable homotopy theory and symplectic geometry.
- The Kervaire invariant one problem was (almost completely) solved, and along with it came a renaissance in equivariant stable homotopy theory (G-spectra), built on the theory of orthogonal spectra discussed in this book. (See [HHR21].)

- Higher algebra now plays an increasingly important role in classical algebra and number theory, building on the fundamental idea that spectra are the next step of the progression after chain complexes.

0.5 Acknowledgments

There are too many mathematicians to thank for this work. The explicit list will have to at least mention Ralph Cohen, Aaron Mazel-Gee, and Eric Peterson for being my first teachers in this subject. I can still remember brushing my teeth in Aaron's bathroom while asking about the difference Adams makes between a function, a map, and a morphism. We've come a long way since then.

The current work is not the first textbook on spectra to appear, and the existing textbooks and expository works have had a significant influence on this one. Frank Adams' classic work [Ada74] has been the standard reference for decades. On a technical level it is now out of date, but Adams' writing never goes out of style. Though it is not focused on spectra, one of the chapters in Switzer's book [Swi75] also had a large influence on the author's perspective. Stefan Schwede's massive project [Sch07] was the book the author used in graduate school, and it continues to be the only place you can go to find a careful treatment of the finer points of symmetric spectra. Urs Schreiber's online book [Sch16] has been a valuable reference to the author during the writing of the present book. David Barnes was kind enough to share drafts of [BR20] while it was being written. The author has also seen expository works by Agnes Beaudry and Jonathan Campbell [BC18], Kirsten Wickelgren [Wic15], and Bruno Stonek [Sto22], along with expository tidbits by many other mathematicians, all of which has informed the perspective of this book. Extra thanks go to Bruno Stonek for numerous helpful comments on the early drafts of this book.

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Lastly, the author thanks his wife and partner Courtney Ignarri for being a loving and supporting presence in his life, and for helping him grow into the best version of himself. We cannot teach without some kind of love for ourselves, our students, and the world. It is the author's hope that his love for the subject, and for his students and colleagues, will be felt in these pages.

Thank you to everyone who made this book possible.

Cary Malkiewich

Chapter 1

Preliminaries

This chapter is a fast review and a reference for concepts from algebraic topology and homotopy theory. We give only definitions and theorem statements, no proofs. More details can be found in standard algebraic topology texts such as [Hat02, May99, tD08, Swi75, Spa66]. We expect that the reader has seen most of this already, but this chapter might be helpful in filling in some gaps.

1.1 Basic operations on topological spaces

1.1.1 Elementary topology, CGWH spaces

A **topological space** X is a set equipped with a choice of which subsets are "open," subject to the axioms that open sets are preserved by arbitrary unions (including empty unions, hence $\emptyset$ is open) and finite intersections (including empty intersections, hence X itself is open). A set is closed precisely when its complement is open.

A map $f: X \to Y$ is **continuous** if the preimage of every open set $U \subseteq Y$ is open in X. f is a **homeomorphism** if it is a bijection and its inverse is also continuous. f is an inclusion if it is a homeomorphism onto its image; when the image is open or closed we call f an **open inclusion**, respectively a **closed inclusion**.

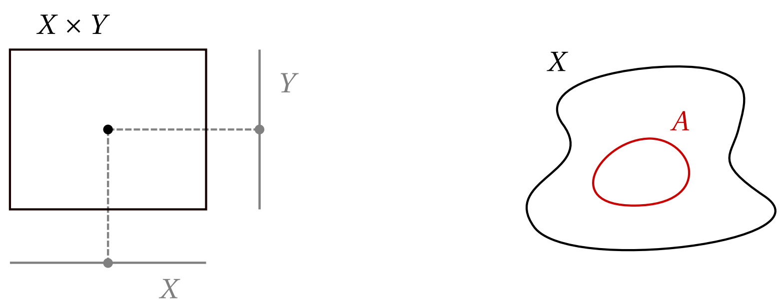
There exist continuous bijections that are not homeomorphisms – for instance, the map that winds the half-open interval $[0,1)$ bijectively around the circle S^1. However, if X is compact and Y is Hausdorff then every continuous bijection $X \to Y$ is a homeomorphism, and therefore also every continuous injection is an inclusion.

A topological space X is **compact** if every open cover has a finite subcover, **connected** if there does not exist a separation $X = A \amalg B$ into nonempty disjoint open sets, and **Hausdorff** if every pair of points can be separated by a pair of disjoint open sets.

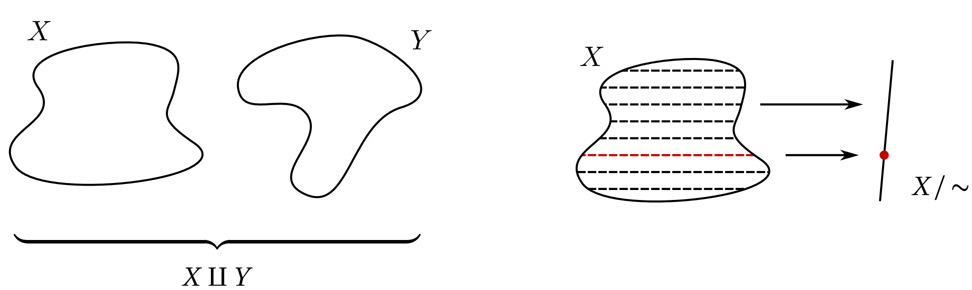
A topological space X is **compactly generated** if for each subset $C \subseteq X$, C is closed iff $f^{-1}(C) \subseteq K$ is closed for every compact Hausdorff space K and continuous map $f: K \to X$. Furthermore X is **weak Hausdorff** if for each such f, the image $f(K)$ is closed. Open and closed subspaces of $\mathbb{R}^n$, along with just about every other space you usually encounter, are compactly generated weak Hausdorff (CGWH). For those that aren't, there are standard operations that make them CGWH. See [Lew78, App A] and [Str09] for a detailed treatment of the properties of these spaces. Or don't. Most people seem to just take on faith that these spaces behave the way you expect.

Convention: All spaces in this book are compactly generated weak Hausdorff (CGWH).

A **product** $X \times Y$ is the Cartesian product with a topology that ensures that a map $Z \to X \times Y$ is continuous iff the two factors $Z \to X$ and $Z \to Y$ are continuous. A **subspace** $A \subseteq X$ is a subset topologized so that a map $Z \to A$ is continuous iff the resulting map $Z \to X$ is continuous.



Dually, a **coproduct** or disjoint union $X \amalg Y$ is the disjoint union with a topology that ensures that $X \amalg Y \to Z$ is continuous iff each of the two summands $X \to Z$ and $Y \to Z$ is continuous. A **quotient** $X \to Y$ is a surjective map (or what is the same thing, an equivalence relation on X), with Y topologized so that $Y \to Z$ is continuous iff the resulting map $X \to Z$ is continuous. The universal properties of these constructions are shown in Figure 1.1.1.



In CGWH spaces, the product and subspace have the same underlying set as the above, but the topology is sometimes a little finer. The coproduct is the same. The quotient is different – if the equivalence relation on X is not closed as a subspace of $X \times X$ (with the CGWH product), then we have to take its closure first before modding out by the relation. This ensures the quotient is CGWH.

$$
\begin{tikzcd}
Z \arrow[rr, dashed, "\exists!"] \arrow[dr] \arrow[drrr] & & X \times Y \arrow[dl] \arrow[dr] & \\
& X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr] \arrow[drrr] & & Y \arrow[dl] \arrow[dr] & \\
& X \amalg Y \arrow[rr, dashed, "\exists!"] & & Z
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & X \arrow[dd, squiggly] \\
Z \arrow[r, dashed, "\exists!"'] \arrow[urr, bend left] \arrow[dd, squiggly] & A \arrow[ur] \arrow[dd, squiggly] & \\
& & X^{\mathrm{set}} \\
Z^{\mathrm{set}} \arrow[r] \arrow[urr, bend left] & A^{\mathrm{set}} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left] \arrow[dd, squiggly] & & \\
& Y \arrow[r, dashed, "\exists!"'] \arrow[dd, squiggly] & Z \arrow[dd, squiggly] \\
X^{\mathrm{set}} \arrow[dr] \arrow[drr, bend left] & & \\
& Y^{\mathrm{set}} \arrow[r] & Z^{\mathrm{set}}
\end{tikzcd}
$$

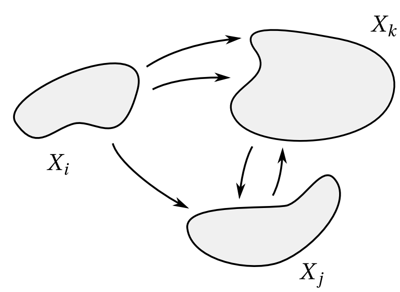
Figure 1.1.1: The defining properties of products, coproducts, subspaces, and quotients.

1.1.2 Colimits and limits

Topological spaces (that are CGWH) form a **category**, that we refer to as **Top**. This means there is a collection of objects (the spaces), between every pair of spaces a set of morphisms (the continuous maps), and a composition rule for the morphisms that is associative and has identity morphisms.

The rule **Top** $\to$ **Set** that assigns each space to its underlying set is an example of a **functor**. That means that it assigns each object to an object (each space to a set), between two objects it assigns each map to a map (forget that the map is continuous), and respects composition of maps and identity maps. A functor is very much like a homomorphism of groups, thinking of the maps in the category as the group elements. See [Mac71, Rie17] for more discussion of categories in general.

Those familiar with categories will recognize the definitions in Figure 1.1.1 as universal properties for products, cartesian arrows (over the forgetful functor to sets), coproducts, and cocartesian arrows.



From these elementary operations we can define all limits and colimits of diagrams of spaces. A **diagram** of spaces is informally a collection of spaces and continuous maps between them. Formally, it is a functor $X\colon \mathbf{I} \to$ **Top** where $\mathbf{I}$ is any small category, which we call an **indexing category**. So we get a space $X(i)$ for each object $i \in \operatorname{ob} \mathbf{I}$ and a map $X(i) \to X(j)$ for each morphism $i \to j$, respecting composition. For instance, if $\mathbf{I}$ has four objects and just enough morphisms to form a commuting square

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
$$

Indexing category	Colimit is called...	Limit is called...
$\bullet \qquad \bullet$	coproduct	product
$\bullet \rightrightarrows \bullet$	coequalizer	equalizer
$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$	pushout	(trivial)
$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$	(trivial)	pullback
$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots$	sequential colimit	(trivial)
$\dots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$	(trivial)	sequential limit

Figure 1.1.2

then a diagram on $\mathbf{I}$ is the same thing as a commuting square of spaces. We illustrate some common choices of indexing category $\mathbf{I}$ in Figure 1.1.2. In some of these cases the colimit or limit is just one of the spaces in the diagram – see exercise 2.

The **limit** of a diagram $X : \mathbf{I} \to \mathbf{Top}$ is the universal space Y that is equipped with maps $Y \to X(i)$ for all $i \in \mathrm{ob}\,\mathbf{I}$, commuting with each of the maps of the diagram. In other words, if Y' is another such space, then there is a unique map $Y' \to Y$ commuting with the map to $X(i)$ for each $i \in \mathrm{ob}\,\mathbf{I}$.

The limit can always be defined as the product $\prod_{i \in \mathrm{ob}\,\mathbf{I}} X(i)$, restricted to the subspace of points that agree along all of the maps in the diagram:

$$\lim_{i \in \mathbf{I}} X(i) = \left\{ \, x_i \in X(i) \quad \forall i \in \mathrm{ob}\,\mathbf{I} \, : \, f(x_i) = x_j \quad \forall f : i \to j \, \right\}. \tag{1.1.3}$$

The **colimit** of a diagram $X : \mathbf{I} \to \mathbf{Top}$ is the universal space Z that is equipped with maps $X(i) \to Z$, commuting with all the maps of the diagram. It can always be defined as the coproduct (disjoint union) of the $X(i)$, modulo an identification along each morphism f:

$$\operatorname*{colim}_{i \in \mathbf{I}} X(i) = \left(\coprod_{i \in \mathrm{ob}\,\mathbf{I}} X(i) \right) \Big/ \Big(x \sim f(x) \quad \forall f : i \to j, \; x \in X(i) \Big). \tag{1.1.4}$$

In particular, a point in a colimit is defined by a point in *one* space $X(i)$, while a point in a limit requires a point in *every* X_i. For this reason colimits are often much simpler than limits.

Remark 1.1.5. In CGWH spaces, limits commute with the forgetful functor to sets. In other words, you can define them by taking the limit of the underlying sets first, then giving that limit a topology. For colimits, this is unfortunately not always true. However it is true for the following kinds of colimits:

- coproducts,
- pushouts along a closed inclusion,
- sequential colimits along closed inclusions, and
- orbits under the action of a compact Hausdorff topological group.

So in these cases, you can argue in terms of the underlying set. Outside of these cases, you should argue instead using the universal property. (It's usually easier to use the universal property anyway.)

Pushouts and pullbacks have special names. For the two diagrams below, we denote the pushout (colimit) of first by $X \cup_A Y$, and the pullback (limit) of the second by $X \times_B Y$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

The further special case $X \cup_A \{*\}$ is called the quotient or cofiber X/A. Note that $X/\emptyset = X_+$ is X with a disjoint point added.

In CGWH spaces, so long as one of the two maps $A \to X$ or $A \to Y$ is a closed inclusion, the pushout is the same as in ordinary spaces (and so it is a pushout on the underlying set). The pullback is always what you expect on the underlying set.

Example 1.1.6. Let D^n be the closed unit disc in $\mathbb{R}^n$ and S^{n-1} the closed unit sphere. Then the pushout $D^n \cup_{S^{n-1}} D^n$ along the inclusions $S^{n-1} \subset D^n$ is homeomorphic to S^n. To prove this, you note that the pushout is compact and S^n is Hausdorff, then you define the continuous bijection directly.

Definition 1.1.7. Given a commuting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"'] & D, \end{tikzcd}$$

we say it is a **pushout square** if the induced map $B \cup_A C \to D$ is a homeomorphism. We also say that g is the pushout of the map f, and sometimes place the symbol $\ulcorner$ inside the square.

Similarly, the above square is a **pullback square** if the induced map $A \to B \times_D C$ is a homeomorphism. In this case we say that f is the pullback of the map g, and sometimes place the symbol $\lrcorner$ inside the square.

1.1.3 Cell complexes and CW complexes

Definition 1.1.8. A **cell complex** is a space X that is the sequential colimit of a sequence of spaces

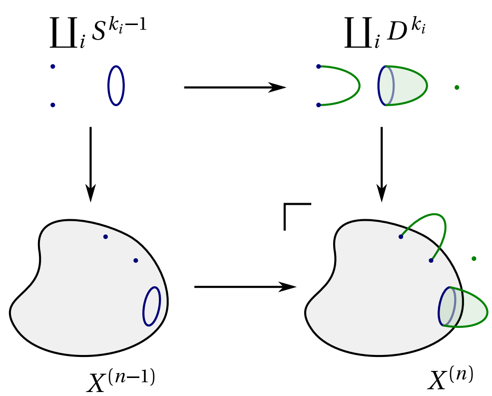
$$\emptyset = X^{(-1)} \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \ldots \longrightarrow X^{(n-1)} \longrightarrow X^{(n)} \longrightarrow \ldots \longrightarrow X.$$

We also call $X^{(-1)} \to X$ the **countable composition** of the smaller maps $X^{(n-1)} \to X^{(n)}$ for $n \geq 0$. Furthermore, we ask that each of the maps $X^{(n-1)} \longrightarrow X^{(n)}$ is a pushout of a coproduct of "cells" $S^{k-1} \to D^k$, with varying $k \geq 0$:

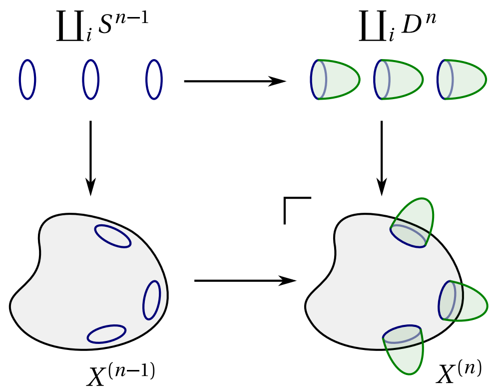
$$\begin{tikzcd} \coprod_i S^{k_i-1} \arrow[r] \arrow[d] & \coprod_i D^{k_i} \arrow[d] \\ X^{(n-1)} \arrow[r] & X^{(n)}. \end{tikzcd}$$

In the case that $k = 0$, the sphere S^{0-1} is defined to be the empty set, and D^0 is a point. So, attaching a 0-cell has the effect of taking disjoint union with a point.

Pictorially, we can imagine this process as follows.



We say that X is a **CW complex** if $k_i = n$ for all i. In other words, we only attach cells of dimension n at stage n:



A **relative cell complex** or relative CW complex $A \to X$ is defined in the same way, except that we take $X^{(-1)} = A$. We sometimes abbreviate this by saying that "$A \to X$ is a cell complex."

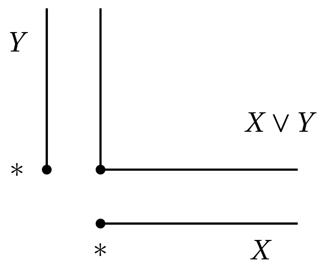
Proposition 1.1.9. *A product $X \times Y$ of CW (or cell) complexes is a CW (or cell) complex with an $(m+n)$-cell for each pair of an m-cell in X and an n-cell in Y.*

Remark 1.1.10. This proposition is actually not true in ordinary spaces, only in CGWH spaces. This is one reason why the product in CGWH spaces is better than the ordinary product topology.

1.1.4 Based spaces

A **based space** is a space X with a chosen basepoint x_0. A based map $f: X \to Y$ is a map sending x_0 to y_0. We often use the symbol $*$ to refer to the basepoint of every space, and also to refer to a generic one-point space. A based cell complex is just a cell complex relative to $*$. This is the same thing as a cell complex with a chosen 0-cell as the basepoint.

Based spaces and based maps form a category $\mathbf{Top}_*$. The limits in this category are the same as those in unbased spaces. The colimits are slightly different – the colimit as computed in unbased spaces may have multiple basepoints, and they have to be identified back together in order to get the colimit in based spaces.



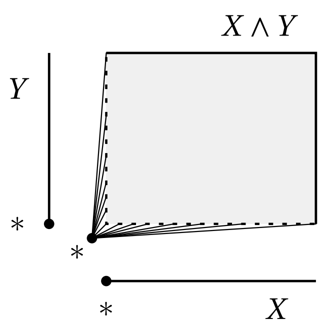
As a special case, the coproduct in based spaces is the **wedge sum**

$$X \vee Y = (X \amalg Y)/(x_0 \sim y_0).$$

The general formula for a colimit in based spaces is the same as (1.1.4) but with the disjoint unions replaced by wedge sums:

$$\operatorname*{colim}_{\mathbf{I}}{}^{(b)} X(i) = \left(\bigvee_{i \in \mathrm{ob}\,\mathbf{I}} X(i) \right) \Big/ \big(x \sim f(x) \quad \forall f : i \to j,\ x \in X(i) \big). \tag{1.1.11}$$

Usually this is just written as colim, without the (b) decoration, even though the based and unbased colimits are not homeomorphic in general. See exercise 4.



Definition 1.1.12. The wedge sum $X \vee Y$ is a subspace of the product $X \times Y$ by including into $X \times \{y_0\}$ and $\{x_0\} \times Y$. The quotient $(X \times Y)/(X \vee Y)$ is called the **smash product** $X \wedge Y$. The smash product is essentially "Cartesian product of the complements of the basepoints."

1.1.5 Mapping spaces

For spaces X and Y the **mapping space** $\mathrm{Map}(X, Y) = Y^X$ is the set of continuous maps $X \to Y$, topologized so that continuous maps $K \to \mathrm{Map}(X, Y)$ are in bijection with continuous maps $K \times X \to Y$. In fact, this correspondence is continuous, giving a homeomorphism

$$\mathrm{Map}(K \times X, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y)). \tag{1.1.13}$$

If X and Y have basepoints then the based mapping space $\mathrm{Map}_*(X, Y) = (Y, *)^{(X,*)}$ is the closed subspace of $\mathrm{Map}(X, Y)$ of all maps preserving the basepoint. Then based (continuous) maps $K \to \mathrm{Map}_*(X, Y)$ correspond to based (continuous) maps $K \wedge X \to Y$. Again, this passes to a homeomorphism

$$\mathrm{Map}_*(K \wedge X, Y) \cong \mathrm{Map}_*(K, \mathrm{Map}_*(X, Y)). \tag{1.1.14}$$

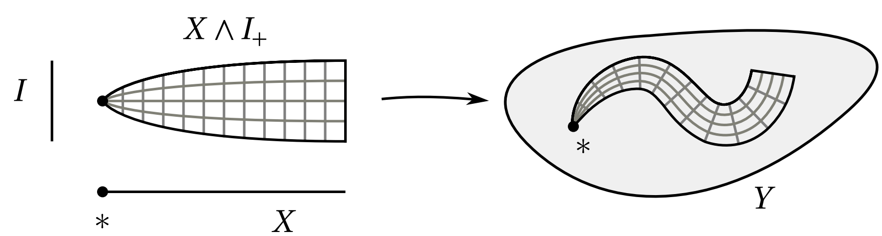
1.2 Homotopies, cofibrations, and fibrations

1.2.1 Homotopies

Given two continuous maps $f, g\colon X \rightrightarrows Y$, a **homotopy** is a (continuous) map

$$h\colon X \times I \to Y$$

such that $h(-,0) = f$ and $h(-,1) = g$. This homotopy could equivalently be given by a map $X \to Y^I$ from X to the space of paths in Y. If X and Y are based, a based homotopy is one in which the basepoint is constant. It can equivalently be described as a based map $X \wedge I_+ \to Y$.



Homotopy is an equivalence relation on the maps from X to Y. We let $[X, Y]$ denote the set of homotopy classes of maps. If X and Y are based we let $[X, Y]_*$ denote based-homotopy classes of based maps. In that case there is a map $[X, Y]_* \to [X, Y]$ that is not always a bijection.

Example 1.2.1. $[S^1, S^1]_* \to [S^1, S^1]$ is a bijection and both are in bijection with $\mathbb{Z}$ by the winding number.

Homotopy respects composition of maps. Therefore the sets $[X, Y]$ form the morphisms of a category, the (classical) homotopy category $h\mathbf{Top}$. Isomorphisms in this category are the same thing as homotopy equivalences. Similarly the sets $[X, Y]_*$ are the morphisms of a category, the based (classical) homotopy category $h\mathbf{Top}_*$. We typically only consider the subcategory of these on the CW complexes, and denote them $h\mathbf{CW} = \mathrm{Ho}\,\mathbf{CW}$ and $h\mathbf{CW}_* = \mathrm{Ho}\,\mathbf{CW}_*$, respectively. The "Ho" decoration refers to the fact that these categories have a universal property, see Definition 3.1.9 and Proposition 3.1.26.

1.2.2 Cofibrations

A map $A \to X$ is called a **cofibration** if it has the homotopy extension property: for any map $X \to Y$, and any homotopy of the map on A, the homotopy can be extended to X.

In other words, for each diagram as below, a dotted lift exists.

$$\begin{tikzcd}
(A\times I)\cup_{A\times\{0\}}(X\times\{0\}) \arrow[r] \arrow[d] & Y \\
X\times I \arrow[ur, dashed] &
\end{tikzcd}$$

It suffices to check this for $Y=(A\times I)\cup_{A\times\{0\}}(X\times\{0\})$, so being a cofibration is equivalent to having a retract of $X\times I$ back onto $(A\times I)\cup_{A\times\{0\}}(X\times\{0\})$.

Cofibrations are sometimes called “Hurewicz cofibrations” in order to distinguish them from a different notion that appears later (Definition 5.1.18).

Lemma 1.2.2. *Every relative cell complex is a cofibration. Any coproduct, pushout, countable composition, or retract of a cofibration is again a cofibration.*

Every map $f\colon A\to X$ can be replaced by an equivalent cofibration. Simply replace X by the mapping cylinder

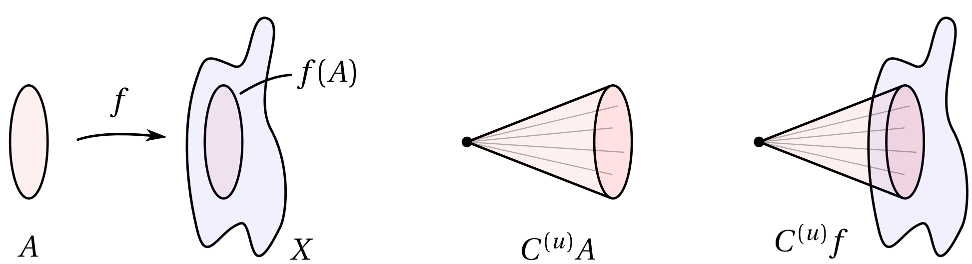
$$(A\times I)\cup_{A\times\{1\}}(X\times\{1\}),$$

and include A by the front end $A\times\{0\}$.

The **cofiber** of $A\to X$ is just the quotient X/A. The unbased **homotopy cofiber** $C^{(u)}f$ is the cofiber of $A\to X$, after replacing the map by a cofibration:

$$C^{(u)}f = *\cup_{A\times\{0\}}(A\times I)\cup_{A\times\{1\}}X = C^{(u)}A\cup_A X.$$

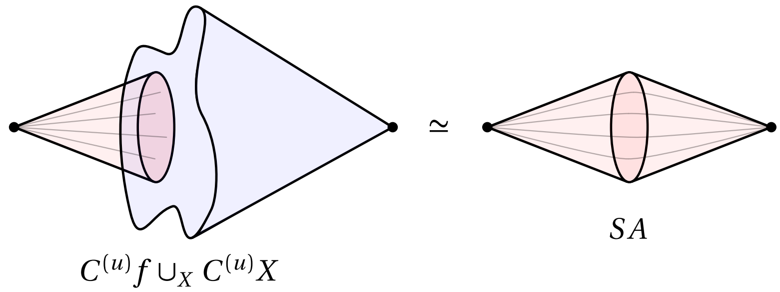
Here $C^{(u)}A=(A\times I)/(A\times\{0\})$ is the unbased cone on A.



Lemma 1.2.3. *If $f\colon A\to X$ is a cofibration then the collapse map $C^{(u)}f\to X/A$ is a homotopy equivalence.*

If we then replace $X\to C^{(u)}f$ by a cofibration and take its cofiber, the result is equivalent to the suspension

$$SA = C^{(u)}A\cup_A C^{(u)}A \cong (A\times I).$$



Repeating this process indefinitely produces the unbased cofiber Puppe sequence

$$A \xrightarrow{f} X \longrightarrow C^{(u)}f \longrightarrow SA \xrightarrow{-Sf} SX \longrightarrow SC^{(u)}f \longrightarrow S^2A \xrightarrow{S^2 f} S^2X \longrightarrow \dots$$

Here $-Sf$ refers to the map that applies f but then flips the two cones around.

The following lemma about products of cofibrations is especially important. To state it, we recall that cofibrations $A \to X$ are always closed inclusions, so without loss of generality we think of them as subspaces $A \subseteq X$.

Lemma 1.2.4. *[May99, §6.4] If $A \subseteq X$ and $B \subseteq Y$ are cofibrations, then*

$$(A \times Y) \cup_{(A\times B)} (X \times B) \subseteq X \times Y$$

is also a cofibration.

Definition 1.2.5. Suppose X has a basepoint. We say X is **well-based** or nondegenerately based if the inclusion of the basepoint $* \to X$ is a cofibration.

Remark 1.2.6. By Lemma 1.2.4, if X and Y are well-based then $(X \vee Y) \subseteq (X \times Y)$ is a cofibration. Therefore the smash product $X \wedge Y$ is equivalent to the homotopy cofiber of $(X \vee Y) \subseteq (X \times Y)$.

There is a based version of the homotopy extension property, where all spaces are based and the homotopies preserve the basepoint. Every unbased cofibration of based spaces is a based cofibration. Every based cofibration of *well-based* spaces is an unbased cofibration.

Let CA be the based cone on A, the quotient of the unbased cone $C^{(u)}A$ by the interval $\{*\} \times I$ consisting of the basepoint of A times I:

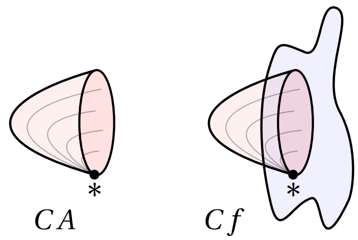
$$CA = \left(* \cup_{A\times\{0\}} (A \times I)\right) \big/ (\{*\} \times I)$$

Alternatively, it is the reduced cylinder $A \wedge I_+$ modulo one endpoint, or just the smash product $A \wedge I$ if we give I the basepoint 0:

$$CA \cong * \cup_{A\wedge I_+} (A \wedge I_+) \cong A \wedge I.$$

Let Cf denote the based version of the homotopy cofiber,

$$Cf = CA \cup_{A \wedge \{1\}_+} X = (C^{(u)}f)/(\{*\} \times I).$$



Lemma 1.2.7. *If $f : A \to X$ is a cofibration (in either sense) and both A and X are well-based, then the collapse map $Cf \to X/A$ is a homotopy equivalence.*

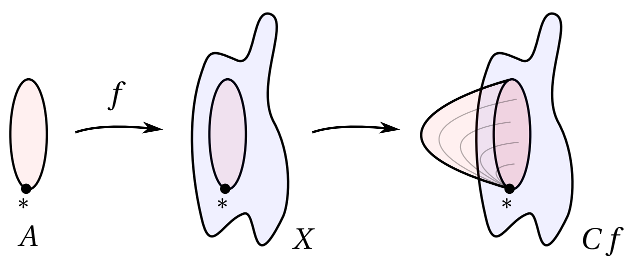
Definition 1.2.8. A **cofiber sequence** is any diagram of based spaces of the form

$$A \xrightarrow{f} X \longrightarrow Cf,$$

or anything weakly equivalent to it. In particular, if $A \to X$ is a cofibration, then

$$A \longrightarrow X \longrightarrow X/A$$

is a cofiber sequence.



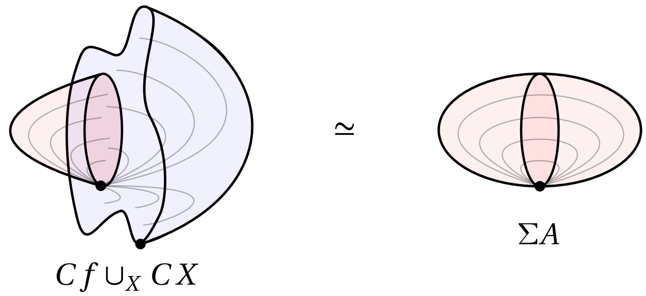
The based version of the cofiber Puppe sequence is the same, except that the cones and suspensions are reduced, i.e. the segment $* \times I$ is collapsed to the basepoint. This produces

$$A \xrightarrow{f} X \longrightarrow Cf \longrightarrow \Sigma A \xrightarrow{-\Sigma f} \Sigma X \longrightarrow \Sigma Cf \longrightarrow \Sigma^2 A \xrightarrow{\Sigma^2 f} \Sigma^2 X \longrightarrow \ldots$$

where ΣA is the reduced suspension

$$\Sigma A = S^1 \wedge A \cong SA/S(*).$$

We only consider this sequence when the spaces A and X are well-based.



Proposition 1.2.9. *For any based space Z, taking based homotopy classes of maps from the Puppe sequence into Z produces a long exact sequence of pointed sets*

$$[A,Z]_* \longleftarrow [X,Z]_* \longleftarrow [Cf,Z]_* \longleftarrow [\Sigma A,Z]_* \longleftarrow [\Sigma X,Z]_* \longleftarrow [\Sigma Cf,Z]_* \longleftarrow \dots$$

By "long exact sequence of pointed sets," we mean that every set has a basepoint $*$, and that the "kernel" of each map (those points mapping to $*$) coincides with the image of the previous map.

Remark 1.2.10. Starting at the term $[\Sigma A,Z]_*$, the sets are all groups and the maps are group homomorphisms. The group operation pinches the suspension in the middle and adds two maps together:

$$\Sigma A \xrightarrow{\text{pinch}} \Sigma A \vee \Sigma A \xrightarrow{(f,g)} Z.$$

Starting at $[\Sigma^2 A,Z]_*$, these group structures are all abelian.

1.2.3 Fibrations

A map $E \to B$ is a **Hurewicz fibration** if it has the homotopy lifting property: any map $X \to E$ and homotopy of the composite $X \to E \to B$, can be lifted to a homotopy of $X \to E$. In other words, for each diagram as below, a dotted lift exists.

$$\begin{tikzcd}
 & E^I \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & E \times_B B^I
\end{tikzcd}$$

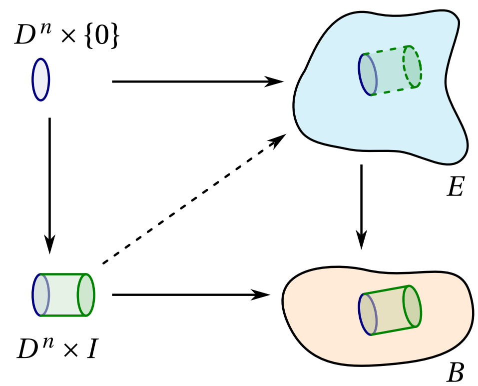
It suffices to check this for $X = E \times_B B^I$. So being a fibration is equivalent to having a map $E \times_B B^I \to E^I$ that lifts the identity of $E \times_B B^I$.

The lifting property could also be rearranged as follows:

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\
X \times I \arrow[r] \arrow[ur, dashed] & B.
\end{tikzcd}$$

In other words, each path in B with a lift of the starting point to E, can be lifted to E. Furthermore, the same can be done for an "Xs worth" of paths in B, and lifts of the starting point to E.

A **Serre fibration** is a map $E \to B$ that has the homotopy lifting property for $X = D^n$, $n \geq 0$, but not necessarily in general. So every square of the following form has a lift.



Recall that $E \to B$ is a fiber bundle if it is locally a product, i.e. B is covered by open sets U_α on which the map is isomorphic to a projection $U_\alpha \times F \to U_\alpha$. A covering space is a fiber bundle in which the fiber F is discrete.

Lemma 1.2.11. *Every fiber bundle, in particular every covering space, is a Hurewicz fibration. Every Hurewicz fibration is a Serre fibration. Any product, pullback, countable inverse composition, or retract of a fibration is a fibration. (This statement applies to both kinds of fibrations.)*

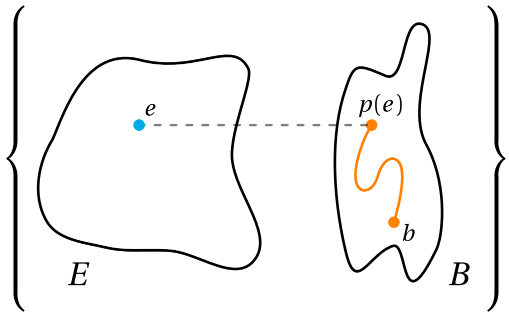
For covering spaces, the path-lifting is actually unique. For general fibrations, the path lifting turns out to be unique up to homotopy, but not unique on the nose.

Every map $p\colon E \to B$ can be replaced by an equivalent Hurewicz fibration. Simply replace E by the mapping co-cylinder $E \times_B B^I$, and project to B by evaluating at the opposite end of I. For each $b \in B$, the **homotopy fiber** $F_b p$ is the fiber over b of this new map:

$$F_b p = E \times_B B^I \times_B \{b\}.$$

It is the space of points $e \in E$ with paths from $p(e) \in B$ to $b \in B$. If B is a based space, we denote the homotopy fiber over the basepoint by Fp:

$$Fp = E \times_B B^I \times_B \{*\}.$$



Lemma 1.2.12. *If p is a Hurewicz fibration then each inclusion $p^{-1}(b) \to F_b p$ is a homotopy equivalence. If p is a Serre fibration then $p^{-1}(b) \to F_b p$ is a weak homotopy equivalence.*

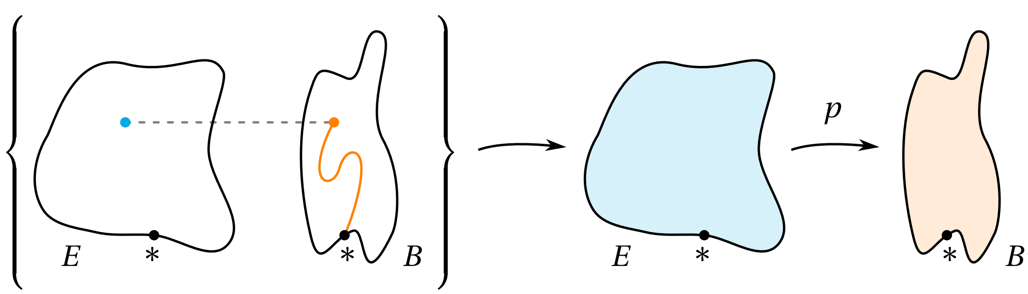
Definition 1.2.13. A **fiber sequence** is any diagram of based spaces of the form

$$Fp \longrightarrow E \xrightarrow{p} B,$$

or anything weakly equivalent to it. In particular, if $E \to B$ is a fibration of either kind, then

$$p^{-1}(*) \longrightarrow E \xrightarrow{p} B$$

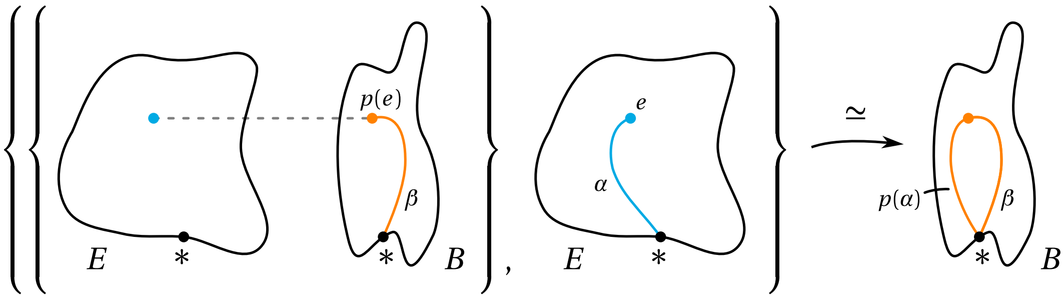
is a fiber sequence.



Assume that $E \to B$ is a map of based spaces, and when we take homotopy fiber, we always take it over the basepoint.[1] If we then replace $Fp \to E$ by a fibration and take its fiber, the result is equivalent to the space of based loops in B,

$$\Omega B = \mathrm{Map}_*(S^1, B).$$

[1]However our notion of "fibration" will remain unbased. There is a based notion of "fibration," but it rarely comes up in practice.



Repeating this process indefinitely produces the fiber Puppe sequence

$$\dots \longrightarrow \Omega^2 E \xrightarrow{\Omega^2 p} \Omega^2 B \longrightarrow \Omega F p \longrightarrow \Omega E \xrightarrow{-\Omega p} \Omega B \longrightarrow F p \longrightarrow E \xrightarrow{p} B.$$

Proposition 1.2.14. *For any based space Z, taking based homotopy classes of maps from Z into the Puppe sequence produces a long exact sequence of pointed sets*

$$\dots \longrightarrow [Z,\Omega F p]_* \longrightarrow [Z,\Omega E]_* \longrightarrow [Z,\Omega B]_* \longrightarrow [Z,F p]_* \longrightarrow [Z,E]_* \longrightarrow [Z,B]_*.$$

Remark 1.2.15. Starting at the term $[Z,\Omega B]_*$, the sets are all groups and the maps are group homomorphisms. The group operation concatenates the loops in B:

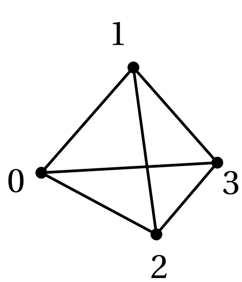
$$Z \xrightarrow{(f,g)} \Omega B \times \Omega B \xrightarrow{\text{concatenate}} \Omega B.$$

Starting at $[Z,\Omega^2 B]_*$, these group structures are all abelian.

Remark 1.2.16. The long exact sequences of Proposition 1.2.9 and Proposition 1.2.14 are to higher algebra what the snake lemma and zig-zag lemma are to homological algebra. They are the basic tool that we use to build our long exact sequences.

1.3 Homology and cohomology

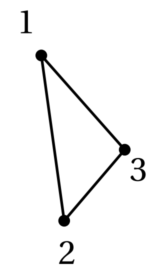
1.3.1 Homology



Let Δ^n be the convex hull of the standard basis vectors in $\mathbb{R}^n$:

$$\Delta^n = \left\{ (t_0,\dots,t_n) \in \mathbb{R}^{n+1} \ : \ \sum_i t_i = 1,\ t_i \geq 0\ \forall i \right\}$$

The case of $n=3$ is pictured to the left.



The ith face of Δ^n is the subspace of points in which $t_i = 0$, which we identify with Δ^{n-1} in the obvious way (preserving the order of the coordinates). The 0th face of Δ^3 is pictured to the right.

Given a continuous map $\sigma\colon \Delta^n \to X$, we define its ith face $d_i\sigma\colon \Delta^{n-1} \to X$ to be the restriction of σ to this subspace.

Let G be any abelian group. (The most common examples are $\mathbb{Z}$, $\mathbb{Z}/p$, and $\mathbb{Q}$.) A singular n-chain in the space X is a formal linear combination of continuous maps $\sigma\colon \Delta^n \to X$, with coefficients in G. In other words, the group

$$C_n(X;G) = \bigoplus_{\Delta^n \to X} G.$$

We define the boundary map

$$\partial_n\colon C_n(X;G) \to C_{n-1}(X;G)$$

by sending each simplex σ to the alternating sum of its faces,

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i d_i \sigma,$$

and extending to $C_n(X;G)$ by linearity. These form a **chain complex**, meaning that $\partial_{n-1}\partial_n = 0$. The **homology** of this chain complex is the abelian group

$$H_n(X;G) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

When $G = \mathbb{Z}$ we simply write the chains as $C_n(X)$ and the homology as $H_n(X)$.

A pair (X, A) is a space X with a subspace $A \subseteq X$. A map of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subseteq B$. We define the relative chains of (X, A) with G coefficients to be the quotient

$$C_n(X, A; G) = C_n(X;G)/C_n(A;G) = \bigoplus_{\sigma\colon \Delta^n \to X,\ \sigma(\Delta^n) \not\subseteq A} G.$$

This forms a chain complex and we denote its homology by $H_n(X, A; G)$, or just $H_n(X, A)$ if $G = \mathbb{Z}$. When $A = \emptyset$, this coincides with the "absolute" homology groups defined before. When $A = *$, we refer to the homology $H_n(X, *; G)$ as the **reduced homology** with G coefficients.

Homology satisfies the following properties, called the **Eilenberg-Steenrod axioms**.

Proposition 1.3.1 (Homotopy)**.** *$H_n(-,-;G)$ defines a functor from pairs of spaces to abelian groups. Homotopic maps of pairs induce the same map on homology.*

Proposition 1.3.2 (Exactness)**.** *For each pair there is a long exact sequence*

$$\dots \xrightarrow{\partial} H_n(A;G) \xrightarrow{i} H_n(X;G) \xrightarrow{j} H_n(X,A;G) \xrightarrow{\partial} H_{n-1}(A;G) \xrightarrow{i} \dots$$

ending at $H_0(X,A;G)$. The maps i and j are induced by the maps of pairs $(A,\emptyset) \to (X,\emptyset) \to (X,A)$. Every map in the sequence is natural with respect to maps of pairs.

Proposition 1.3.3 (Excision)**.** *If $A \to X$ is a cofibration then the collapse map $(X,A) \to (X/A,*)$ induces an isomorphism on homology.*

Proposition 1.3.4 (Additivity)**.** *If $X = \coprod_\alpha X_\alpha$ is a decomposition into subspaces that are unions of path components, then the map*

$$\bigoplus_\alpha H_k(X_\alpha) \longrightarrow H_k(X)$$

is an isomorphism.

Proposition 1.3.5 (Dimension)**.** *$H_0(*;G) \cong G$, and $H_k(*;G) \cong 0$ for all $k \neq 0$.*

Remark 1.3.6. It follows that every cofiber sequence $A \to X \to Cf$ induces a long exact sequence on reduced homology groups.

Amazingly, the Eilenberg-Steenrod axioms determine $H_*(X,A;G)$ completely, at least on pairs (X,A) where X is a CW complex and A is a subcomplex. For instance, it follows from the exactness axiom that $H_k(S^n,*;G)$ is G when $k=n$ and 0 otherwise.

An **ordinary homology theory** is any functor from CW pairs to sequences of groups $h_n(X,A)$, along with natural snake maps $h_n(X,A) \to h_{n-1}(A)$, satisfying the above axioms. For the dimension axiom, we fix the choice of isomorphism $h_0(*) \cong G$ and ask that any map between two such theories respect this choice.

Theorem 1.3.7. *Any two ordinary homology theories with coefficients in the same group G, are canonically naturally isomorphic.*

See e.g. [May99]. The proof proceeds by showing that each such theory is isomorphic to **cellular homology**. Cellular homology for a CW complex X is constructed by taking an existing ordinary homology theory (such as singular homology), defining a new chain complex by

$$C_n^{CW}(X;G) = H_n(X^{(n)}, X^{(n-1)};G) \cong \bigoplus_{n\text{-cells}} G,$$

with boundary maps given by

$$H_n(X^{(n)}, X^{(n-1)};G) \xrightarrow{\delta} H_{n-1}(X^{(n-1)};G) \xrightarrow{j} H_{n-1}(X^{(n-1)}, X^{(n-2)};G).$$

We then show that the starting homology theory is canonically isomorphic to cellular homology. However, the description of cellular homology turns out to be independent of which homology theory we used to construct it, and so all ordinary homology theories are canonically isomorphic to each other.

If X is any well-based space then the cofiber sequence $X \to CX \to CX/X \simeq \Sigma X$ gives a long exact sequence on homology. Since the homology of $(CX,*)$ vanishes in every degree, we therefore get a natural isomorphism

$$H_{n+1}(\Sigma X,*;G) \cong H_n(X,*;G).$$

For abelian groups A and B, take any presentation of A, i.e. a short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

and define

$$\operatorname{Tor}(A,B) := \ker(P_1 \otimes B \to P_0 \otimes B).$$

This is independent of the choice of presentation and $\operatorname{Tor}(A,B) \cong \operatorname{Tor}(B,A)$. It is zero if either A or B is free, and otherwise describes their “common torsion.”

Theorem 1.3.8 (Universal coefficient theorem for homology)**.** *There is a natural short exact sequence*

$$0 \longrightarrow H_n(X;\mathbb{Z}) \otimes G \longrightarrow H_n(X;G) \longrightarrow \operatorname{Tor}(H_{n-1}(X;\mathbb{Z}),G) \longrightarrow 0$$

and similar sequences with the reduced and relative homology groups. The sequence splits, but not naturally.

Theorem 1.3.9 (Künneth theorem for homology)**.** *For any pair of spaces X and Y there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \xrightarrow{\times} H_n(X \times Y) \longrightarrow \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(X),H_j(Y)) \longrightarrow 0.$$

that splits, but not naturally. For based spaces we also get

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X,*) \otimes H_j(Y,*) \xrightarrow{\times} H_n(X \wedge Y,*) \longrightarrow \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(X,*),H_j(Y,*)) \longrightarrow 0.$$

For field coefficients, there are isomorphisms

$$\bigoplus_{i+j=n} H_i(X;k) \otimes_k H_j(Y;k) \xrightarrow{\times} H_n(X \times Y;k),$$

$$\bigoplus_{i+j=n} H_i(X,*;k) \otimes_k H_j(Y,*;k) \xrightarrow{\times} H_n(X \wedge Y,*;k).$$

The maps labeled $\times$ are called the **cross product** maps on homology. They are easy to define on the cellular chain complex, using the fact that a product of CW complexes is a CW complex. If X has a multiplication $\mu\colon X \times X \to X$ that is associative and unital up to homotopy, we define the Pontryagin product on $H_*(X)$ as the composite

$$H_i(X) \otimes H_j(X) \xrightarrow{\times} H_{i+j}(X \times X) \xrightarrow{\mu_*} H_{i+j}(X)$$

and this makes $H_*(X)$ into a graded ring. For instance, concatenation of loops makes $H_*(\Omega X)$ into a ring.

1.3.2 Cohomology

We define the **cohomology** of X with G coefficients by applying $\mathrm{Hom}(-, G)$ to the chain complex $C_n(X)$ to get

$$C^n(X; G) := \mathrm{Hom}(C_n(X), G).$$

As n varies, this forms a cochain complex, which is just a chain complex whose differential raises degree. Taking kernel mod image gives the cohomology groups $H^n(X; G)$. Doing the same to the chain complexes $C_n(X, A) = C_n(X)/C_n(A)$ gives the relative cohomology groups $H^n(X, A; G)$.

Cohomology satisfies a dual version of the above axioms for homology.

Proposition 1.3.10 (Homotopy). *$H^n(-,-; G)$ defines a contravariant functor from pairs to abelian groups. Homotopic maps induce the same map on cohomology.*

Proposition 1.3.11 (Exactness). *For each pair there is a long exact sequence*

$$\dots \xleftarrow{\partial} H^n(A; G) \xleftarrow{i} H^n(X; G) \xleftarrow{j} H^n(X, A; G) \xleftarrow{\partial} H^{n-1}(A; G) \xleftarrow{i} \dots$$

ending at $H^0(X, A; G)$. The maps i and j are induced by the maps of pairs $(A, \emptyset) \to (X, \emptyset) \to (X, A)$. Every map in the sequence is natural with respect to maps of pairs.

Proposition 1.3.12 (Excision). *If $A \to X$ is a cofibration then the collapse map $(X, A) \to (X/A, *)$ induces an isomorphism on cohomology.*

Proposition 1.3.13 (Additivity). *If $X = \coprod_\alpha X_\alpha$ is a decomposition into subspaces that are unions of path components, then the map*

$$H^k(X) \longrightarrow \prod_\alpha H^k(X_\alpha)$$

is an isomorphism.

Proposition 1.3.14 (Dimension). *$H^0(*; G) \cong G$, and $H^k(*; G) \cong 0$ for all $k \neq 0$.*

Any two such theories on CW complexes are canonically naturally isomorphic, because they are isomorphic to cellular cohomology. We again get, as consequence of homotopy and exactness, a suspension isomorphism

$$H^{n+1}(\Sigma X, *; G) \cong H^n(X, *; G).$$

For abelian groups A and B, take any presentation of A as above

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

and define

$$\mathrm{Ext}(A, B) := \mathrm{Hom}(P_1, B)/\mathrm{im}\,\mathrm{Hom}(P_0, B).$$

Theorem 1.3.15 (Universal coefficient theorem for cohomology)**.** *There is a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \longrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0.$$

and similar sequences with the reduced and relative cohomology groups. The sequence splits, but not naturally.

We say that a space Y is finite type if its chain complex is equivalent to one that is a finitely generated free abelian group at each level. In particular, a CW complex with finitely many cells in each dimension will be of finite type.

Theorem 1.3.16 (Künneth theorem for cohomology)**.** *For any space X and any finite type space Y, there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\times} H^n(X \times Y) \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X), H^j(Y)) \longrightarrow 0.$$

that splits, but not naturally. For based spaces we also get

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X,*) \otimes H^j(Y,*) \xrightarrow{\times} H^n(X \wedge Y,*) \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X,*), H^j(Y,*)) \longrightarrow 0.$$

For field coefficients, there are isomorphisms

$$\bigoplus_{i+j=n} H^i(X;k) \otimes_k H^j(Y;k) \xrightarrow{\times} H^n(X \times Y;k).$$

$$\bigoplus_{i+j=n} H^i(X,*;k) \otimes_k H^j(Y,*;k) \xrightarrow{\times} H^n(X \wedge Y,*;k).$$

Again the maps labeled $\times$ are called the **cross product** on cohomology. They are defined for all X and Y, but the theorem only holds if Y is finite type. For any space X, we define the **cup product** as the composite of the cross product and the diagonal map $\Delta\colon X \to X \times X$:

$$H^i(X) \otimes H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X).$$

It is defined on reduced cohomology similarly. This makes the cohomology of any space $H^*(X)$, and also with ring coefficients $H^*(X;R)$, into a graded ring in a natural way, and the cross product map $H^i(X) \otimes H^j(Y) \to H^n(X \times Y)$ is a ring homomorphism. This is because the diagonal map is "coassociative" and "counital."

1.3.3 Poincaré duality

There is another product of this flavor called the **cap product**:

$$H^i(X) \otimes H_j(X) \xrightarrow{\cap} H_{j-i}(X).$$

It is defined, not at the homology level, but at the chain level, by applying the diagonal map to homology and then pairing off one of the resulting factors with cohomology. The way to remember how $\cap$ interacts with $\cup$ is, $\cup$ makes H^* into a ring, and $\cap$ makes H_* into a module over that ring.

A n-dimensional **manifold** M is a Hausdorff space that can be covered with countably many open sets, each homeomorphic to an open subset of $\mathbb{R}^n$. A manifold is **closed** if it is compact. It also has to have no boundary, but already none of the manifolds here have boundary.[2]

By excision, the homology groups $H_n(M, M-\{x\})$ are all isomorphic to $H_n(D^n, D^n - \{0\}) \cong \mathbb{Z}$. These can be linked together for nearby points, so they form a bundle of abelian groups over M with fiber $\mathbb{Z}$ that we call $\tilde{\mathbb{Z}}$. An **orientation** of M is a continuous choice of generator for each group in this bundle.

Proposition 1.3.17. *Orientations of M are in natural bijection with generators of $H_n(M)$.*

Therefore we represent an orientation by a class $[M] \in H_n(M)$ called the **fundamental class**.

Theorem 1.3.18 (Poincaré duality)**.** *If M is a closed connected n-dimensional manifold, with orientation $[M]$, then cap product with $[M]$ gives an isomorphism*

$$H^i(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-i}(M;\mathbb{Z}).$$

Even when M fails to have an orientation, there is still a mod 2 fundamental class $[M] \in H_n(M;\mathbb{Z}/2)$ and capping with $[M]$ gives isomorphisms

$$H^i(M;\mathbb{Z}/2) \xrightarrow{\cong} H_{n-i}(M;\mathbb{Z}/2).$$

1.4 Homotopy groups and homotopy theory

1.4.1 Homotopy groups and relative homotopy groups

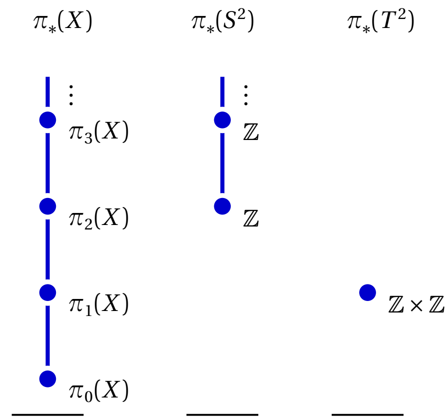
[2] A more general definition would permit M to be locally homeomorphic to $\mathbb{R}^{n-1} \times [0,\infty)$, and the points identified with $\mathbb{R}^{n-1} \times \{0\}$ would be the boundary points.

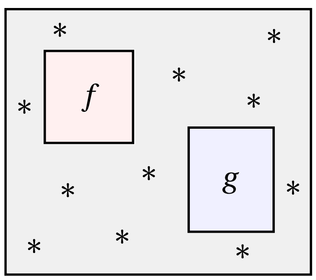
For any based space X, the nth **homotopy group** $\pi_n(X)$ is the set of homotopy classes of based maps $S^n \to X$. Equivalently, homotopy classes of maps of pairs

$$(I^n, \partial I^n) \to (X, *)$$

where $I^n = I \times \ldots \times I$ is the n-dimensional cube and ∂I^n is its boundary.

When $n = 0$, $\pi_0(X)$ is the set of path components. For $n \geq 1$ we can make $\pi_n(X)$ into a group as follows.





$f + g$

Say that an embedding $I^n \to I^n$ is nice if it is an n-fold product of linear embeddings $i(x) = ax + b$ with $a > 0$. Intuitively, it is an embedding that scales and translates each coordinate, but does not otherwise rotate or stretch.

We add two elements $f, g \in \pi_n(X)$ maps by picking any two disjoint nice embeddings $i, j \colon I^n \to I^n$, and defining the map $f + g$ to be $f \circ i^{-1}$ inside the image of i, $g \circ j^{-1}$ in the image of j, and $*$ at all other points. This definition varies continuously as we vary the embeddings.

When $n \geq 2$, all pairs of disjoint embeddings are homotopic, so this gives a well-defined operation on $\pi_n(X)$ that is commutative. It is also associative and unital, making $\pi_n(X)$ into an abelian group. When $n = 1$, the space of pairs of disjoint embeddings is not connected. In other words, when we compose paths, the order matters. Therefore $\pi_1(X)$ is a not-necessarily-abelian group, the **fundamental group** of X.

Proposition 1.4.1. *Any two based maps that are based homotopic induce the same map on π_n. Homotopy equivalences (not necessarily respecting basepoints) induce isomorphisms on π_n.*

Theorem 1.4.2 (van Kampen)**.** *If X is a union of path-connected open sets (or subcomplexes) A_α containing the basepoint, and if the double intersections $A_\alpha \cap A_\beta$ and triple intersections $A_\alpha \cap A_\beta \cap A_\gamma$ are path-connected, then $\pi_1(X)$ is the free group on the groups $\pi_1(A_\alpha)$ modulo the obvious relations coming from each double intersection $A_\alpha \cap A_\beta$.*

Corollary 1.4.3. *For a connected CW complex X, $\pi_1(X)$ is computed by contracting a spanning tree of $X^{(1)}$ to a point, then taking a generator for each remaining 1-cell and a relation for each 2-cell.*

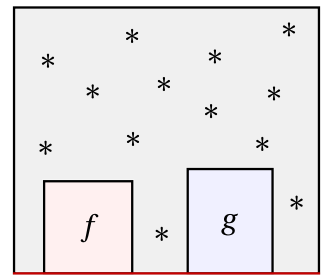
There is no such neat formula for π_n in general, making these groups hard to compute for finite complexes, even for spheres.

For any subspace $A \subseteq X$ containing the basepoint we define $\pi_n(X,A)$ to be homotopy classes of maps of triples $(D^n, S^{n-1}, *) \to (X, A, *)$. Equivalently, maps of triples

$$(I^n, \partial I^n, J^{n-1}) \to (X, A, *)$$

where $J^{n-1} \subset \partial I^n$ is the boundary with the interior of the bottom face removed. The group operation here is parametrized by embeddings of I^{n-1} into I^{n-1}, so it is an abelian group if $n \geq 3$, a group if $n = 2$, a set if $n = 1$, and undefined if $n = 0$. Note that $\pi_n(X,*) \cong \pi_n(X)$.

$f + g$ in $\pi_n(X,A)$



bottom goes to A

More generally, for any based map $f : A \to X$ we define $\pi_n(X,A)$ to be commuting squares of based maps

$$\begin{tikzcd} S^{n-1} \arrow[r, "\subseteq"] \arrow[d] & D^n \arrow[d] \\ A \arrow[r, "f"] & X \end{tikzcd}$$

up to based homotopies of the vertical maps that make the square commute at each time in the homotopy.

Theorem 1.4.4. *There is a natural long exact sequence of sets*

$$\dots \xrightarrow{\partial} \pi_n(A) \xrightarrow{f} \pi_n(X) \longrightarrow \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{f} \dots$$

The maps in this sequence come from the maps of pairs

$$(A,*) \longrightarrow (X,*) \longrightarrow (X,A),$$

and the restriction of a map

$$(D^n, S^{n-1}) \longrightarrow (X,A)$$

to $S^{n-1} \to A$. The maps in the sequence are group homomorphisms up until $\pi_1(A) \to \pi_1(X)$, after which $\pi_1(X,A)$ is a set.

There is an action of $\pi_1(A)$ on every term in this long exact sequence, by "attaching a string to each balloon." One might call this action "conjugation" since the resulting action of $\pi_1(A)$ on itself is in fact the conjugation action

$$g \cdot a := gag^{-1}.$$

Every map of the long exact sequence commutes with $\pi_1(A)$-conjugation, and so is a map of $\mathbb{Z}[\pi_1(A)]$-modules.

Remark 1.4.5. More generally, for maps $B \to A \to X$ there is a natural long exact sequence of sets

$$\dots \xrightarrow{\partial} \pi_n(A,B) \xrightarrow{f} \pi_n(X,B) \longrightarrow \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A,B) \xrightarrow{f} \dots$$

Theorem 1.4.6. *If $E \to B$ is a Serre fibration with fiber F, there is natural long exact sequence of sets*

$$\dots \xrightarrow{\partial} \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \xrightarrow{f} \dots$$

The connecting homomorphism is through the identification $\pi_n(E,F) \cong \pi_n(B)$ induced by $(E,F) \to (B,*)$.

When $E \to B$ is a covering space, F is discrete and this sequence becomes

$$\pi_n(E) \xrightarrow{\cong} \pi_n(B) \qquad \forall n \geq 2, \qquad\qquad 0 \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \longrightarrow F \longrightarrow *.$$

Setting $G = \pi_1(E)$ and H to be the image of $\pi_1(B)$ in G, we therefore get that the set G/H is isomorphic to the fiber F of the covering space. When H is normal, this is a group, and is isomorphic to the group of deck transformations of the covering space. (A deck transformation is a homeomorphism $E \cong E$ commuting with the map to B.)

A **universal cover** of a connected CW complex X is a simply-connected covering space $p: \widetilde{X} \to X$. These exist and are unique up to isomorphism over X. By the previous results, $\pi_1(X)$ can be identified with both the fiber $p^{-1}(x_0)$ and with the deck group of $\widetilde{X}$. Note that along these identifications the fiber is $G = \pi_1(X)$, the deck group is multipication on one side, and the path-lifting action is multiplication on the *other* side.

1.4.2 Cells and connectivity

A map of CW complexes $f: X \to Y$ is **cellular** if $f(X^{(n)}) \subseteq Y^{(n)}$ for all n.

Proposition 1.4.7 (Cellular approximation)**.** *Every map of CW complexes is homotopic to a cellular map.*

Corollary 1.4.8. $\pi_i(S^n) = 0$ *for $i < n$. More generally, attaching an n-cell does not affect π_i for $i < (n-1)$.*

A map $A \to X$ is n**-connected** if it is surjective on π_n, and an isomorphism on π_i for $i < n$. Equivalently, it is surjective on π_0 and the relative groups $\pi_i(X,A)$ vanish for $i \leq n$.

A space is n-connected if the map $* \to X$ is; equivalently, $\pi_i(X)$ vanishes for $i \leq n$. A 0-connected space is path-connected, and a 1-connected space is simply-connected.

A **weak homotopy equivalence**, or just weak equivalence, is a map $A \to X$ inducing isomorphisms on π_n for all n. Equivalently, it is n-connected for all n. Two spaces are weakly equivalent if they are connected by a zig-zag of weak equivalences. Every homotopy equivalence is a weak equivalence, but not vice-versa.

Proposition 1.4.9 (Cell attachment)**.** *For any map $A \to X$ and any element*

$$\alpha \in \pi_n(X, A),$$

one can attach an n-cell to A to get a factorization $A \to A' \to X$, such that

$$\pi_i(X, A) \to \pi_i(X, A')$$

is an isomorphism for $i < n$, and surjective for $i = n$, with kernel containing α.

Informally, this means that attaching an n-cell to A makes

$\pi_n(A) \to \pi_n(X)$	more surjective,
$\pi_{n-1}(A) \to \pi_{n-1}(X)$	more injective, and
$\pi_k(A) \to \pi_k(X)$	unchanged for $k < n-1$.

See also Remark 2.6.21.

Corollary 1.4.10. *$A \to X$ is n-connected iff it is weakly equivalent to a relative CW complex $A \to X'$ in which every cell has dimension $\geq (n+1)$.*

Corollary 1.4.11 (Cofibrant replacement)**.** *Every map $A \to X$ factors into a relative CW complex $A \to B$ and a weak equivalence $B \to X$.*

This replacement can be done in a functorial way, so each commuting square as shown on the left gives a commuting diagram as on the right.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ A' \arrow[r] & X' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dashed] & X \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & X' \end{tikzcd}$$

In particular, there is a functor $Q\colon \mathbf{Top} \to \mathbf{Top}$ taking each space X to a weakly equivalent CW complex $QX \xrightarrow{\sim} X$.

Theorem 1.4.12 (Whitehead)**.** *Every weak equivalence between CW complexes is a homotopy equivalence. If $A \subseteq X$ is the inclusion of a weakly equivalent subcomplex, then X deformation retracts onto A.*

Taken together, these results tell us that studying all spaces up to weak equivalence is essentially the same as studying CW complexes up to homotopy equivalence.

Cell attachment is also used to build the **Postnikov tower**, a functorial sequence of spaces under each path-connected space X

$$X \longrightarrow \cdots \longrightarrow P_n X \longrightarrow P_{n-1} X \longrightarrow \cdots \longrightarrow P_2 X \longrightarrow P_1 X \longrightarrow P_0 X = *$$

such that $X \to P_n X$ is an isomorphism on π_i for $i \leq n$, and the remaining homotopy groups of $P_n X$ vanish.

Finally, we relate homotopy groups back to homology and cohomology groups. There is a natural **Hurewicz homomorphism** $\pi_n(X) \to H_n(X)$ which takes each element $S^n \to X$ to the image of the generator of $H_n(S^n)$ in $H_n(X)$. We also get a similar map for pairs $\pi_n(X, A) \to H_n(X, A)$, which is a homomorphism when $n \geq 2$.

Proposition 1.4.13. *Every weak equivalence induces an isomorphism on singular homology and cohomology.*

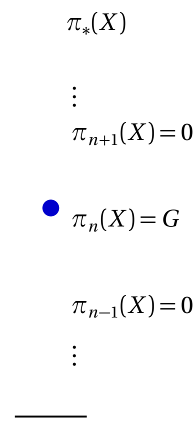
Proposition 1.4.14 (Hurewicz)**.** *The Hurewicz map induces*

- *an isomorphism $\pi_1(X)^{\mathrm{ab}} \cong H_1(X)$ if X is path-connected,*
- *an isomorphism $\pi_n(X) \cong H_n(X)$ if X is $(n-1)$-connected, and*
- *an isomorphism $\pi_n(X, A) \cong H_n(X, A)$ if (X, A) is $(n-1)$-connected, $n \geq 2$, and A is 1-connected.*

As a result, for simply-connected spaces, the lowest nonzero homotopy group of a space or map is controlled by the lowest nonzero homology group.

Corollary 1.4.15. *A map of simply-connected CW complexes $X \to Y$ is a homotopy equivalence iff $H_*(X) \to H_*(Y)$ is an isomorphism. A map of connected CW complexes $X \to Y$ is a homotopy equivalence iff $\pi_1(X) \to \pi_1(Y)$ and $H_*(\widetilde{X}) \to H_*(\widetilde{Y})$ are both isomorphisms.*

1.4.3 Eilenberg-Maclane spaces and the Yoneda lemma



Definition 1.4.16. An **Eilenberg-Maclane space** $K(G,n)$ is a CW complex with only one nonzero homotopy group, $\pi_n = G$. We fix the identification between $\pi_n(K(G,n))$ and G.

Lemma 1.4.17. *Eilenberg-Maclane spaces are unique up to homotopy equivalence. Taking homotopy groups gives a bijection between based homotopy classes of maps $[K(G,n),K(H,n)]_*$ and group homomorphisms $G \to H$.*

Corollary 1.4.18. *We have canonical weak equivalences*

$$K(G,n) \simeq \Omega K(G,n+1).$$

Theorem 1.4.19 (Brown representability)**.** *There is a natural isomorphism*

$$H^n(X,A;G) \cong [X/A, K(G,n)]_*$$

for each CW complex X and subcomplex A, each abelian group G, and each $n \geq 0$. (See Theorem 2.5.24 for a proof.)

Moreover, we can reconstruct the long exact sequence for the cohomology of a cofiber sequence $A \to X \to Cf$ from Proposition 1.2.9 and the isomorphisms

$$[\Sigma X, K(G,n)]_* \cong [X, \Omega K(G,n)]_* \cong [X, K(G,n-1)]_*.$$

Brown representability is powerful when combined with the Yoneda Lemma. For any category $\mathbf{C}$ and object $X \in \operatorname{ob}\mathbf{C}$, let $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be the functor represented by X:

$$F(Y) = \mathbf{C}(Y,X).$$

Let $G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be any other functor to sets.

Lemma 1.4.20 (Yoneda)**.** *Natural transformations $\eta\colon F \Rightarrow G$ correspond bijectively to elements of the set $G(X)$. The bijection takes each η to the element $\eta(\mathrm{id}_X)$.*

Corollary 1.4.21. *If F is represented by X and G is represented by Y, natural transformations $\eta\colon F \Rightarrow G$ correspond to morphisms $X \to Y$ in $\mathbf{C}$. Natural isomorphisms $F \cong G$ correspond to isomorphisms $X \cong Y$ in $\mathbf{C}$.*

We apply this to the category $h\mathbf{CW}$ of CW complexes and homotopy classes of maps. Cohomology $H^n(-;G)$ is the functor $h\mathbf{CW}_* \to \mathbf{Ab}$ represented by $K(G,n)$:

$$H^n(X;G) \cong H^n(X_+,*;G) \cong [X_+, K(G,n)]_* \cong [X, K(G,n)].$$

Therefore, any natural transformation from cohomology to cohomology, must arise from a map of Eilenberg-Maclane spaces in the homotopy category $h\mathbf{CW}$. For instance, the cup product

$$H^m(-;R)\times H^n(-;R) \xrightarrow{\cup} H^{m+n}(-;R)$$

is represented by a map of spaces

$$K(R,m)\times K(R,n) \longrightarrow K(R,m+n).$$

1.5 Basic homotopy colimits and limits

We get a long exact sequence on homology for every short exact sequence of spaces

$$A \longrightarrow X \longrightarrow X/A,$$

provided the quotient X/A is nice. What does "nice" mean? In Section 1.2, we phrased it by saying that $f\colon A\to X$ is a cofibration. This implies that X/A is homotopy equivalent to the homotopy cofiber $Cf = CA\cup_A X$.

The same phenomenon happens for other kinds of colimits. There is a "correct" homotopy type for the colimit, that allows us to compute its homology groups from the homology groups of the pieces $X(i)$. But the colimit doesn't always have this correct homotopy type.

It is possible to "fix" the colimit by thickening it up so that it always has the correct homotopy type. The resulting space is called the **homotopy colimit**. The ordinary colimit is "correct" if it is equivalent to the homotopy colimit.

To be more specific, recall from Section 1.1 that the colimit of a diagram $X\colon \mathbf{I}\to\mathbf{Top}$ is constructed as the disjoint union $\coprod_{i\in\mathrm{ob}\,\mathbf{I}} X(i)$, modulo an identification $x\sim f(x)$ for each of the maps f in the diagram. In the homotopy colimit, this construction is thickened up by creating a *path* from x to $f(x)$, instead of gluing them directly together. If the diagram is sufficiently complicated, we have to create additional paths between these paths.

In this section, we restrict our attention to the three basic homotopy colimits:

- coproducts,
- homotopy pushouts, and
- homotopy sequential colimits.

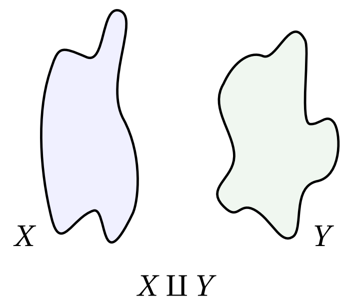
These three cases are especially important, in part because all other homotopy colimits can be built out of them. The reader interested in the case of a general homotopy colimit may skip ahead to Definition 5.3.20 or **??**. The expository paper [Dug08] is also an excellent reference.

1.5.1 Homotopy colimits of unbased spaces

As mentioned above, there are three basic homotopy colimits.

The first is the coproduct, or disjoint union, $\coprod_\alpha X_\alpha$. This is the colimit of a diagram of spaces $\{X_\alpha\}$ that has no nontrivial morphisms. It is also the homotopy colimit of $\{X_\alpha\}$ – since there are no additional morphisms, there are no homotopies to add.

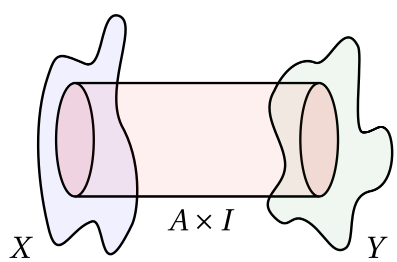
So in this case, the homotopy colimit and the actual colimit are the same thing. In other words, coproducts are always correct.



The next example is the pushout.

Definition 1.5.1. The **homotopy pushout** of X and Y along A is defined as the double mapping cylinder

$$X \cup_A^h Y := X \cup_{(A\times\{0\})} (A \times I) \cup_{(A\times\{1\})} Y.$$



Collapsing away the cylinder gives a map to the ordinary pushout $X \cup_A Y$.

Lemma 1.5.2. *This collapse map is a weak equivalence if*

- *X and Y are open subsets of a topological space and A is their intersection, or*
- *X and Y are CW complexes sharing a common subcomplex A, or more generally*
- *either $A \to X$ or $A \to Y$ is a cofibration.*

So the ordinary pushout is correct under any of these assumptions.

Example 1.5.3. The unbased homotopy cofiber $C^u f$ and unreduced suspension SX from Section 1.2 can be expressed as homotopy pushouts

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ * \arrow[r] & C^u f \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & SX. \end{tikzcd}$$

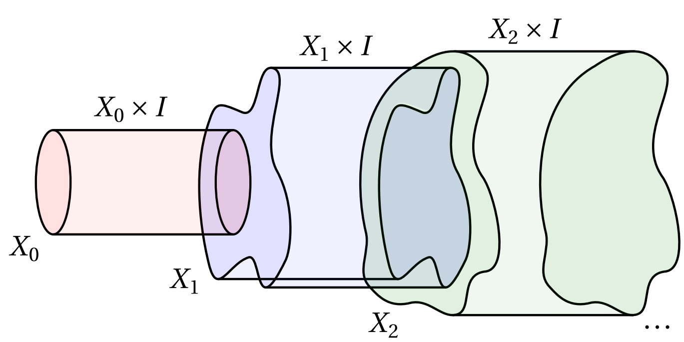
The last basic colimit is the colimit of a sequence of maps

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \dots$$

We take the homotopy colimit in this case by forming successive mapping cylinders.

Definition 1.5.4. The **mapping telescope** is formed by taking a cylinder on each space X_n and gluing the ends together along the maps f_n:

$$\operatorname*{hocolim}_{n\to\infty} X_n := \coprod_{n\geq 0} (X_n \times [0,1]) \Big/ \Big((x_n, 1) \sim (f_{n+1}(x_n), 0)\Big)$$



Collapsing down the telecope gives a map to the ordinary colimit, $\operatorname*{colim}_{n\to\infty} X_n$.

Lemma 1.5.5. *This collapse map is a weak equivalence if*

- *the X_n are all open subsets of a topological space,*
- *the X_n are all subcomplexes of a CW complex, or more generally*
- *the maps $f_n \colon X_n \to X_{n+1}$ are closed inclusions.*[3]

What properties does the "correct" homotopy type have? We can give more detail in each of these three examples:

[3]This relies on the assumption that we are in CGWH spaces. Without this, we'd have to make the stronger assumption that the maps f_n are cofibrations.

- For coproducts $\coprod_\alpha X_\alpha$, the correct homotopy type gives a direct sum on homology (Proposition 1.3.4), and on homotopy gives the homotopy groups of each of the spaces X_α, depending on where we pick our basepoint.

  On cohomology, and more generally on homotopy classes of maps $[-, Y]$, we get a product:

$$\left[\coprod_\alpha X_\alpha, Y\right] \cong \prod_\alpha [X, Y].$$

- For pushouts $X \cup_A Y$, the correct homotopy type gives the Mayer-Vietoris exact sequence on homology

$$\dots \xrightarrow{\partial} H_n(A) \xrightarrow{i} H_n(X) \oplus H_n(Y) \xrightarrow{j} H_n(X \cup_A^h Y) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i} \dots$$

  where the map i is induced by the given maps $A \to X$ and $A \to Y$, and j is similar except that one of the two maps is negated. On cohomology we get the same sequence with maps reversed, while on $[-, Z]$ we get the weaker statement that the map to the fiber product

$$[X \cup_A^h Y, Z] \twoheadrightarrow [X, Z] \times_{[A,Z]} [Y, Z]$$

  is surjective. On the fundamental group π_1, the van Kampen theorem applies. In the case where A, X, and Y are connected, this gives $\pi_1(X \cup_A^h Y)$ as the free product $\pi_1(X) * \pi_1(Y)$ modulo relations from $\pi_1(A)$. On π_n, we get the homotopy excision or Blakers-Massey theorem, which says that the map

$$\pi_n(X, A) \to \pi_n(X \cup_A^h Y, Y)$$

  is an isomorphism in a range, depending on the connectivity of the maps $A \to X$ and $A \to Y$.

- For sequential colimits, the correct homotopy type is the one in which the colimit passes to the homology and homotopy groups:

$$H_k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} H_k(X_n), \qquad \pi_k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \pi_k(X_n).$$

  More generally, the colimit commutes with $[Y, -]$ if Y is finite CW, see Exercise 23. On the other hand, for cohomology we get a $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1 H^{k-1}(X_n) \longrightarrow H^k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty} H^k(X_n) \longrightarrow 0. \tag{1.5.6}$$

Recall that $\lim^1$ is the first derived functor of the limit:

Definition 1.5.7. For any inverse system of abelian groups

$$\dots \longrightarrow A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

we define the map of products

$$\prod_n A_n \xrightarrow{1-f} \prod_n A_n$$

by sending the infinite tuple $(a_n)_{n\geq 0}$ to $(a_n - f_{n+1}(a_{n+1}))_{n\geq 0}$. Essentially, it is the identity map, minus the map that applies f_n to each A_n. The limit of the system is the kernel of $(1-f)$. We define $\lim^1$ to be its cokernel:

$$0 \longrightarrow \lim_{n\to\infty} A_n \longrightarrow \prod_n A_n \xrightarrow{1-f} \prod_n A_n \longrightarrow \lim_{n\to\infty}{}^1 A_n \longrightarrow 0.$$

It is well known that $\lim_{n\to\infty}{}^1 A_n$ vanishes:

- when the maps f_n are isomorphisms for all sufficiently large n, or more generally
- when the maps f_n are surjective for all sufficiently large n, or more generally
- when for each m, the image of the map $A_n \to A_m$ becomes constant for all sufficiently large n (the *Mittag-Leffler condition*).

Remark 1.5.8. If the term "correct homotopy type" seems irritatingly vague, you can make it more precise by saying that we seek the *left-derived functor* of the colimit. See Definition 3.3.15 and Definition 3.4.10.

Remark 1.5.9. Recall that the homotopy category of spaces $h\mathbf{Top}$ is the one in which the morphisms are homotopy classes of maps $[X, Y]$. This category does *not* have pushouts or sequential colimits. As a result, the homotopy colimit is *not* the colimit of the diagram in the homotopy category.

The following is a fundamental theorem about homotopy colimits. For the three basic homotopy colimits, it can be deduced from our description of how the homotopy colimit interacts with homology and homotopy groups.

Theorem 1.5.10. *Any map of diagrams $X \to Y$ that gives a weak equivalence*

$$X(i) \xrightarrow{\sim} Y(i)$$

for each i, gives a weak equivalence of homotopy colimits

$$\operatorname{hocolim}_i X(i) \xrightarrow{\sim} \operatorname{hocolim}_i Y(i).$$

Corollary 1.5.11. *The smash product of spaces* $(X, Y) \mapsto X \wedge Y$ *preserves weak homotopy equivalences, if* X *and* Y *are well-based topological spaces.*

In other words, if $X \to X'$ and $Y \to Y'$ are weak equivalences of well-based spaces, then $X \wedge Y \to X' \wedge Y'$ is a weak equivalence.

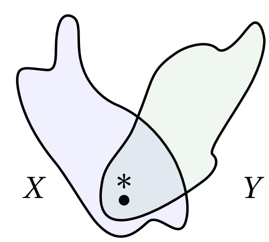
Proof. By Remark 1.2.6, since X and Y are well-based, the smash product is equivalent to the homotopy cofiber of $X \vee Y \to X \times Y$. By Theorem 1.5.10, this homotopy cofiber preserves equivalences. □

1.5.2 Homotopy colimits of based spaces

Suppose now that we have a diagram X in which the spaces $X(i)$ are based, and the maps preserve the basepoints.

Definition 1.5.12. The **based homotopy colimit** is defined by first making the spaces $X(i)$ well-based, in other words, making $* \to X(i)$ is a cofibration. Then we take the quotient of the unbased homotopy colimit by the subspace of all the basepoints and the paths between them:

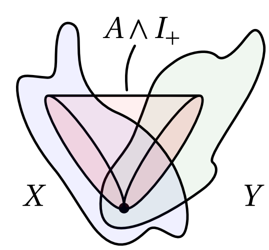
$$\operatorname{hocolim}^{(b)} X(i) = \left(\operatorname{hocolim}^{(u)} X(i)\right) \Big/ \left(\operatorname{hocolim}^{(u)} *\right).$$



If the spaces $X(i)$ are not well-based, then the above construction is not the homotopy colimit, because it may have the wrong homotopy type.

Theorem 1.5.10 applies to based homotopy colimits as well – they preserve all weak equivalences of diagrams.

In particular, the based homotopy coproduct is the wedge sum $\bigvee_\alpha X_\alpha$, assuming X_α is well-based. The special case of two spaces $X \vee Y$ is is illustrated to the right.



The based homotopy pushout is

$$X \cup_{(A\times\{0\})} [(A \times I)/(\{*\} \times I)] \cup_{(A\times\{1\})} Y, \tag{1.5.13}$$

assuming A, X, and Y are well-based. This is also illustrated to the right – it is the double mapping cylinder with the subspace $\{*\} \times I$ in the cylinder collapsed to a point.

Similarly, the based mapping telescope is

$$\coprod_{n\geq 0} [(X_n \times [0,1])/(\{*\} \times [0,1])] \Big/ \big((x_n, 1) \sim (f_{n+1}(x_n), 0)\big). \tag{1.5.14}$$

Lemma 1.5.15.
- *If A, X, and Y are well-based then the collapse map from the unbased homotopy pushout of Definition 1.5.1 to the based homotopy pushout in* (1.5.13) *is a homotopy equivalence.*
- *The collapse map from the unbased mapping telescope of Definition 1.5.4 to the based mapping telescope in* (1.5.14) *is always a weak equivalence.*[4] *(Exercise 24)*

Example 1.5.16. The homotopy cofiber Cf and reduced suspension ΣX from Section 1.2 are the based homotopy pushouts of the same diagrams from Example 1.5.3:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ * \arrow[r] & Cf \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X. \end{tikzcd}$$

Remark 1.5.17. It is an unfortunate fact that we use the same notation to denote both the based and unbased version of homotopy colimits. We sometimes use decorations (u) to distinguish the unbased ones, as we did with $C^{(u)}f$, but for the most part we follow historical convention and use the same notation for both. We are at least partly justified in doing this by Lemma 1.5.15.

1.5.3 Homotopy limits

The dual of a homotopy colimit is a **homotopy limit**. Since the universal property of a limit is stated using maps *into* $X(i)$, we add homotopies to this limit by using path spaces $X(i)^I = \mathrm{Map}(I, X(i))$ instead of cylinders $X(i) \times I$. As with colimits, we have three basic homotopy limits:

- products,
- homotopy pullbacks, and
- homotopy sequential limits.

The first of these is the product $\prod_\alpha X_\alpha$. As before, this is already a homotopy limit, so products are always correct.[5]

The next case is the pullback. Suppose we have a pullback diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & B. \end{tikzcd}$$

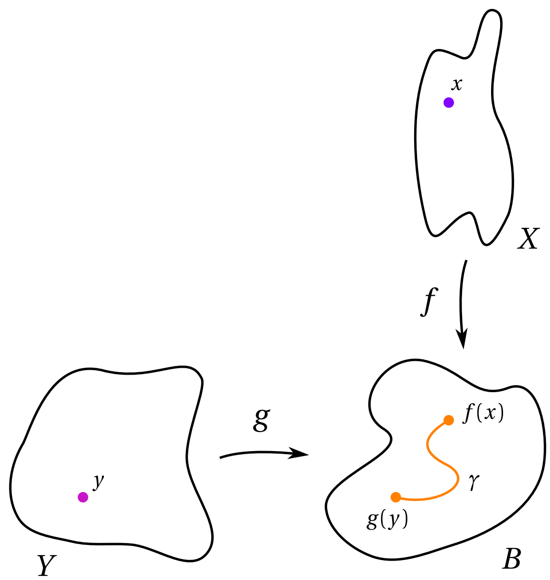
[4] This is isn't true if we move away from (CGWH) spaces. If phrased for general spaces, we also need the X_n to be well-based.

[5] It is a bit of a surprise that in spectra, infinite products are not always correct – the factors have to be made into Ω-spectra first.

Definition 1.5.18. The **homotopy pullback** is the homotopy fiber product

$$\begin{aligned} X \times_B^h Y &= X \times_B B^I \times_B Y \\ &= \{\, x \in X,\ y \in Y, \gamma \colon I \to B \ :\ f(x) = \gamma(0),\ g(y) = \gamma(1) \,\}. \end{aligned}$$

Here the two maps $B^I \rightrightarrows B$ are the evaluation at the two endpoints of I.



The inclusion of constant paths $B \to B^I$ gives a map from the ordinary fiber product $X \times_B Y$ to homotopy pullback $X \times_B^h Y$.

Lemma 1.5.19. *The inclusion $X \times_B Y \to X \times_B^h Y$ is a weak equivalence if $X \to B$ or $Y \to B$ is a Serre fibration.*

Example 1.5.20. The homotopy fiber Fp and based loopspace ΩX from Section 1.2 can be expressed as homotopy pullbacks

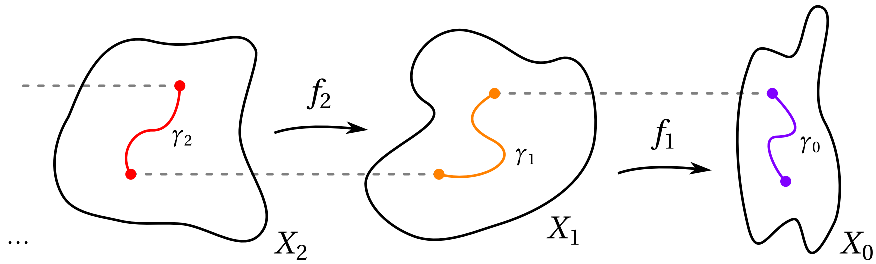
$$\begin{tikzcd} Fp \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X. \end{tikzcd}$$

Definition 1.5.21. We take the **sequential homotopy limit** of a sequence of maps

$$\ldots \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

as the infinite homotopy fiber product

$$\begin{aligned} \operatorname*{holim}_{n\to\infty} X_n &= \ldots \times_{X_2} X_2^I \times_{X_1} X_1^I \times_{X_0} X_0^I \\ &= \{\, \gamma_n \colon I \to X_n \quad \forall n \geq 0 \ :\ f_n(\gamma_n(1)) = \gamma_{n-1}(0) \,\}. \end{aligned}$$



The ordinary limit includes into this homotopy limit, as the subspace in which all of the paths are constant.

Lemma 1.5.22. *This inclusion map is a weak equivalence if all of the maps f_n are Serre fibrations.*

In each of these cases, the homotopy limit captures the correct homotopy type:

- For products, the correct homotopy type is the one that gives a product on homotopy groups, and more generally on $[Y,-]$:

$$\left[Y, \prod_\alpha X_\alpha\right] \cong \prod_\alpha [Y,X].$$

  On homology, we get the Künneth formula (Theorem 1.3.9).

- For pullbacks, the correct homotopy type is the one in which we get the homotopy Mayer-Vietoris sequence

$$\dots \longrightarrow \pi_n(X \times^h_B Y) \longrightarrow \pi_n(X) \oplus \pi_n(Y) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(X \times^h_B Y) \longrightarrow \dots ,$$

  see exercise 18. More generally, we get a surjective map

$$[Z, X \times^h_B Y] \twoheadrightarrow [Z,X] \times_{[Z,B]} [Z,Y]$$

  and similarly with based homotopy classes of maps. On homology and cohomology, we get the Eilenberg-Moore spectral sequence, and when $Y = *$ we get the Serre spectral sequence (see **??**).

- For sequential homotopy limits, the correct homotopy type is the one in which we get a $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1 \pi_{k+1}(X_n) \longrightarrow \pi_k\left(\operatorname*{holim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty} \pi_k(X_n) \longrightarrow 0. \qquad (1.5.23)$$

  See Definition 1.5.7 for a recollection of $\lim^1$.

The homology of the homotopy limit has no relation to the homology of the spaces X_n in general. The most we can say is that if the maps $f_n \colon X_n \to X_{n-1}$ have connectivity increasing to ∞, then the sequential limit commutes with homology and cohomology. So,

$$H_k\left(\operatorname*{holim}_{n\to\infty} X_n\right) \cong \lim_{n\to\infty} H_k(X_n) \quad \text{if the connectivity of } f_n \text{ goes to } \infty.$$

The dual of Theorem 1.5.10 is the following.

Theorem 1.5.24. *Any map of diagrams $X \to Y$ that gives a weak equivalence*

$$X(i) \xrightarrow{\sim} Y(i)$$

for each i, gives a weak equivalence of homotopy limits

$$\operatorname*{holim}_i X(i) \xrightarrow{\sim} \operatorname*{holim}_i Y(i).$$

Remark 1.5.25. There is no based version of homotopy limits. Or rather, there is, but it is always the same as the unbased version, so there is no need to distinguish them, as for homotopy colimits.

1.6 Adjunctions

Recall from Section 1.1 that there is a bijection between continuous maps

$$K \times X \to Y \quad \text{and} \quad K \to \mathrm{Map}(X, Y).$$

This is an example of an "adjunction" from category theory.

Definition 1.6.1. An **adjunction** between categories **C** and **D** is a pair of functors

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "L"] & \mathbf{D}, \arrow[l, bend left, "R"] \end{tikzcd}$$

and for each object X in **C** and Y in **D**, a bijection between the sets of morphisms

$$\{f\colon LX \to Y\} \longleftrightarrow \{\widetilde{f}\colon X \to RY\},$$

or in other words $\mathbf{D}(LX, Y) \cong \mathbf{C}(X, RY)$, that are natural in X and Y. The functor L is called the **left adjoint** and R is called the **right adjoint**. The maps f and $\widetilde{f}$ are **adjunct**. We often say that $(L \dashv R)$ or (L, R) is an **adjoint pair**.

Adjunctions are a simple manipulation: whenever you want, you can pull the L off the source of a map and stick an R to the target, without gaining or losing information.

$$\mathbf{L}X \to Y$$
$$X \to \mathbf{R}Y$$

The naturality means that if we have another map $g \colon X' \to X$, then the maps

$$LX' \xrightarrow{L(g)} LX \xrightarrow{f} Y$$
$$X' \xrightarrow{g} X \xrightarrow{\tilde{f}} RY$$

are adjunct – in other words $(\widetilde{f \circ L(g)}) = (\tilde{f}) \circ g$. We also have a similar condition for composing with maps $h \colon Y \to Y'$.

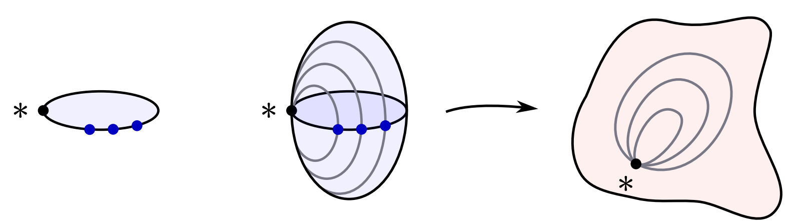
Remark 1.6.2. You can remember the naturality statement by thinking of the adjunction as giving you a natural notion of "a map across categories" from $X \in \mathbf{C}$ to $Y \in \mathbf{D}$. This notion is captured either by a map $LX \to Y$ or by $X \to RY$. They mean the same thing. The "action" of an arrow in $\mathbf{C}$ on this map has two possible interpretations, but those two interpretations produce the same "map across categories." That is what the naturality statement really says.

Example 1.6.3. In unbased spaces **Top**, we have an adjoint pair in which $K \times -$ is the left adjoint and $\mathrm{Map}(K, -)$ is the right adjoint.

In based spaces $\mathbf{Top}_*$, we have an adjoint pair in which $K \wedge -$ is the left adjoint and $\mathrm{Map}_*(K, -)$ is the right adjoint.

Corollary 1.6.4. *Suspension and loops are adjoint: based maps $\Sigma X \to Y$ are in canonical correspondence with based maps $X \to \Omega Y$.*

This means any time we have a map $\Sigma X \to Y$, we can immediately substitute for it a map $X \to \Omega Y$. This particular case can be visualized explicitly: each point in $x \in X$ gives a loop through ΣX, that traverses the segment $I \times \{x\}$ inside $I \times X$. Giving a based map $\Sigma X \to Y$ is the same thing as sending each of these loops to a based loop in Y, in a continuous way. That's the same thing as a based map $X \to \Omega Y$.



X ΣX Y

Corollary 1.6.5. *For all based spaces B,*

$$\pi_n(B) \cong \pi_{n-1}(\Omega B) \cong \cdots \cong \pi_0(\Omega^n B).$$

This could also be deduced from Proposition 1.2.14 or Theorem 1.4.6 by taking the map $* \to B$ and making it into a fibration – the homotopy fiber is then ΩB.

Example 1.6.6. The forgetful functor from based spaces to unbased spaces

$$\mathbf{Top}_* \to \mathbf{Top}$$

is a right adjoint. Its left adjoint is the functor that adds a disjoint basepoint, $(-)_+$. In other words, based maps $X_+ \to Y$ correspond to unbased maps $X \to Y$.

Example 1.6.7. The forgetful functor from unbased spaces to sets

$$\mathbf{Top} \to \mathbf{Set}$$

is a right adjoint. Its left adjoint is the functor that gives each set S the discrete topology. If X is a space, then continuous maps $S \to X$ with this topology correspond to maps of sets $S \to X$.

Example 1.6.8. The forgetful functor from abelian groups to sets

$$\mathbf{Ab} \to \mathbf{Set}$$

is a right adjoint. Its left adjoint is the free abelian group functor, taking each set S to the direct sum $\bigoplus_{s \in S} \mathbb{Z}$. Maps of abelian groups

$$\bigoplus_{s \in S} \mathbb{Z} \to A$$

correspond to maps of sets $S \to A$.

Theorem 1.6.9. *Left adjoints preserve colimits. Given any left adjoint $L\colon \mathbf{C} \to \mathbf{D}$ and diagram $X\colon \mathbf{I} \to \mathbf{C}$, we have a canonical isomorphism*

$$\operatorname{colim} L(X) \cong L(\operatorname{colim} X).$$

Similarly, right adjoints preserve limits:

$$\lim R(X) \cong R(\lim X).$$

Example 1.6.10. This theorem explains why limits are the same in unbased spaces, based spaces, sets, and abelian groups, but colimits are different. It also explains why in spaces, the based colimit and unbased colimit don't agree, but they do commute with the functor $(-)_+$ that adds a disjoint basepoint.

Example 1.6.11. Taking Cartesian product $K \times -$ commutes with colimits of unbased spaces:

$$K \times \operatorname{colim}^{(u)} X(i) \cong \operatorname{colim}^{(u)} (K \times X(i)).$$

Taking smash product $K \wedge -$ commutes with colimits of based spaces:

$$K \wedge \operatorname{colim}^{(b)} X(i) \cong \operatorname{colim}^{(b)} (K \wedge X(i)).$$

Theorem 1.6.12. *Left adjoints are unique. If $L_1, L_2 \colon \mathbf{C} \to \mathbf{D}$ are two functors that are both left adjoint to the same functor $R \colon \mathbf{D} \to \mathbf{C}$, then there is a unique natural isomorphism $L_1 \cong L_2$ that agrees with the identifications*

$$\mathbf{D}(L_1X, Y) \cong \mathbf{C}(X, RY) \cong \mathbf{D}(L_2X, Y).$$

The dual statement for right adjoints also holds.

This is an incredibly useful result because it allows us to *define* something as a left adjoint, and know that the definition is unique. See exercise 27 for a proof.

The **unit** of an adjunction $(L \dashv R)$ is the map

$$\eta \colon X \to RLX$$

adjunct to the identity map $LX \to LX$. The **counit** is the map

$$\epsilon \colon LRY \to Y$$

adjunct to the identity $RY \to RY$. These are natural transformations, satisfying the "triangle identities"

$$\begin{tikzcd} LX \arrow[r, "L\eta"] \arrow[dr, equal] & LRLX \arrow[d, "\epsilon L"] \\ & LX, \end{tikzcd} \qquad \begin{tikzcd} RY \arrow[r, "\eta R"] \arrow[dr, equal] & RLRY \arrow[d, "R\epsilon"] \\ & RY. \end{tikzcd}$$

If you know the unit and counit, you can reconstruct the adjunction: the map $f : LX \to Y$ is adjunct to the composite

$$X \xrightarrow{\eta} RLX \xrightarrow{Rf} RY,$$

and $\tilde{f} \colon X \to RY$ is adjunct to the composite

$$LX \xrightarrow{L\tilde{f}} LRY \xrightarrow{\epsilon} Y.$$

Going back and forth gives the same map back! See [Rie17, Ch. 4] for an extensive discussion.

1.7 Exercises

1. Verify that the formulas (1.1.4) and (1.1.3) for the colimit and limit of a diagram of spaces, have the required universal properties.

2. (a) Show that the limit of a diagram of the form

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots$$

   is isomorphic to the first term X_0. Therefore the limit is not very interesting, only the colimit is interesting.

   (b) Explain in a similar way the trivial colimits and limits that appear in Figure 1.1.2.

   (c) How would you generalize these examples? (Recall that an object a in a category $\mathbf{I}$ is **initial** if a admits a unique map to each object $b \in \mathbf{I}$, and a is **terminal** if it admits a unique map *from* each object $b \in \mathbf{I}$.)

3. Suppose G is a group and X is a space with a continuous left action by G. The space of orbits X_G is the quotient of X by the relation $x \sim gx$ for every $x \in X$ and $g \in G$. Explain how the orbit space is an example of a colimit. What is the corresponding limit?

4. Suppose $X: \mathbf{I} \to \mathbf{Top}_*$ is a diagram of based spaces and basepoint-preserving maps between them. By forgetting the basepoints, it also defines a diagram of unbased spaces $X: \mathbf{I} \to \mathbf{Top}$.

   (a) Explain why the colimit of X could change when the universal property is phrased in the category of based spaces $\mathbf{Top}_*$, instead of the category of unbased spaces $\mathbf{Top}$.

   (b) Prove that it does change in the case of a coproduct: the based and unbased coproducts are not homeomorphic to each other.

   (c) Prove that the unbased colimit and based colimit agree if $\mathbf{I}$ is **connected**, meaning every pair of objects is related by a finite zig-zag of morphisms

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \dots \longleftarrow \bullet.$$

   In particular, the definition of a pushout and of a sequential colimit has the same formula in based spaces that it had in unbased spaces.

   (d) Prove that, on the other hand, limits always have the same formula in both based spaces and unbased spaces.

5. A topological space X is *discrete* if its points are open; equivalently, if every subset is open.

   (a) Prove that the colimit of any diagram of discrete spaces $X(i)$ is discrete. In particular, this is true for sequential colimits.

(b) Show by example that the limit of a diagram of discrete spaces $X(i)$ may not be discrete. In other words, "inverse limits can create a topology."

You might consider the 2-adic integers, defined as the limit of the sets $\mathbb{Z}/(2^n\mathbb{Z})$ along the surjective homomorphisms

$$\mathbb{Z}/(2^n\mathbb{Z}) \xrightarrow{1} \mathbb{Z}/(2^{n-1}\mathbb{Z}).$$

Here the "1" means that the map multiplies by 1, i.e. it sends $a \mapsto a$ for each $a \in \mathbb{Z}/(2^n\mathbb{Z})$.

6. Let **Ab** denote the category of abelian groups and group homomorphisms. If $A\colon \mathbf{I} \to \mathbf{Ab}$ is a diagram of abelian groups and homomorphisms, show that its colimit is given by the same formula as (1.1.4), except the disjoint unions are replaced by direct sums:

$$\operatorname*{colim}_{\mathbf{I}} A(i) = \left(\bigoplus_{i \in \mathrm{ob}\,\mathbf{I}} A(i) \right) \Big/ \big(a \sim f(a) \quad \forall f : i \to j,\ a \in A(i) \big).$$

On the other hand, its limit is given by the same formula as for spaces, (1.1.3).

7. Suppose we have a sequential diagram in an arbitrary category **C**

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots$$

and that the diagram stabilizes, i.e. the maps f_i are isomorphisms from X_n onwards. Prove that the colimit of this diagram is isomorphic to X_n.

8. The system of abelian groups

$$\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \longrightarrow \dots \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \dots$$

does not stabilize. Prove that its colimit in abelian groups is isomorphic to the rational numbers $\mathbb{Q}$.

9. Recall that the forgetful functors $\mathbf{Top}_* \to \mathbf{Top} \to \mathbf{Set}$ and $\mathbf{Ab} \to \mathbf{Set}$ are right adjoints. Use this to explain why the limits are preserved in exercises 4 and 6 but not in exercise 5.

10. Let $X, Y, Z \in \mathbf{Top}_*$ be based topological spaces.

(a) Let $S(X, Y)$ be the collection of set maps (not necessarily continuous) from X to Y. Explain why there is a bijection between set maps $Z \to S(X, Y)$ and set maps $Z \times X \to Y$.

(b) The defining feature of the mapping space topology is that this correspondence passes to continuous maps $Z \to \mathrm{Map}(X, Y)$ and continuous maps $Z \times X \to Y$. Prove that along this correspondence, the based continuous maps $Z \to \mathrm{Map}_*(X, Y)$ landing in the subspace $\mathrm{Map}_*(X, Y)$ of based continuous maps, correspond to the continuous maps $Z \times X \to Y$ that factor through the smash product $Z \wedge X$ (Definition 1.1.12).

(c) Explain why a homotopy of maps $X \rightrightarrows Y$ preserving the basepoint is the same thing as a map $X \wedge I_+ \to Y$.

11. Explain how the definition of a fibration is similar to the definition of a cofibration. We informally say that the definitions are dual to each other.

12. Prove that for any map $f\colon A \to X$, the inclusion of $A \times \{0\}$ into the mapping cylinder

$$A \times I \cup_{A \times \{1\}} X$$

is a cofibration. Dually, for any map $p\colon E \to B$, prove that the projection

$$E \times_B B^I \to B, \qquad \{ (e, \gamma) \in E \times B^I \ : \ p(e) = \gamma(0) \} \mapsto \gamma(1)$$

is a fibration.

13. Recall that $h\mathbf{CW}$ is defined to have objects the CW complexes and morphisms the *homotopy classes* of maps. Prove that every homotopy equivalence is an isomorphism in $h\mathbf{CW}$. Conversely, prove that any functor $\mathbf{CW} \to \mathbf{D}$ sending homotopy equivalences to isomorphisms must factor through a functor $h\mathbf{CW} \to \mathbf{D}$. In other words, inverting homotopy equivalences is the same thing as passing to homotopy classes of maps. (See also Proposition 3.1.26).

14. Prove that the pushout or pullback of an isomorphism is an isomorphism. Prove that the pushout of a cofibration (Definition 1.1.7) is a cofibration. Prove that the pullback of a fibration is a fibration.

15. Prove that in the definition of a fibration, the dotted lift is in general not unique, but it is unique up to homotopy. (The homotopy should be through lifts that make the diagram commute!)

16. If Cf is the based homotopy cofiber of $f\colon A \to X$, and everything is well-based, prove that the based homotopy cofiber of $X \to Cf$ is equivalent to ΣA. Dually, prove that the homotopy fiber of $Fp \to E$ is equivalent to ΩB.

17. Give an example of two spaces that are weakly equivalent but not homotopy equivalent. (Hint: One approach is to use the topologist's sine curve.)

18. Prove the following claim from Section 1.5. Suppose the commuting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a homotopy pullback square. In other words, the induced map from A to the homotopy pullback

$$A \to B \times_D C \to B \times_D^h C$$

is a weak equivalence. Prove that there is a homotopy Mayer-Vietoris sequence

$$\dots \longrightarrow \pi_{n+1}(D) \xrightarrow{\partial} \pi_n(A) \xrightarrow{(f_*,h_*)} \pi_n(B) \oplus \pi_n(C) \xrightarrow{(k_*,-g_*)} \pi_n(D) \longrightarrow \dots$$

(Hint: You might try computing the homotopy fiber of $A \to B \times C$.)

19. Prove that $E \to B$ is both a Serre fibration and a weak equivalence, if and only if a lift exists for every commuting square of the following form.

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "\subseteq"] & E \arrow[d] \\ D^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

20. Prove the claim that if the connectivity of the maps $f_n : X_n \to X_{n-1}$ increases to ∞, then the formation of the homotopy limit commutes with homology and cohomology. You might want to use the $\lim^1$ exact sequence for homotopy groups.

21. (a) Suppose $f : A \to X$ is a cofibration and Y is any space. Composing with f gives a continuous map

$$(-) \circ f : \mathrm{Map}(X, Y) \to \mathrm{Map}(A, Y).$$

Prove that this map is a fibration.

(b) Suppose $p : E \to B$ is a fibration and Y is any space. Composing with p gives a continuous map

$$p \circ (-) : \mathrm{Map}(Y, E) \to \mathrm{Map}(Y, B).$$

Prove that this map is a fibration.

22. Use the Yoneda Lemma to prove the homeomorphism (1.1.13) from the bijection on the underlying sets. (Hint: What functor is represented by each side? Can you show that the two functors are isomorphic?)

23. Suppose we have an infinite sequence of closed inclusions (of CGWH spaces)

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \dots$$

and Y is any compact topological space.

   (a) Prove that every map $Y \to \operatorname*{colim}_{n\to\infty} X_n$ factors through some finite stage X_n. (Recall that all spaces in this book are assumed weak Hausdorff, and this implies they are T_1. In other words, their points are closed.)

   (b) Prove that homotopy classes of maps commute with the colimit,

$$\left[Y, \operatorname*{colim}_{n\to\infty} X_n\right] \cong \operatorname*{colim}_{n\to\infty} [Y, X_n].$$

   Similarly if Y and all X_n are based we get $[Y, \operatorname*{colim}_{n\to\infty} X_n]_* \cong \operatorname*{colim}_{n\to\infty} [Y, X_n]_*$.

   (c) Use the Yoneda lemma (Lemma 1.4.20) to get a homeomorphism

$$\operatorname{Map}\left(Y, \operatorname*{colim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \operatorname{Map}(Y, X_n)$$

   and in the based case

$$\operatorname{Map}_*\left(Y, \operatorname*{colim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \operatorname{Map}_*(Y, X_n).$$

   In particular, based loopspace commutes with the colimit,

$$\Omega\left(\operatorname*{colim}_{n\to\infty} X_n\right) \cong \operatorname*{colim}_{n\to\infty} \Omega X_n.$$

   (d) Show by counterexample that these fail if Y is not compact.

24. As in the previous exercise, suppose we have an infinite sequence of maps, not necessarily closed inclusions this time

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \dots$$

and Y is any compact topological space.

   (a) Explain how the unbased mapping telescope $\operatorname*{hocolim}_{n\to\infty} X_n$ is the colimit of a sequence of closed inclusions. Conclude that

$$\left[Y, \operatorname*{hocolim}_{n\to\infty} X_n\right] \cong \operatorname*{colim}_{n\to\infty} [Y, X_n].$$

(b) If Y and all X_n are based, explain how both the unbased and based mapping telescopes are colimits along closed inclusions. Use this to prove

$$\left[Y, \operatorname*{hocolim}_{n\to\infty} X_n\right]_* \cong \operatorname*{colim}_{n\to\infty} [Y, X_n]_*$$

using either the based or unbased version of the mapping telescope. Conclude that the collapse map from the unbased telescope to the based telescope is always a weak equivalence, proving part of Lemma 1.5.15.

(c) Show that there is a canonical weak equivalence

$$\operatorname*{hocolim}_{n\to\infty} \Omega X_n \xrightarrow{\sim} \Omega\left(\operatorname*{hocolim}_{n\to\infty} X_n\right),$$

using the based version of the mapping telescope.

25. Similarly to the previous exercises, but easier, assume we have a sequence of maps

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \ldots$$

and Z is any topological space, not necessarily compact.

(a) Prove that there is a homeomorphism

$$\operatorname{Map}\left(\operatorname*{colim}_{n\to\infty} X_n, Z\right) \cong \lim_{n\to\infty} \operatorname{Map}(X_n, Z)$$

and in the based case

$$\operatorname{Map}_*\left(\operatorname*{colim}_{n\to\infty} X_n, Z\right) \cong \lim_{n\to\infty} \operatorname{Map}_*(X_n, Z).$$

(b) Explain why this does not in general give a bijection on homotopy classes of maps

$$\left[\operatorname*{colim}_{n\to\infty} X_n, Z\right] \ncong \lim_{n\to\infty} [X_n, Z].$$

(It does give a $\lim^1$ sequence if we take based maps and Z is a loop space, so that the maps form an abelian group. In particular, this gives the $\lim^1$ sequence for cohomology of a homotopy colimit.)

26. Adapt Example 1.6.11 from strict colimits to the homotopy colimits defined in Section 1.5: $K \times -$ commutes with unbased homotopy colimits, and $K \wedge -$ commutes with based homotopy colimits.

27. Prove that left adjoints are unique (Theorem 1.6.12). (This follows almost immediately from the Yoneda Lemma, Lemma 1.4.20, which you've hopefully been able to practice in the last few exercises.)

Chapter 2

Spectra

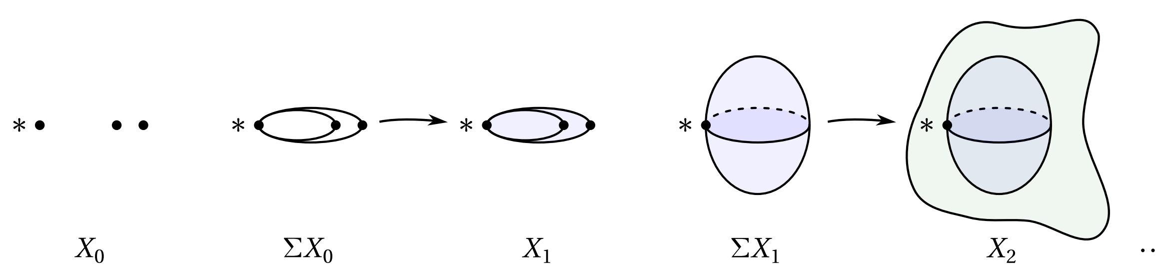
In this chapter we'll define spectra and prove their fundamental properties. We study spectra up to stable equivalence, just like how in homotopy theory, we study spaces up to weak homotopy equivalence. We'll save the discussion of the homotopy category Ho **Sp** until the next chapter.

2.1 Definition and basic examples

2.1.1 Spectra and stable homotopy groups

Definition 2.1.1. A **spectrum** X is a sequence of based spaces $\{X_n\}_{n\geq 0}$ and bonding maps (or structure maps)

$$\xi_n \colon \Sigma X_n \longrightarrow X_{n+1}.$$



A **map of spectra** $f\colon X \to Y$ is a sequence of maps $f_n\colon X_n \to Y_n$ such that each square of the following form commutes.

$$\begin{tikzcd}
\Sigma X_n \arrow[r, "\xi_n"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma Y_n \arrow[r, "\upsilon_n"] & Y_{n+1}
\end{tikzcd}$$

It is important to say right away that we only care about the *limiting behavior* of the spaces X_n as $n \to \infty$. To capture this idea, we define the homotopy groups of a spectrum as a colimit (direct limit) of the homotopy groups of the spaces X_n. Because of the suspensions, the homotopy groups shift in degree as we go from each level to the next.

Definition 2.1.2. If X is a spectrum, for each $k \in \mathbb{Z}$, the kth **stable homotopy group** $\pi_k(X)$ is defined as the colimit of the system of abelian groups

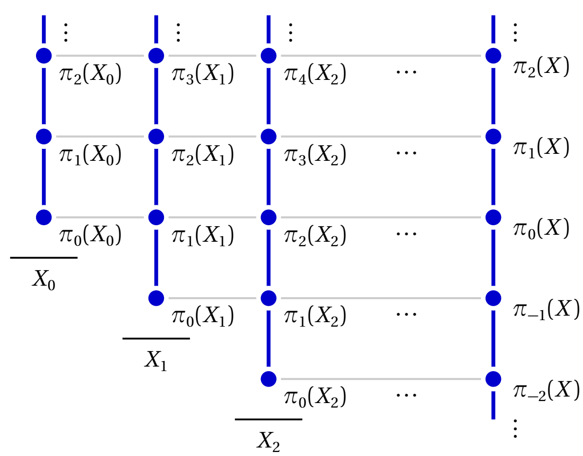
$$\cdots \longrightarrow \pi_{k+n}(X_n) \xrightarrow{\sigma} \pi_{k+n+1}(\Sigma X_n) \xrightarrow{\xi_n} \pi_{k+n+1}(X_{n+1}) \longrightarrow \cdots$$

where each homotopy group is taken at the basepoint. The operation σ suspends each map $S^{k+n} \to X_n$ to produce a map $S^{k+n+1} \to X_{n+1}$. When k is negative, the system is only defined for $n \geq |k|$.

(See Section 1.7, exercise 6 for a discussion of colimits of abelian groups.)

Remark 2.1.3. In the literature, spectra of this form are sometimes called **sequential spectra** or **prespectra**. In this book, we will call them sequential spectra when we want to distinguish them from the diagram spectra that occur in Chapter 6.

Intuitively, what is going on in this definition is that every sphere in X_n becomes a sphere in X_{n+1} of *one dimension higher*. The stable homotopy groups are counting these spheres up to homotopy, in the limit where $n \to \infty$. The dimension of the sphere is therefore going to ∞, but the *difference* between this dimension and the index of X_n stays at a constant value $k \in \mathbb{Z}$.



We'll sometimes draw the homotopy groups of each level X_n as blue dots, connected by vertical blue lines. The bonding maps of the spectrum X create maps between these homotopy groups that shift degree by one. We depict this by drawing the levels of a spectrum in staggered formation, so that the maps between the homotopy groups go straight to the right. We then draw the stable homotopy groups all the way on the right-hand side.

Before proceeding further, it's helpful to fix some conventions. There are many homeomorphic models for the n-sphere, but we'll define it as the one-point compactification of $\mathbb{R}^n$:

$$S^n := \mathbb{R}^n \cup \{\infty\}.$$

Let $X \wedge Y$ denote the smash product (Definition 1.1.12). We define the reduced suspension to be $\Sigma X = X \wedge S^1$, with the S^1 on the right-hand side of X. Similarly, the n-fold suspension is $\Sigma^n X = X \wedge S^n$. We'll always use this convention for the suspensions that occur in the bonding maps of a spectrum.

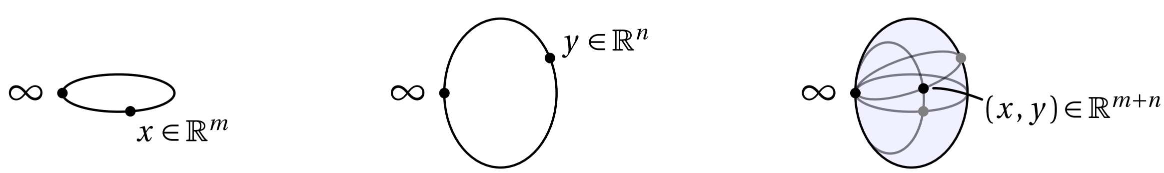
The smash product of two spheres is always a sphere, $S^m \wedge S^n \cong S^{m+n}$. To see this, we recall that the smash product of two spaces is, away from the basepoint, just the Cartesian product:

$$(X \wedge Y) - \{*\} \cong (X - \{*\}) \times (Y - \{*\}).$$

Therefore $S^m \wedge S^n$ minus its basepoint is identified with the product $\mathbb{R}^m \times \mathbb{R}^n$.

Definition 2.1.4. The canonical isomorphism $S^m \wedge S^n \cong S^{m+n}$ is the one that, after removing the basepoints, becomes the concatenation map

$$\begin{aligned} \mathbb{R}^m \times \mathbb{R}^n &\xrightarrow{\cong} \mathbb{R}^{m+n} \\ (x_1, \ldots, x_m), (y_1, \ldots, y_n) &\longmapsto (x_1, \ldots, x_m, y_1, \ldots, y_n). \end{aligned}$$



$S^m = \mathbb{R}^m \cup \{\infty\}$ $S^n = \mathbb{R}^n \cup \{\infty\}$ $S^{m+n} = \mathbb{R}^{m+n} \cup \{\infty\}$

We can now add more precision to the definition of the stable homotopy groups in Definition 2.1.2. We define σ to be the operation that takes each map $\phi \colon S^{k+n} \to X_n$ to the composite map

$$\begin{tikzcd} S^{k+n+1} \arrow[r, "\cong"] \arrow[rr, bend right, "\sigma(\phi)"'] & S^{k+n} \wedge S^1 \arrow[r, "\phi \wedge \mathrm{id}"] & X_n \wedge S^1, \end{tikzcd}$$

where the isomorphism is the canonical isomorphism of Definition 2.1.4.

Remark 2.1.5. We might say that an **element** of X of degree k is a map $\phi \colon S^{k+n} \to X_n$, up to the relation that ϕ is identified with $\sigma(\phi)$. This is the correct notion of "degree k points" in the spectrum X. The kth homotopy group $\pi_k(X)$ is the set of degree k elements up to homotopy.

2.1.2 Suspension and desuspension spectra

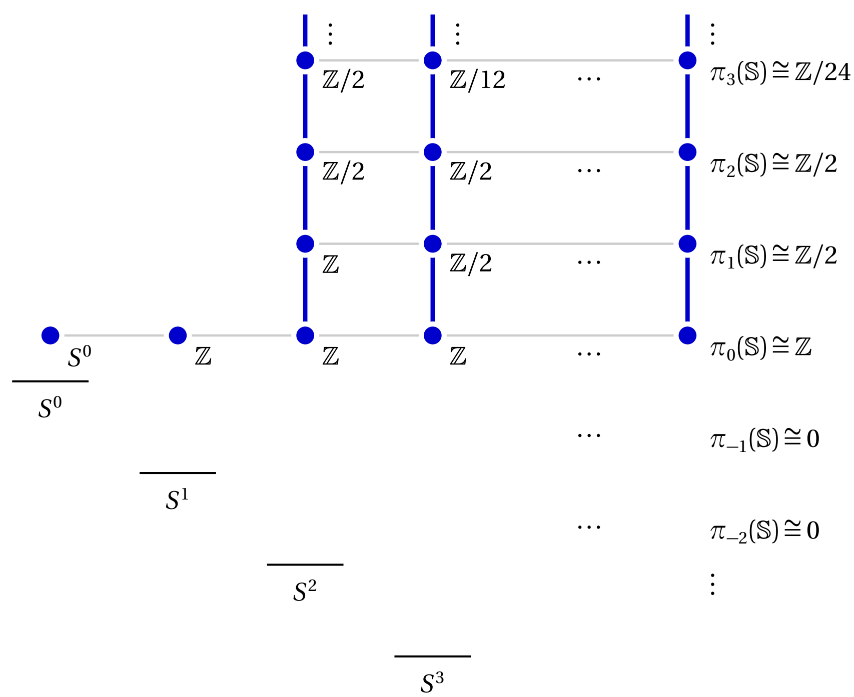
Example 2.1.6. The **sphere spectrum** $\mathbb{S}$ has nth level S^n and bonding maps the canonical homeomorphisms

$$S^n \wedge S^1 \cong S^{n+1}.$$

Following Definition 2.1.4, this isomorphism is given away from the basepoint by the formula

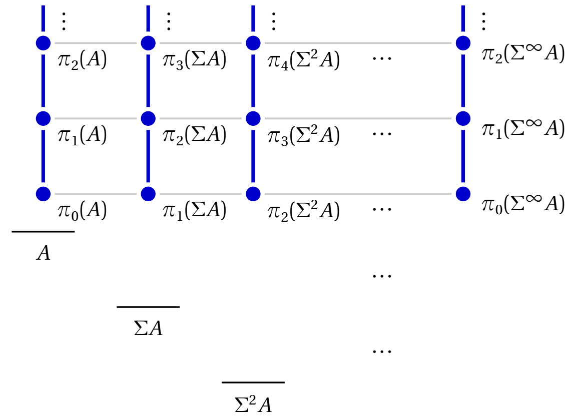
$$\begin{aligned} \mathbb{R}^n \times \mathbb{R} &\xrightarrow{\cong} \mathbb{R}^{n+1} \\ (x_1, \ldots, x_n), (y_1) &\longmapsto (x_1, \ldots, x_n, y_1). \end{aligned}$$

The first few stable homotopy groups of $\mathbb{S}$ are shown below.



The negative homotopy groups are all zero, since $\pi_{k+n}(S^n) = 0$ when $k < 0$ (Corollary 1.4.8). But, *we don't know all of its positive homotopy groups!* Computing them is one of the central problems of homotopy theory, and to this day, it continues to be an open problem. It's strange, but in homotopy theory, we work frequently with groups that we cannot compute. It adds an air of mystery and adventure to the subject.

Example 2.1.7. If A is any based space, the **suspension spectrum** $\Sigma^\infty A$ is a spectrum whose nth level is $\Sigma^n A = A \wedge S^n$.



The bonding map is the canonical homeomorphism $A \wedge S^n \wedge S^1 \cong A \wedge S^{n+1}$. Its first two homotopy groups (assuming A is well-based) are:

$$\pi_0(\Sigma^\infty A) \cong H_0(A, *; \mathbb{Z}),$$
$$\pi_1(\Sigma^\infty A) \cong H_0(A, *; \mathbb{Z}/2) \oplus H_1(A, *; \mathbb{Z}).$$

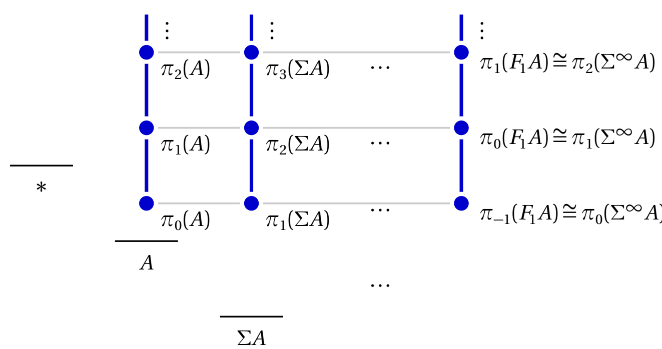
The negative homotopy groups are zero. Note that the sphere spectrum $\mathbb{S}$ is the suspension spectrum $\Sigma^\infty S^0$.

If B is an unbased space, we define its **unbased suspension spectrum** by adding a disjoint basepoint and then taking the suspension spectrum. The resulting spectrum is denoted either $\Sigma^\infty B_+$ or $\Sigma^\infty_+ B$. So we have $\mathbb{S} = \Sigma^\infty_+(*)$, and

$$\pi_0(\Sigma^\infty_+ B) \cong H_0(B; \mathbb{Z}), \qquad \pi_1(\Sigma^\infty_+ B) \cong H_0(B; \mathbb{Z}/2) \oplus H_1(B; \mathbb{Z}).$$

Returning to the case where A is a based space, suppose we replace A by its d-fold suspension $\Sigma^d A$ for some $d \geq 0$. Then the suspension spectrum $\Sigma^\infty(\Sigma^d A)$ has the same homotopy groups as $\Sigma^\infty A$, but shifted up d slots. In spectra, we can give an inverse to this operation:

Example 2.1.8. For any integer $d \geq 0$, the **shift desuspension spectrum** $F_d A$ is a spectrum whose nth level is the one-point space $*$ when $n < d$, and $\Sigma^{n-d} A$ when $n \geq d$. The bonding maps are again canonical homeomorphisms. The spectrum $F_1 A$ is illustrated to the right.



The spectrum $F_d A$ is also sometimes denoted $\Sigma^{-d} A$ or $\Sigma^{\infty-d} A$. Its homotopy groups are those of $\Sigma^\infty A$, but shifted down d slots.

Example 2.1.9. As a special case, the $(-d)$-sphere spectrum $\mathbb{S}^{-d} := F_d S^0$ is the spectrum that at level n is the sphere S^{n-d}, so long as n is large enough that $n - d \geq 0$. For positive d we define the d-sphere spectrum as $\mathbb{S}^d = F_0 S^d = \Sigma^\infty S^d$.

The category of spectra is delightful – it allows us to formally define negative-dimensional objects, without having access to any actual negative-dimensional geometry. The $(-d)$-

sphere is an object that, after suspending d times, becomes the 0-sphere. This is philosophically similar to how the integers $\mathbb{Z}$ are defined from the natural numbers $\mathbb{N}$: an integer is an object that, if you add a large enough natural number to it, becomes a natural number.

2.1.3 Stable equivalences

Let **Sp** denote the category of spectra defined by Definition 2.1.1. For each $k \in \mathbb{Z}$, the stable homotopy group π_k forms a functor from spectra to abelian groups,

$$\mathbf{Sp} \xrightarrow{\pi_k} \mathbf{Ab}.$$

To spell this out: each map of spectra $f: X \to Y$ induces a map of colimit systems

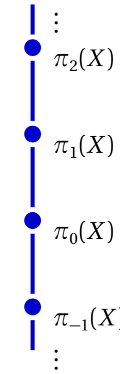
$$\begin{tikzcd}
\dots \arrow[r] & \pi_{k+n}(X_n) \arrow[r, "\sigma"] \arrow[d, "f_n"] & \pi_{k+n+1}(\Sigma X_n) \arrow[r, "\xi_n"] \arrow[d, "\Sigma f_n"] & \pi_{k+n+1}(X_{n+1}) \arrow[r] \arrow[d, "f_{n+1}"] & \dots \\
\dots \arrow[r] & \pi_{k+n}(Y_n) \arrow[r, "\sigma"] & \pi_{k+n+1}(\Sigma Y_n) \arrow[r, "\upsilon_n"] & \pi_{k+n+1}(Y_{n+1}) \arrow[r] & \dots
\end{tikzcd}$$

which in turn gives a map on the colimits, $f_*: \pi_k(X) \to \pi_k(Y)$. It is easy to check this rule respects identity maps of spectra, and composition of maps of spectra. This is what we mean when we say that π_* is a functor.

Definition 2.1.10.

- A **stable equivalence**, or **π_*-isomorphism**, is a map of spectra $X \to Y$ inducing an isomorphism $\pi_k(X) \xrightarrow{\cong} \pi_k(Y)$ for all $k \in \mathbb{Z}$.
- A **level equivalence** is a map of spectra $X \to Y$ that at each level $X_n \to Y_n$ is a weak homotopy equivalence. So it induces isomorphisms on the homotopy groups $\pi_{k+n}(X_n)$ at all basepoints of X_n.
- A **homotopy equivalence** is a map of spectra $f: X \to Y$ such that there exists another map $g: Y \to X$ and homotopies of maps of spectra $g \circ f \sim \mathrm{id}_X$, $f \circ g \sim \mathrm{id}_Y$. (See Definition 2.3.11.)
- Spectra X and Y are **stably equivalent** if they are connected by any zig-zag of stable equivalences. Level equivalent and homotopy equivalent spectra are defined similarly.

Stable homotopy theory is the study of spectra up to stable equivalence. We consider two stably equivalent spectra to be "essentially the same." This is similar to homological algebra, where we consider chain complexes up to quasi-isomorphism, i.e. maps inducing isomorphisms on homology. In the pictures we been drawing, this means that we focus on the blue line at the far right. It is a mathematical object that can have homotopy groups in every degree.



We have the implications

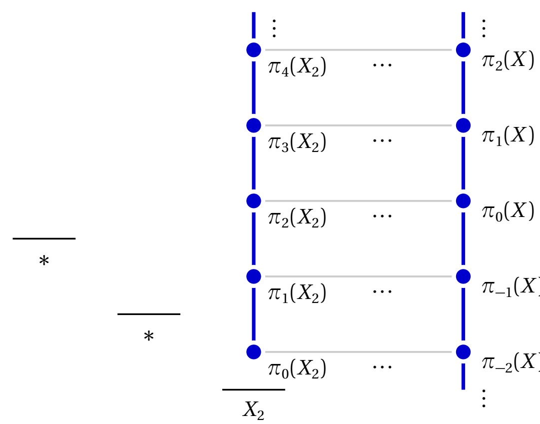
$$\text{homotopy equivalence} \Rightarrow \text{level equivalence} \Rightarrow \text{stable equivalence},$$

so homotopy and level equivalences are a stricter notion than stable equivalences. For certain classes of nice spectra, the implications can be reversed, see Lemma 2.2.5 and Corollary 2.6.17.

Example 2.1.11. If X is any spectrum, we can cut off everything before level d to make a new spectrum X'. To be precise,

$$X'_n = \begin{cases} * & \text{if} \quad n < d \\ X_n & \text{if} \quad n \geq d, \end{cases}$$

and bonding maps the same as X, starting at level d. The inclusion $X' \subseteq X$ is a stable equivalence. In other words, we can always throw away finitely many of the levels of X, without changing its stable homotopy type.



Example 2.1.12. The **zero spectrum** $*$ has every level the one-point space $*$, and every bonding map is the unique map

$$\Sigma(*) \cong * \longrightarrow *.$$

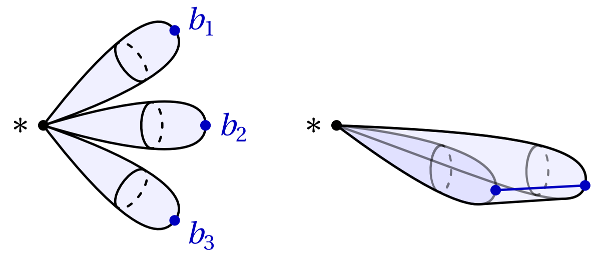
Every spectrum X admits a unique maps of spectra $* \to X \to *$. We say X is **weakly contractible** if one (equivalently both) of these maps are stable equivalences. This is also equivalent to having vanishing homotopy groups, $\pi_k(X) \cong 0$ for all $k \in \mathbb{Z}$.

Given two spectra X and Y, the **zero map** $X \to Y$ is the unique map of spectra that factors through $*$. At level n it is the map $X_n \to Y_n$ sending everything to the basepoint.

2.1.4 Thom spectra

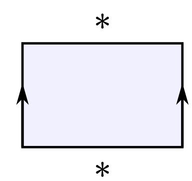
The next few examples will use several concepts from the theory of fiber bundles, see e.g. [Ste51, MS74, Coh98, Hat03].

If B is an unbased space, the n-fold suspension $\Sigma^n_+ B = \Sigma^n B_+ = B_+ \wedge S^n$ can be visualized as a collection of n-spheres, one for each $b \in B$, that share a common basepoint.

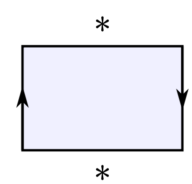


$\Sigma^n_+\{b_1, b_2, b_3\}$ $\Sigma^n_+ I$

For example, the one-fold suspension of $B = S^1$ can be written as a square with edges identified as shown, and the top and bottom edges collapsed to the basepoint $*$. Each point in B gives a circle, which is a single vertical line in this picture, but the circles are glued together continuously, giving the whole square. The resulting space $\Sigma_+ S^1$ is homeomorphic to S^2 with two points identified, and homotopy equivalent to $S^1 \vee S^2$.



One might imagine trying to modify this suspension by spinning the spheres around as we rove around B. For instance, the one-fold suspension of $B = S^1$ can be twisted like a Möbius strip. The resulting "twisted suspension" of S^1 is written as a square with edges identified as shown, and is homeomorphic to the projective plane $\mathbb{RP}^2$.

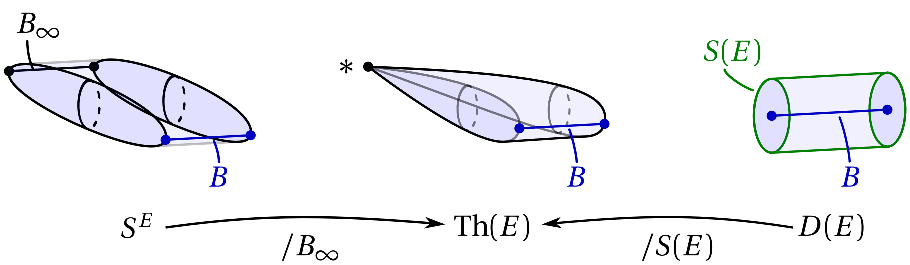


To accomplish this in general, we pick a vector bundle $E \to B$. This is a continuous map whose fibers are vector spaces, and that is locally a product projection $U \times \mathbb{R}^n \to U$. We define the E-suspension of B to be like the n-fold suspension, except that the spheres twist according to the bundle E:

Definition 2.1.13. The **Thom space** of the bundle $E \to B$ is denoted $\mathrm{Th}(E)$, $\Sigma^E B$, or B^E. We construct it by taking one-point compactification of the fibers, giving a bundle $S^E \to B$ whose fibers are spheres S^n. Then, we collapse to a point the subspace $B_\infty \subseteq S^E$ containing all the newly-created points at infinity.

Intuitively, $\mathrm{Th}(E)$ is just the vector bundle E with a single additional point, which we can reach by going to ∞ in *any* of the fibers of $E \to B$.

Lemma 2.1.14. *There is a homeomorphism* $\mathrm{Th}(E) \cong D(E)/S(E)$*, where* $D(E)$ *is the unit* *disc bundle and* $S(E) \subseteq D(E)$ *is the unit sphere bundle, for any fixed choice of metric on* E.



Example 2.1.15. When E is a trivial bundle $B \times \mathbb{R}^n$, the Thom space is

$$(B \times S^n)/(B \times \{\infty\}) \cong (B \times D^n)/(B \times S^{n-1}) \cong \Sigma^n_+ B.$$

So when E is nontrivial, we think of the Thom space as a twisted suspension of B.

We observe that adding a trivial bundle to E has the effect of suspension,

$$\mathrm{Th}(E \oplus \epsilon^1) \cong \mathrm{Th}(E \times \mathbb{R}) \cong \Sigma \mathrm{Th}(E), \tag{2.1.16}$$

see exercise 5. Therefore, if we make a spectrum by formally de-suspending $\mathrm{Th}(E)$, this should correspond to *subtracting off* a trivial bundle.

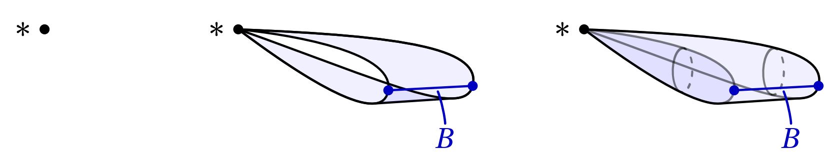
Example 2.1.17. Suppose we have a virtual bundle ζ, that is, a formal difference of vector bundles $E - E'$, over a compact space B. Without loss of generality, it is a nontrivial bundle E minus a d-dimensional trivial bundle $\epsilon^d = B \times \mathbb{R}^d$:

$$\zeta = E - \epsilon^d. \tag{2.1.18}$$

We define the **Thom spectrum** of ζ as

$$\mathrm{Th}(\zeta) := F_d(\mathrm{Th}(E)).$$

That is, we take the Thom space for E, and then take its shift desuspension spectrum. This spectrum is also denoted $\Sigma^\zeta B$ or B^ζ.



$\mathrm{Th}(\zeta)_{d-1} = *$ $\quad$ $\mathrm{Th}(\zeta)_d = \mathrm{Th}(E)$ $\quad$ $\mathrm{Th}(\zeta)_{d+1} = \Sigma\mathrm{Th}(E)$

Intuitively, ζ is a mathematical object that is not a vector bundle, but if we add enough trivial line bundles to it, it becomes a vector bundle. We can visualize the definition of its Thom spectrum as an iterative process: at each step, add a trivial line bundle. If the result is not a vector bundle yet, take the zero space $*$. If the resulting object is a vector bundle, take its Thom space $\mathrm{Th}(\zeta \oplus \epsilon^n)$.

Put another way, the "Thom space" $\mathrm{Th}(\zeta)$ does not exist as a geometric object, but it does exist if suspended enough times, and this is enough to define a spectrum out of it.

Example 2.1.19. Suppose M is a closed manifold with tangent bundle τ. The virtual bundle $-\tau$ can be presented as $\nu - \epsilon^d$, where ν is the normal bundle of some embedding $M \to \mathbb{R}^d$. Therefore its Thom spectrum is

$$\mathrm{Th}(-\tau) = \Sigma^{-\tau} M = M^{-\tau} := F_d(\mathrm{Th}(\nu)).$$

Example 2.1.20. There is a universal Thom spectrum MO whose base is the space BO classifying all stable real vector bundles. To build it, we use space $BO(n)$ at spectrum level n, so that as $n \to \infty$ we get BO as the base space.

In more detail, let $BO(n)$ denote the space of n-dimensional subspace of $\mathbb{R}^\infty$, and $\gamma^n \to BO(n)$ be the tautological bundle whose fiber over each point consists of all the vectors in that n-dimensional subspace.

We define the spectrum MO by letting $MO_n = \mathrm{Th}(\gamma^n)$. To define the bonding maps, we take the maps $BO(n) \to BO(n+1)$ that add a fixed line to the subspaces. The pullback of γ^{n+1} along this map is $\gamma^n \oplus \epsilon^1$, so we get maps

$$\Sigma MO_n = \Sigma \mathrm{Th}(\gamma^n) \cong \mathrm{Th}(\gamma^n \oplus \epsilon^1) \to \mathrm{Th}(\gamma^{n+1}) = MO_{n+1}.$$

These define the bonding maps of MO. Intuitively, MO is a suspension spectrum of $BO(n)$ with $n \to \infty$, but the spheres are twisted according to the vector bundles γ^n.

The Pontryagin-Thom isomorphism says that for each $d \geq 0$, the stable homotopy group $\pi_d(MO)$ is isomorphic to the group of d-dimensional closed manifolds M^d up to cobordism. See [Wes96, Kup17, Mil94, Sto68, Coh20] for more details.

Warning 2.1.21. The bonding maps matter! Given any sequence $\{X_n\}$, we can form the spectrum X in which the bonding maps are all zero,

$$\Sigma X_n \longrightarrow * \longrightarrow X_{n+1}.$$

Then X is weakly contractible, i.e. $X \to *$ is a stable equivalence.

2.2 Ω-spectra and infinite loop spaces

Recall from Corollary 1.6.4 that the reduced suspension $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$ and the based loopspace $\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$ are adjoint. This is a fancy way of saying that continuous based maps $\Sigma X \to Y$ are the same thing as continuous based maps $X \to \Omega Y$.

Applying this to Definition 2.1.1, we see that a spectrum is equivalently described as a sequence of based spaces X_n and based maps

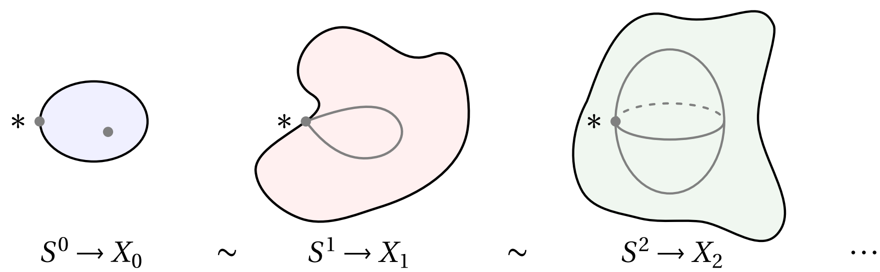
$$\widetilde{\xi}_n : X_n \to \Omega X_{n+1}.$$

In other words, each point in X_n creates a loop in the next space X_{n+1}. We call $\widetilde{\xi}_n$ the **adjunct bonding map**.

Definition 2.2.1. A **fibrant spectrum** or **Ω-spectrum** is a spectrum X in which the adjunct bonding maps $X_n \to \Omega X_{n+1}$ are weak homotopy equivalences.

The zeroth space X_0 of an Ω-spectrum is also called an **infinite loop space**.

Intuitively, this means that not only does a sphere in X_n create a sphere in X_{n+1} of one dimension higher, but having a sphere in X_n is *equivalent* to having a sphere in X_{n+1} of one dimension higher.



When $\Sigma X_n \to X_{n+1}$ is a weak equivalence, its adjunct $X_n \to \Omega X_{n+1}$ is usually not a weak equivalence, and vice-versa. For instance, the adjunct of $S^2 \cong S^2$ is a map $S^1 \to \Omega S^2$ that is not a weak equivalence. Having a two-sphere in S^1 is *not* equivalent to having a three-sphere in S^2.

So the sphere spectrum $\mathbb{S}$ is not an Ω-spectrum. In fact, almost none of the suspension spectra and Thom spectra of the previous section are Ω-spectra.

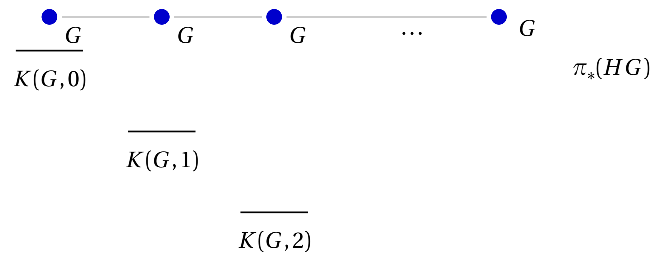
Example 2.2.2. Let G be an abelian group. Recall that an Eilenberg-Maclane space $K(G,n)$ is a CW complex whose nth homotopy group is identified with G and all other homotopy groups are zero (Definition 1.4.16). Define an **Eilenberg-Maclane spectrum** HG by

$$(HG)_n = K(G,n),$$

and bonding maps the canonical equivalences $K(G,n) \simeq \Omega K(G,n+1)$.

By construction, this is an Ω-spectrum with only one stable homtopy group,

$$\pi_k(HG) = \begin{cases} G & k = 0 \\ 0 & k \neq 0, \end{cases}$$



just like the Eilenberg-Maclane spaces $K(G,n)$. In the exercises we will see that this property defines HG uniquely, up to stable equivalence. Since $(HG)_0 = G$, we conclude that every abelian group is an infinite loop space.

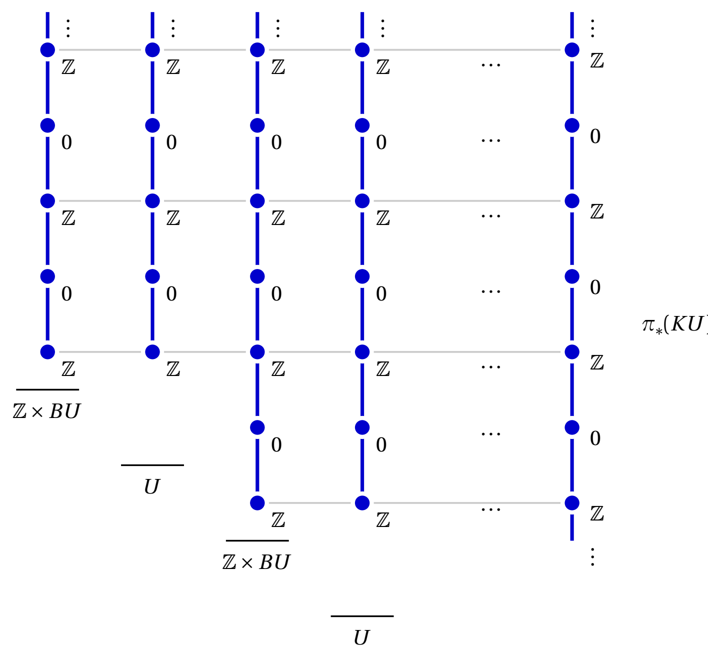
Example 2.2.3. Let $U = \operatorname{colim}_{n\to\infty} U(n)$ be the infinite unitary group and $BU = \operatorname{colim}_{n\to\infty} BU(n)$ its classifying space. The space $\mathbb{Z}\times BU$ classifies stable complex vector bundles, and the Bott periodicity theorem [Bot59] tells us that

$$\Omega(\mathbb{Z}\times BU) \simeq U, \qquad \Omega U \simeq \mathbb{Z}\times BU.$$

The **complex K-theory spectrum** KU has every even level $KU_{2n} = \mathbb{Z}\times BU$, every odd level $KU_{2n+1} = U$, and adjunct bonding maps given just above. The stable homotopy groups follow a periodic pattern, alternating between $\mathbb{Z}$ and 0:

$$\pi_k(KU) = \begin{cases} \mathbb{Z} & k = 2n \\ 0 & k = 2n+1. \end{cases}$$

Therefore $\mathbb{Z}\times BU$ is also an infinite loop space.



Lemma 2.2.4. *When X is an Ω-spectrum, the map from each group $\pi_{k+n}(X_n)$ to the colimit $\pi_k(X)$ is always an isomorphism.*

Proof. The definition of $\pi_k(X)$ can be rearranged to the colimit of

$$\dots \longrightarrow \pi_{k+n}(X_n) \xrightarrow{\tilde{\xi}_n} \pi_{k+n}(\Omega X_{n+1}) \xrightarrow{\cong} \pi_{k+n+1}(X_{n+1}) \longrightarrow \dots$$

Here the isomorphism comes from the adjunction between Σ and Ω, see Corollary 1.6.5. If X is an Ω-spectrum then all the maps of this system are isomorphisms, hence the map from any one term to the colimit is also an isomorphism. □

Lemma 2.2.5. *Between Ω-spectra, every stable equivalence is a level equivalence.*

Proof. By the previous lemma, if $f: X \to Y$ is a stable equivalence of Ω-spectra, then for any $n \geq 0$, $f_{n+1}: X_{n+1} \to Y_{n+1}$ is an isomorphism on all homotopy groups at the basepoint. Therefore it is a weak equivalence on basepoint components, so $\Omega f_{n+1}: \Omega X_{n+1} \to \Omega Y_{n+1}$ is a weak equivalence. Since X and Y are Ω-spectra, this is equivalent to $f_n: X_n \to Y_n$ so f_n is a weak equivalence for every $n \geq 0$. □

We next show that every spectrum is stably equivalent to an Ω-spectrum. We first do the sphere spectrum as a special case.

Example 2.2.6. The fibrant sphere spectrum $f\mathbb{S}$ has nth level equal to the colimit of the spaces

$$S^n \xrightarrow{\widetilde{\xi}_n} \Omega S^{n+1} \xrightarrow{\Omega\widetilde{\xi}_{n+1}} \Omega^2 S^{n+2} \xrightarrow{\Omega^2\widetilde{\xi}_{n+2}} \Omega^3 S^{n+3} \xrightarrow{\Omega^3\widetilde{\xi}_{n+3}} \dots$$

where each map is the adjunct of the identity map of a sphere. We call this colimit $\Omega^\infty\Sigma^\infty S^n$. It is also sometimes called QS^n after Quillen. Intuitively, $\Omega^\infty\Sigma^\infty S^n$ is the space of based maps $S^k \to S^{n+k}$, but stabilized so that $k \to \infty$. It is also the space of "elements" of degree n in $\mathbb{S}$, in the sense of Remark 2.1.5.

(We could have also taken the homotopy colimit, or mapping telescope, instead of the strict colimit. The homotopy colimit is equivalent to the strict colimit, because of Lemma 1.5.5 and the fact that the maps $\Omega^k\widetilde{\xi}_{n+k}$ are closed inclusions.)

To define the adjunct bonding map, we use Section 1.7 exercise 23 to rewrite $\Omega^\infty\Sigma^\infty S^{n+1}$ as the colimit along the bottom row of the following diagram. We form a homeomorphism between this colimit and $\Omega^\infty\Sigma^\infty S^n$ using identity maps:

$$\begin{tikzcd}
S^n \arrow[r, "\widetilde{\xi}_n"] \arrow[d, "\widetilde{\xi}_n"'] & \Omega S^{n+1} \arrow[r, "\Omega\widetilde{\xi}_{n+1}"] \arrow[dl, "\mathrm{id}"] & \Omega^2 S^{n+2} \arrow[r, "\Omega^2\widetilde{\xi}_{n+2}"] \arrow[dl, "\mathrm{id}"] & \dots \arrow[r] \arrow[dl, "\mathrm{id}"] & \Omega^\infty\Sigma^\infty S^n \arrow[d, "\cong"] \\
\Omega S^{n+1} \arrow[r, "\Omega\widetilde{\xi}_{n+1}"'] & \Omega^2 S^{n+2} \arrow[r, "\Omega^2\widetilde{\xi}_{n+2}"'] & \Omega^3 S^{n+3} \arrow[r, "\Omega^3\widetilde{\xi}_{n+3}"'] & \dots \arrow[r] & \Omega\Omega^\infty\Sigma^\infty S^{n+1}
\end{tikzcd}$$

This defines the adjunct bonding maps for $f\mathbb{S}$. The inclusion of the first term of each colimit system gives maps $S^n \to \Omega^\infty\Sigma^\infty S^n$, commuting with these bonding maps, giving a map of spectra $\mathbb{S} \to f\mathbb{S}$. We will see shortly that it is a stable equivalence.

Remark 2.2.7. In the previous example, it is possible to follow several different conventions when defining the maps $\Omega^k\widetilde{\xi}_{n+k}$. We adopt the following one. The k-fold loop space is defined as $\Omega^k X := \mathrm{Map}_*(S^k, X)$. When we take Ω^k of a map $X_{n+k} \to \Omega X_{n+k+1}$, the extra loop in the target goes to the *right* of the existing k loops. The resulting map

$$\Omega^k \tilde{\xi}_{n+k} : \Omega^k X_{n+k} \to \Omega^{k+1} X_{n+k+1}$$

takes $\left(\phi : S^k \to X_{n+k}\right) \in \Omega^k X_{n+k}$ to the composite

$$\begin{tikzcd} S^{k+1} \arrow[r, "\cong"] \arrow[rr, bend right, "\sigma(\phi)"'] & S^k \wedge S^1 \arrow[r, "\phi\wedge\mathrm{id}"] & X_{n+k} \wedge S^1 \arrow[r, "\xi_n"] & X_{n+k+1}. \end{tikzcd}$$

Example 2.2.8. Similarly to the last example, we may define a fibrant suspension spectrum of a space A by setting level n to be $\Omega^\infty \Sigma^\infty \Sigma^n A$, the colimit of

$$\Sigma^n A \xrightarrow{\tilde{\xi}_n} \Omega\Sigma^{n+1} A \xrightarrow{\Omega\tilde{\xi}_{n+1}} \Omega^2\Sigma^{n+2} A \xrightarrow{\Omega^2\tilde{\xi}_{n+2}} \Omega^3\Sigma^{n+3} A \xrightarrow{\Omega^3\tilde{\xi}_{n+3}} \dots$$

This is stably equivalent to $\Sigma^\infty A$.

We can generalize the construction of the last two examples to any spectrum X. The only problem is that the colimit is not always a homotopy colimit (Section 1.5), so it may have the wrong homotopy type. The following proof fixes this by using the homotopy colimit instead.

Proposition 2.2.9. *To each spectrum X, we may assign an Ω-spectrum RX and a stable equivalence $X \to RX$ in a natural way.*

We sometimes call RX a **fibrant replacement** of X. The naturality part of the statement is that R is a functor and the maps $X \to RX$ define a natural transformation. So for each map $X \to Y$, the following square commutes.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & RX \arrow[d] \\ Y \arrow[r] & RY \end{tikzcd}$$

Proof. We define $(RX)_n$ to be the based homotopy colimit of the spaces

$$X_n \xrightarrow{\tilde{\xi}_n} \Omega X_{n+1} \xrightarrow{\Omega\tilde{\xi}_{n+1}} \Omega^2 X_{n+2} \xrightarrow{\Omega^2\tilde{\xi}_{n+2}} \Omega^3 X_{n+3} \longrightarrow \dots$$

where $\Omega^k \tilde{\xi}_{n+k}$ is defined as in Remark 2.2.7. We map $X_n \to (RX)_n$ by including the first term into the colimit. The commuting grid of maps

$$\begin{tikzcd} X_n \arrow[r, "\tilde{\xi}_n"] \arrow[d, "\tilde{\xi}_n"] & \Omega X_{n+1} \arrow[r, "\Omega\tilde{\xi}_{n+1}"] \arrow[d, "\Omega\tilde{\xi}_{n+1}"] \arrow[dl, dashed, "\mathrm{id}"] & \Omega^2 X_{n+2} \arrow[r, "\Omega^2\tilde{\xi}_{n+2}"] \arrow[d, "\Omega^2\tilde{\xi}_{n+2}"] \arrow[dl, dashed, "\mathrm{id}"] & \Omega^3 X_{n+3} \arrow[r, "\Omega^3\tilde{\xi}_{n+3}"] \arrow[d, "\Omega^3\tilde{\xi}_{n+3}"] \arrow[dl, dashed, "\mathrm{id}"] & \cdots \\ \Omega X_{n+1} \arrow[r, "\Omega\tilde{\xi}_{n+1}"'] & \Omega^2 X_{n+2} \arrow[r, "\Omega^2\tilde{\xi}_{n+2}"'] & \Omega^3 X_{n+3} \arrow[r, "\Omega^3\tilde{\xi}_{n+3}"'] & \Omega^3 X_{n+4} \arrow[r, "\Omega^4\tilde{\xi}_{n+4}"'] & \cdots \end{tikzcd}$$

induces a map of homotopy colimits. It is homotopic to the map induced by the dashed arrows, and is therefore a weak equivalence. Composing with the weak equivalence from Section 1.7, exercise 24 gives a composite map

$$\operatorname*{hocolim}_{m\to\infty} \Omega^m X_{n+m} \xrightarrow{\sim} \operatorname*{hocolim}_{m\to\infty} \Omega^{1+m} X_{n+1+m} \xrightarrow{\sim} \Omega\Big(\operatorname*{hocolim}_{m\to\infty} \Omega^m X_{n+1+m}\Big).$$

We define the adjunct bonding map $(RX)_n \xrightarrow{\sim} \Omega(RX)_{n+1}$ to be this composite. Since it is a weak equivalence, RX is an Ω-spectrum.

It remains to show that the map $X \to RX$ induces isomorphisms on the stable homotopy groups. To do this we form a commuting grid in which the maps are all induced by $\Omega^m \widetilde{\xi}_{n+m}$:

$$\begin{tikzcd}
\pi_{k+n}(X_n) \arrow[r] \arrow[d] & \pi_{k+n}(\Omega X_{n+1}) \arrow[r] \arrow[d] & \pi_{k+n}(\Omega^2 X_{n+2}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \pi_{k+n}(RX_n) \arrow[d, "\cong"] \\
\pi_{k+n+1}(X_{n+1}) \arrow[r] \arrow[d] & \pi_{k+n+1}(\Omega X_{n+2}) \arrow[r] \arrow[d] & \pi_{k+n+1}(\Omega^2 X_{n+3}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \pi_{k+n+1}(RX_{n+1}) \arrow[d, "\cong"] \\
\pi_{k+n+2}(X_{n+2}) \arrow[r] \arrow[d] & \pi_{k+n+2}(\Omega X_{n+3}) \arrow[r] \arrow[d] & \pi_{k+n+2}(\Omega^2 X_{n+4}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \pi_{k+n+2}(RX_{n+2}) \arrow[d, "\cong"] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d, "\cong"] \\
\pi_k(X) \arrow[r, "\cong"] & A_1 \arrow[r, "\cong"] & A_2 \arrow[r, "\cong"] & \cdots \arrow[r, "\cong"] & \pi_k(RX).
\end{tikzcd}$$

Along the left-hand column, this gives the colimit system defining the stable homotopy groups of X. In each row, it gives the colimit system defining the homotopy groups of a mapping telescope, and therefore gives the homotopy groups of each level $(RX)_{n+k}$. The induced vertical maps between these define the stable homotopy groups of RX itself, so $\pi_k(RX)$ is the colimit of the entire grid.

By the above argument, the maps of colimits on the right-hand side are isomorphisms. By the same argument, the maps along the bottom are isomorphisms. Since the colimits in both directions commute, $\pi_k(RX)$ is the colimit of a system of isomorphisms starting with $\pi_k(X)$. Therefore the map $\pi_k(X) \to \pi_k(RX)$ is an isomorphism. □

Remark 2.2.10. In the literature one sometimes finds **strict Ω-spectra** in which the adjunct maps $X_n \to \Omega X_{n+1}$ are homeomorphisms, see e.g. [LMSM86, EKMM97]. For instance, the fibrant sphere spectrum of Example 2.2.6 is a strict Ω-spectrum. In these cases, X_0 is quite literally a loop space of a loop space of a loop space... and so on ad infinitum. Every spectrum is equivalent to a strict Ω-spectrum, but the replacement requires two steps: we first make the maps $\widetilde{\xi}_n$ into closed inclusions, then we take the strict colimit of the maps $\Omega^m \widetilde{\xi}_{n+m}$, as in Example 2.2.6. Note that this is equivalent to the homotopy colimit construction we used in the proof above.

As in the Whitehead theorem for spaces (Theorem 1.4.12), the results of this section imply that studying spectra up to stable equivalence is the same thing as studying Ω-spectra up to level equivalence. Sometimes this is a good idea, but sometimes it's a better idea to leave our spectra alone – suspension spectra are much easier to think about than their fibrant replacements.

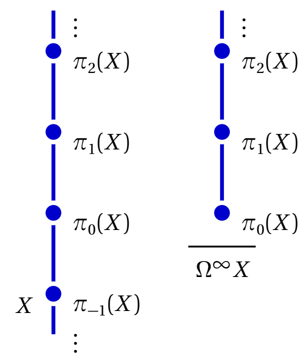
Definition 2.2.11. If X is any spectrum, its **infinite loop space** $\Omega^\infty X$ is the 0th space of any Ω-spectrum RX receiving a stable equivalence from X.

For example, the infinite loop space of the sphere spectrum is

$$\Omega^\infty \mathbb{S} \simeq \Omega^\infty \Sigma^\infty S^0 = \operatorname*{colim}_{n\to\infty} \Omega^n S^n.$$

By the above results,

$$\pi_k(\Omega^\infty X) \cong \pi_k(X), \quad k \geq 0.$$



So the infinite loop space captures everything about X that happens at π_0 and above. It can be visualized by taking the bi-infinite sequence of homotopy groups of a spectrum, and cutting it off below 0 to form a space again.

Remark 2.2.12. The infinite loop space of X is unique up to weak equivalence. We will prove this in Example 3.4.9 after characterizing $\Omega^\infty X$ as the right-derived 0th space of X.

2.3 Operations on spectra

We're ready now to define the basic operations on spectra, just like we did for spaces in Section 1.1.

Definition 2.3.1. Given spectra X and Y, their **wedge sum** or **coproduct** $X \vee Y$ is the spectrum whose nth level is the wedge sum $X_n \vee Y_n$ and whose bonding maps are

$$\Sigma(X_n \vee Y_n) \xleftrightarrow{\cong} (\Sigma X_n) \vee (\Sigma Y_n) \xrightarrow{\xi_n \vee \upsilon_n} X_{n+1} \vee Y_{n+1}.$$

The map on the left is a canonical isomorphism (Example 1.6.11). It arises from the inclusions $\Sigma X_n \to \Sigma(X_n \vee Y_n)$ and $\Sigma Y_n \to \Sigma(X_n \vee Y_n)$.

Similarly, the **product** $X \times Y$ is the spectrum whose nth level is the product $X_n \times Y_n$ and whose bonding maps are

$$\Sigma(X_n \times Y_n) \longrightarrow (\Sigma X_n) \times (\Sigma Y_n) \xrightarrow{\xi_n \times \upsilon_n} X_{n+1} \times Y_{n+1}.$$

The map on the left arises from the projections $\Sigma(X_n \times Y_n) \to \Sigma X_n$ and $\Sigma(X_n \times Y_n) \to \Sigma Y_n$. Alternatively, the adjunct bonding map is the composite

$$X_n \times Y_n \xrightarrow{\widetilde{\xi}_n \times \widetilde{\upsilon}_n} (\Omega X_{n+1}) \times (\Omega Y_{n+1}) \xleftarrow{\cong} \Omega(X_{n+1} \times Y_{n+1}).$$

Example 2.3.2. A wedge sum of suspension spectra is a suspension spectrum,

$$\Sigma^\infty A \vee \Sigma^\infty B \simeq \Sigma^\infty (A \vee B), \qquad \Sigma^\infty_+ A \vee \Sigma^\infty_+ B \simeq \Sigma^\infty_+ (A \amalg B).$$

A product of Eilenberg-Maclane spectra is an Eilenberg-Maclane spectrum,

$$HG_1 \times HG_2 \cong H(G_1 \times G_2).$$

Definition 2.3.3. Given a spectrum X, a **subspectrum** $A \subseteq X$ is a sequence of subspaces $A_n \subseteq X_n$ such that $\xi_n(A_n) \subseteq A_{n+1}$. This makes the spaces A_n into a spectrum as well.

Definition 2.3.4. For any integer $d \in \mathbb{Z}$, the d-fold **shift operator** sends each spectrum X to the spectrum $\operatorname{sh}^d X$ whose nth level is the $(n+d)$th level of X,

$$(\operatorname{sh}^d X)_n = X_{d+n},$$

with the same bonding maps as X. When $n + d < 0$, we define $(\operatorname{sh}^d X)_n = *$. Clearly sh^d has the effect of shifting the homotopy groups,

$$\pi_k(\operatorname{sh}^d X) = \pi_{k-d}(X).$$

Example 2.3.5. The d-sphere $\mathbb{S}^d$ of Example 2.1.9 can be re-interpreted as $\operatorname{sh}^d \mathbb{S}$. Shifting a suspension spectrum $\Sigma^\infty A$ makes a suspension spectrum $\Sigma^\infty \Sigma^d A$ if $d \geq 0$, and a desuspension spectrum $F_d A$ if $d \leq 0$. Shifting an Eilenberg-Maclane spectrum creates a spectrum $\operatorname{sh}^d HG$ in which the only nonzero homotopy group is $\pi_d = G$.

Definition 2.3.6. If X is a spectrum and K is a based space, we form the **tensor** or **smash product** $K \wedge X$ by smashing K with every level of X. So $K \wedge X$ is the spectrum whose nth level is $K \wedge X_n$ and whose bonding maps are

$$K \wedge X_n \wedge S^1 \xrightarrow{\mathrm{id} \wedge \xi_n} K \wedge X_{n+1}.$$

We call this the tensor because it distributes over direct sums, see exercise 10.

As a special case, the **reduced suspension** is $\Sigma X := S^1 \wedge X$. This is just the reduced suspension on each spectrum level, but with the new S^1 is on the left of X, while the bonding maps put an S^1 on the right. If one is less careful and puts them on the same side, it is necessary to apply a shuffle map $S^1 \wedge S^1 \cong S^1 \wedge S^1$ when defining the bonding maps.[1]

[1]If you don't apply the shuffle maps, you do get a well-defined spectrum, it's just hard to relate it to anything else because it's not a special case of the more general construction $K \wedge X$.

We sometimes put the "extra" suspension in bold, to emphasize which one is on the left of X. Then the bonding map is then written as

$$\Sigma\mathbf{\Sigma}X_n \cong \mathbf{\Sigma}\Sigma X_n \xrightarrow{\mathbf{\Sigma}\xi_n} \mathbf{\Sigma}X_{n+1}$$

where the isomorphism is the shuffle that switches the two copies of the circles that are being smashed with X.

Example 2.3.7. The suspension spectrum $\Sigma^\infty A$ is isomorphic to the tensor $A \wedge \mathbb{S}$.

Definition 2.3.8. If X is a spectrum and K is a based space, we form the **cotensor** or **function spectrum** $F(K,X)$ by applying $\mathrm{Map}_*(K,-)$ to every spectrum level of X. So $F(K,X)$ is the spectrum whose nth level is the space of based maps $\mathrm{Map}_*(K,X_n)$. The bonding maps are

$$\mathrm{Map}_*(K,X_n) \xrightarrow{\mathrm{Map}_*(K,\xi_n)} \mathrm{Map}_*(K,\Omega X_{n+1}) \xleftrightarrow{\cong} \Omega\mathrm{Map}_*(K,X_{n+1}).$$

where the last isomorphism is deduced from (1.1.14).

As a special case, the **based loops** are $\Omega X := F(S^1,X)$. This is just the based loops on each spectrum level. As in the suspension example, the bonding maps involve a shuffle:

$$\mathbf{\Omega}X_n \xrightarrow{\mathbf{\Omega}\tilde{\xi}_n} \mathbf{\Omega}\Omega X_{n+1} \cong \Omega\mathbf{\Omega}X_{n+1}.$$

It is an exercise to check that the operations $K \wedge -$ and $F(K,-)$ are adjoint functors on spectra. (Exercise 15.)

Warning 2.3.9. The group A_1 from the end of the proof of Proposition 2.2.9 is *not* canonically identified with $\pi_k(\mathbf{\Omega}X)$. The two colimit systems differ by shuffles applied to the spheres that map to X_{k+n}. The colimits are still isomorphic, but choosing an isomorphism requires us to make choices about how to add in shuffle maps. See also exercise 19c.

Example 2.3.10. For each based finite CW complex X, its **Spanier-Whitehead dual** is the function spectrum $F(X,\mathbb{S})$. This is formally like the linear dual of a vector space, $V^* = \mathrm{Hom}_k(V,k)$, only using the sphere spectrum as the ground ring. It is a theorem of Spanier and Whitehead that when M is a smooth closed manifold, the Thom spectrum $\Sigma^{-\tau}M$ from Example 2.1.19 is stably equivalent to the dual $F(M_+,\mathbb{S})$, see Theorem 4.2.18.

Definition 2.3.11. A **homotopy** of maps of spectra $X \to Y$ is a map $h\colon I_+ \wedge X \to Y$, where $\wedge$ is the tensor from Definition 2.3.6, or equivalently $X \to F(I_+,Y)$, where F is the cotensor from Definition 2.3.8. This is the same as asking for homotopies of based maps $X_n \to Y_n$ that agree along the bonding maps.

Definition 2.3.12. For spectra X and Y, we define the **mapping space** $\mathrm{Map}_*(X,Y)$ as the subspace

$$\mathrm{Map}_*(X,Y) = \left\{ f_n \in \mathrm{Map}_*(X_n, Y_n)\ \forall n\ :\ f_{n+1}\circ\xi_n = \upsilon_n \circ (\Sigma f_n) \right\} \subseteq \prod_{n\geq 0} \mathrm{Map}_*(X_n, Y_n)$$

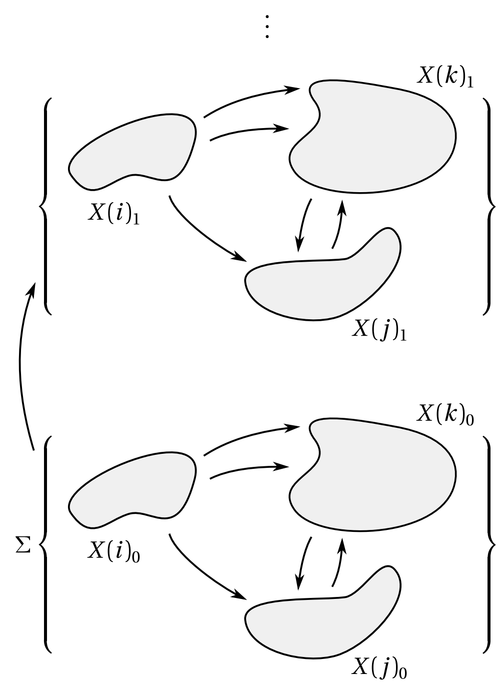
of all tuples of maps (f_n) that commute with the bonding maps of X and Y. This can also be written as an equalizer (i.e. the limit of a diagram of the following form):

$$\mathrm{Map}_*(X,Y) \longrightarrow \prod_{n\geq 0} \mathrm{Map}_*(X_n, Y_n) \rightrightarrows \prod_{m\geq 0} \mathrm{Map}_*(\Sigma X_m, Y_{m+1})$$

Of the two parallel maps, one composes with the bonding map for X, and the other suspends and then composes with the bonding map for Y.

It is an exercise to check that a point in $\mathrm{Map}(X,Y)$ is precisely a map of spectra $X \to Y$, and a path in $\mathrm{Map}(X,Y)$ is a homotopy. More generally, a map $K \to \mathrm{Map}(X,Y)$ corresponds to both a map from the tensor $K \wedge X \to Y$ and a map to the cotensor $X \to F(K,Y)$ (exercise 15).

Let $\mathbf{I}$ be a small category. As in Section 1.1, we say that a **diagram of spectra** indexed by $\mathbf{I}$ is a functor $X\colon \mathbf{I} \to \mathbf{Sp}$. In particular, we get a spectrum $X(i)$ for each object $i \in \mathrm{ob}\,\mathbf{I}$ and a map $X(i) \to X(j)$ for each morphism $i \to j$. This is the same thing as, for each $n \geq 0$, a diagram of spaces $X(i)_n$, along with bonding maps $\Sigma X(i)_n \to X(i)_{n+1}$ commuting every map of the diagram $X(i)_n \to X(j)_n$, as illustrated on the right.



Definition 2.3.13. The **colimit** of the diagram $X\colon \mathbf{I} \to \mathbf{Sp}$ is formed by taking the based colimit from (1.1.11) at each spectrum level n. The structure map is

$$\Sigma \operatorname*{colim}_{\mathbf{I}}{}^{(b)} X(i)_n \xleftrightarrow{\ \cong\ } \operatorname*{colim}_{\mathbf{I}}{}^{(b)} \Sigma X(i)_n \xrightarrow{\operatorname*{colim}_{\mathbf{I}}^{(b)} \xi(i)_n} \operatorname*{colim}_{\mathbf{I}}{}^{(b)} X_{n+1},$$

where the canonical isomorphism is as in Example 1.6.11. Usually we drop the (b) decoration.

Example 2.3.14. The wedge sum of Definition 2.3.1 is an example of a colimit of spectra. Another example is the **pushout** $X \cup_A Y$, formed from two maps of spectra $A \to X$, $A \to Y$ by taking the pushout at each spectrum level, $X_n \cup_{A_n} Y_n$.

Definition 2.3.15. Similarly the **limit** of the diagram $X\colon \mathbf{I} \to \mathbf{Sp}$ is formed by taking the limit (1.1.3) at each spectrum level. The adjunct structure map is

$$\lim_{\mathbf{I}} X(i)_n \xrightarrow{\lim \tilde{\xi}(i)_n} \lim_{\mathbf{I}} \Omega X(i)_{n+1} \xleftrightarrow{\cong} \Omega \lim_{\mathbf{I}} X(i)_{n+1}.$$

Example 2.3.16. The product from Definition 2.3.1 is an example of a limit. Another example is the **pullback** $X \times_B Y$, formed from two maps of spectra $X \to B$, $Y \to B$ by taking the pullback at each spectrum level, $X_n \times_{B_n} Y_n$.

It is an exercise to check that the spectra defined in Definition 2.3.13 and Definition 2.3.13 have the universal property of the colimit, respectively the limit, in the category of spectra (see exercise 13).

Definition 2.3.17. For any diagram of spectra $X\colon \mathbf{I} \to \mathbf{Sp}$, the **homotopy colimit** is formed by taking the based homotopy colimit at each spectrum level n. The bonding map is

$$\Sigma \operatorname*{hocolim}_{\mathbf{I}}^{(b)} X(i)_n \xleftrightarrow{\cong} \operatorname*{hocolim}_{\mathbf{I}}^{(b)} \Sigma X(i)_n \xrightarrow{\operatorname*{hocolim}_{\mathbf{I}}^{(b)} \xi(i)_n} \operatorname*{hocolim}_{\mathbf{I}}^{(b)} X_{n+1}.$$

We usually we drop the (b) decoration.

Since we have not introduced general homotopy colimits yet, we focus on the basic examples. The **homotopy pushout** of a diagram of spectra

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array}$$

is formed at each level as the based mapping cylinder,

$$(X \cup_A^h Y)_n = X_n \cup_A^h Y_n = X_n \cup_{(A_n \wedge \{0\}_+)} (A_n \wedge I_+) \cup_{(A_n \wedge \{1\}_+)} Y_n.$$

The homotopy colimit of a sequence of maps of spectra

$$X(0) \xrightarrow{f_1} X(1) \xrightarrow{f_2} X(2) \longrightarrow \dots$$

is formed at each level as the based mapping telescope,

$$\left(\operatorname*{hocolim}_{\mathbf{I}}^{(b)} X(i)\right)_n = \bigvee_{k \geq 0} (X(k)_n \wedge I_+) / \big((x_k, 1) \sim (f_{k+1}(x_k), 0)\big).$$

Remark 2.3.18. One might worry, because of Lemma 1.5.15, that we need to assume the levels of these spectra are well-based spaces, so that the based homotopy colimit has the correct homotopy type. It is a minor miracle that this isn't necessary: in spectra, this model of the homotopy colimit always has the correct homotopy type, up to *stable* equivalence. This will be proven in **??**.

Example 2.3.19. For any map of spectra $f: X \to Y$, the homotopy pushout $* \cup^h_X Y$ is called the **homotopy cofiber** Cf. At each spectrum level, it is the based mapping cone from Section 1.2,

$$(Cf)_n = C(f_n) = CX_n \cup_{X_n} Y_n.$$

This can also be described as $(I \wedge X_n) \cup_{X_n} Y_n$, if consider the interval I to be a based space with basepoint 0. The bonding maps arise from commuting Σ with the mapping cone.

Definition 2.3.20. For any diagram of spectra $X: \mathbf{I} \to \mathbf{Sp}$, the **homotopy limit** is formed by replacing all the $X(i)$ by Ω-spectra $RX(i)$, then taking the homotopy limit at each spectrum level n. The adjunct bonding map is

$$\operatorname*{holim}_{\mathbf{I}} (RX(i))_n \xrightarrow{\sim} \operatorname*{holim}_{\mathbf{I}} \Omega(RX(i))_{n+1} \xleftarrow{\cong} \Omega \operatorname*{holim}_{\mathbf{I}} (RX(i))_{n+1}.$$

Warning 2.3.21. We could define a spectrum by just taking the homotopy limit at each spectrum level:

$$\operatorname*{holim}_{\mathbf{I}} X(i)_n \xrightarrow{\lim \widetilde{\xi}(i)_n} \operatorname*{holim}_{\mathbf{I}} \Omega X(i)_{n+1} \xleftarrow{\cong} \Omega \operatorname*{holim}_{\mathbf{I}} X(i)_{n+1}.$$

However, this is only the homotopy limit in spectra in the following cases:

- The spectra $X(i)$ are all Ω-spectra, or
- the category $\mathbf{I}$ is homotopy finite, meaning it has finitely many objects and finitely many strings of composable non-identity morphisms.

If neither of these conditions hold, then this is not the correct way to define the homotopy limit. The issue is that *stable* equivalences $X(i) \to Y(i)$ don't necessarily give a stable equivalence on this spectrum. In particular, an infinite product of spectra sometimes has the wrong homotopy type. (See exercise 25.)

In particular, the **homotopy pullback** of a diagram of spectra

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

is formed at each level as the homotopy fiber product,

$$(X \times^h_B Y)_n = X_n \times^h_B Y_n = X_n \times_{B_n} B_n^I \times_{B_n} Y_n.$$

This will always have the correct homotopy type.

Example 2.3.22. For any map of spectra $f: X \to Y$, the homotopy pullback $* \times^h_Y X$ is called the **homotopy fiber** Ff. At each spectrum level, it is the pullback

$$(Ff)_n = F(f_n) = X_n \times_{Y_n} PY_n$$

where $PY_n = \mathrm{Map}_*(I, Y_n)$ is the **based path space**, the space of paths in Y_n whose endpoint $0 \in I$ is sent to the basepoint of Y_n. So a point in Ff_n is a point in X_n, and a path from its image in Y_n to the basepoint of Y_n.

Definition 2.3.23. For spectra X and Y, define the **handicrafted smash product** $X \wedge Y$ by

$$(X \wedge Y)_n = X_{p_n} \wedge Y_{q_n},$$

for any sequence of pairs of the form (p_n, q_n) with the following conditions: $(p_0, q_0) = (0,0)$, at every stage either p_n or q_n increases by one, and both p_n and q_n increase without bound as $n \to \infty$. The bonding maps are as follows:

$$X_p \wedge Y_q \wedge S^1 \xrightarrow{\mathrm{id} \wedge \upsilon_q} X_p \wedge Y_{q+1}$$

$$X_p \wedge Y_q \wedge S^1 \xrightarrow{\cong} X_p \wedge S^1 \wedge Y_q \xrightarrow{(-1)^q} X_p \wedge S^1 \wedge Y_q \xrightarrow{\xi_p \wedge \mathrm{id}} X_{p+1} \wedge Y_q,$$

where $(-1)^q$ is the map $S^1 \to S^1$ that flips the circle if q is odd.

Remark 2.3.24. This version of the smash product is not well-defined – it depends on the choice of sequence (p_n, q_n). Any two sequences turn out to give stably equivalent results, but the proof of this fact is complicated, and will not be used in this book. Instead, in Chapter 6 we will pass to an equivalent model of spectra where this smash product becomes well-defined and independent of choices.

The sign in Definition 2.3.23 can be explained as follows. Consider all possible smash products of the levels of X and Y:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
| & & | & & | & & \\
X_0 \wedge Y_2 & \text{——} & X_1 \wedge Y_2 & \text{——} & X_2 \wedge Y_2 & \text{——} & \dots \\
| & & | & & | & & \\
X_0 \wedge Y_1 & \text{——} & X_1 \wedge Y_1 & \text{——} & X_2 \wedge Y_1 & \text{——} & \dots \\
| & & | & & | & & \\
X_0 \wedge Y_0 & \text{——} & X_1 \wedge Y_0 & \text{——} & X_2 \wedge Y_0 & \text{——} & \dots
\end{array}$$

The suspension of each space in this grid maps to the space above it, and also to the space to its right. However, if we traverse the grid using different choices of sequence (p_n, q_n), the different routes do not commute, for reasons of applying the suspensions in a different order. The sign convention fixes this.

As a result, if we take the maps on homotopy groups induced by these bonding maps, the resulting grid of homotopy groups commutes. (The (−) decorations denote those maps where the flip $S^1 \to S^1$ is applied in the definition of the bonding map. For $k < 0$, the grid is only defined once we move sufficiently far up or to the right.)

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \text{colim} = \pi_k(X \wedge Y) \\
\pi_{k+3}(X_0 \wedge Y_3) \arrow[u, "+"'] \arrow[r, "-"] & \pi_{k+4}(X_1 \wedge Y_3) \arrow[u, "+"'] \arrow[r, "-"] & \pi_{k+5}(X_2 \wedge Y_3) \arrow[u, "+"'] \arrow[r, "-"] & \cdots \\
\pi_{k+2}(X_0 \wedge Y_2) \arrow[u, "+"'] \arrow[r, "+"] & \pi_{k+3}(X_1 \wedge Y_2) \arrow[u, "+"'] \arrow[r, "+"] & \pi_{k+4}(X_2 \wedge Y_2) \arrow[u, "+"'] \arrow[r, "+"] & \cdots \\
\pi_{k+1}(X_0 \wedge Y_1) \arrow[u, "+"'] \arrow[r, "-"] & \pi_{k+2}(X_1 \wedge Y_1) \arrow[u, "+"'] \arrow[r, "-"] & \pi_{k+3}(X_2 \wedge Y_1) \arrow[u, "+"'] \arrow[r, "-"] & \cdots \\
\pi_{k}(X_0 \wedge Y_0) \arrow[u, "+"'] \arrow[r, "+"] & \pi_{k+1}(X_1 \wedge Y_0) \arrow[u, "+"'] \arrow[r, "+"] & \pi_{k+2}(X_2 \wedge Y_0) \arrow[u, "+"'] \arrow[r, "+"] & \cdots
\end{tikzcd}$$

The choice of sequence (p_n, q_n) gives a path through this grid that is cofinal, meaning any other term in the grid maps to something in this path. It follows that $\pi_k(X \wedge Y)$ is isomorphic to the colimit of this grid of abelian groups.

Example 2.3.25. The smash product of a spectrum X with a suspension spectrum $\Sigma^\infty K$ is stably equivalent to the tensor,

$$(\Sigma^\infty K) \wedge X \simeq K \wedge X.$$

The smash product of two suspension spectra is also equivalent to a suspension spectrum,

$$(\Sigma^\infty A) \wedge (\Sigma^\infty B) \simeq \Sigma^\infty (A \wedge B).$$

More generally, the smash product of two shift desuspensions is equivalent to the shift desuspension,

$$(F_d A) \wedge (F_e B) \simeq F_{d+e}(A \wedge B).$$

Remark 2.3.26. The constructions in this section are all functors, meaning that they can also be applied to maps of spectra or maps of diagrams of spectra. For instance, a map $X \to X'$ will also give a map on the suspensions $\Sigma X \to \Sigma X'$, and a pair of maps $X \to X'$, $Y \to Y'$ gives a map $X \vee Y \to X' \vee Y'$. The definition of these maps is usually obvious, so we don't spell them out explicitly.

2.4 Stability theorems

Homotopy theorists often find that spectra are easier to work with than spaces. The reason for this is "stability," a property similar to excision for homology. Stability can be formulated in a few different ways:

- Σ and Ω are inverses up to stable equivalence.
- Cofiber and fiber sequences of spectra are the same, up to stable equivalence.
- Homotopy pushout and homotopy pullback squares of spectra are the same, up to stable equivalence.

In this section, we prove these claims and deduce some standard corollaries.

2.4.1 Suspension and loops

Proposition 2.4.1. *For any spectrum X there are natural isomorphisms*

$$\pi_{k+1}(\mathbf{\Sigma}X) \cong \pi_k(X) \cong \pi_{k-1}(\mathbf{\Omega}X).$$

Proof. We take the colimit defining $\pi_k(X)$ in Definition 2.1.2, and restrict to the terms of the form $\pi_{k+2n+1}(\Sigma X) = \pi_{k+2n+1}(X_{2n} \wedge S^1)$. We similarly take the colimit defining $\pi_{k+1}(S^1 \wedge X)$ and restrict to the terms $\pi_{k+1+2n}(S^1 \wedge X_{2n})$. We form an isomorphism between the colimit systems by the symmetry isomorphism $X_{2n} \wedge S^1 \cong S^1 \wedge X_{2n}$.

To check this isomorphism commutes with the maps of the colimit system, it suffices to show the following diagram commutes up to homotopy.

$$\begin{tikzcd}
X_{2n} \wedge S^1 \wedge S^1 \wedge S^1 \arrow[r, "\xi_{2n}\wedge\mathrm{id}\wedge\mathrm{id}"] \arrow[d, "\cong"] & X_{2n} \wedge S^1 \wedge S^1 \arrow[r, "\xi_{2n}\wedge\mathrm{id}"] & X_{2n} \wedge S^1 \arrow[d, "\cong"] \\
S^1 \wedge X_{2n} \wedge S^1 \wedge S^1 \arrow[r, "\mathrm{id}\wedge\xi_{2n}\wedge\mathrm{id}"] & S^1 \wedge X_{2n} \wedge S^1 \arrow[r, "\mathrm{id}\wedge\xi_{2n}"] & S^1 \wedge X_{2n}.
\end{tikzcd}$$

The two branches do not strictly agree, because different copies of S^1 are "fed" into the bonding maps ξ. However, the two branches differ by a self-map of $S^3 = S^1 \wedge S^1 \wedge S^1$ that applies a 3-cycle to the copies of S^1. As a map $S^3 \to S^3$, this has degree 1, so it is homotopic to the identity. Therefore the diagram commutes up to homotopy, which is enough to conclude it commutes on homotopy groups.

The isomorphism $\pi_k(X) \cong \pi_{k-1}(\mathbf{\Omega}X)$ is constructed in the same way, using the version of the colimit system from Lemma 2.2.4. □

Remark 2.4.2. One can use this to prove that that $\mathbf{\Sigma}X \simeq \mathrm{sh}\, X$. See exercise 19.

Corollary 2.4.3. *For a map of spectra $f : X \to Y$, the following are equivalent:*

- *$f : X \to Y$ is a stable equivalence.*
- *$\Sigma f : \Sigma X \to \Sigma Y$ is a stable equivalence.*
- *$\Omega f : \Omega X \to \Omega Y$ is a stable equivalence.*

Proof. The isomorphisms of Proposition 2.4.1 are natural, meaning that f induces commuting diagrams

$$\begin{tikzcd}
\pi_{n+1}(\mathbf{\Sigma} X) \arrow[d, "(\Sigma f)_*"] & \pi_n(X) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "f_*"] & \pi_{n-1}(\mathbf{\Omega} X) \arrow[d, "(\Omega f)_*"] \\
\pi_{n+1}(\mathbf{\Sigma} Y) & \pi_n(Y) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \pi_{n-1}(\mathbf{\Omega} Y).
\end{tikzcd}$$

Therefore f_* is an isomorphism for all n iff $(\Sigma f)_*$ is an isomorphism for all n, iff $(\Omega f)_*$ is an isomorphism for all n. □

Remark 2.4.4. If $f : A \to B$ is a weak equivalence of spaces, then $\Omega f : \Omega A \to \Omega B$ is also weak equivalence, but $\Sigma f : \Sigma A \to \Sigma B$ is only a weak equivalence if A and B are well-based. So it is surprising that Corollary 2.4.3 doesn't require us to assume that the levels X_n and Y_n are well-based.

Corollary 2.4.5. *There are natural stable equivalences $X \to \Omega\Sigma X$ and $\Sigma\Omega X \to X$.*

In fact, these are the unit and counit maps of the adjunction (Σ, Ω) on the category of spectra.

Proof. If we compose together the two isomorphisms from Proposition 2.4.1

$$\pi_k(X) \cong \pi_{k+1}(\mathbf{\Sigma} X) \cong \pi_k(\mathbf{\Omega\Sigma} X),$$

we see that it takes each map $S^{k+2n} \to X_{2n}$ to its suspension $\Sigma S^{k+2n} \to \Sigma X_{2n}$, then its adjunct $S^{k+2n} \to \Omega\Sigma X_{2n}$. This is the same as the composite

$$S^{k+2n} \longrightarrow X_{2n} \longrightarrow \Omega\Sigma X_{2n}$$

where the second map is the unit of the adjunction (Σ, Ω). Therefore the unit map is a stable equivalence. The proof for the counit map is the same but composes the isomorphisms in a different order, $\pi_k(X) \cong \pi_{k-1}(\mathbf{\Omega} X) \cong \pi_k(\mathbf{\Sigma\Omega} X)$. □

Therefore Σ is invertible up to stable equivalence, and its inverse is Ω.

2.4.2 Cofiber and fiber sequences

Next we show that cofiber and fiber sequences of spectra coincide. We first give the two definitions.

Definition 2.4.6. Informally, a **cofiber sequence** is anything of the form

$$X \xrightarrow{f} Y \longrightarrow Cf,$$

where Cf is the homotopy cofiber from Example 2.3.19.

A **fiber sequence** is anything of the form

$$Fg \longrightarrow Y \xrightarrow{g} Z,$$

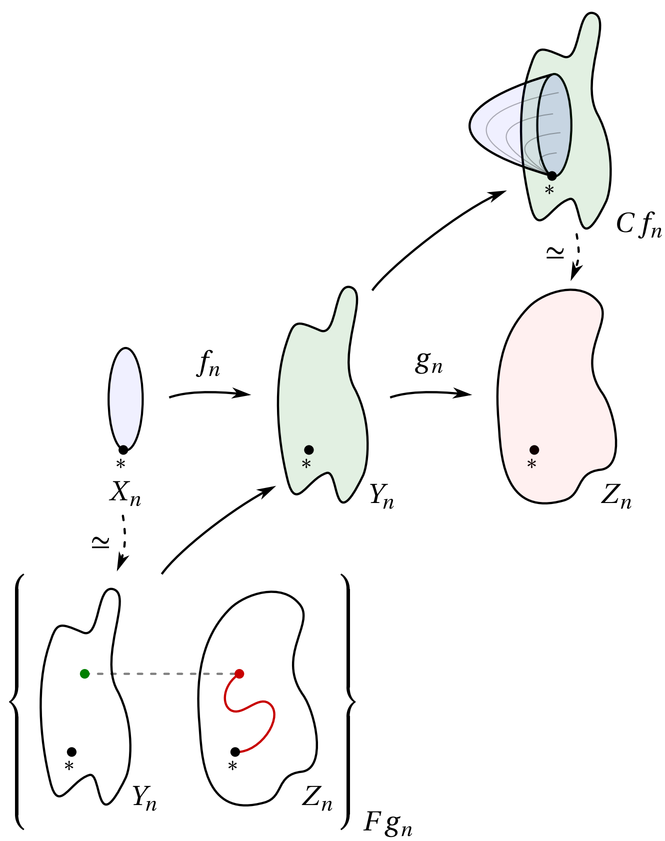
where Fg is the homotopy fiber from Example 2.3.22.

To define these more formally, consider three spectra X, Y, Z, two maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

and a homotopy $h\colon X \wedge I \to Z$ from the zero map to $g \circ f$. This homotopy induces maps

$$h \cup g\colon Cf \longrightarrow Z, \qquad f \times h\colon X \longrightarrow Fg.$$



We say that (X,Y,Z,f,g,h) is a **cofiber sequence** if $Cf \to Z$ is a stable equivalence, and a **fiber sequence** if $X \to Fg$ is a stable equivalence.

A **map of cofiber or fiber sequences** consists of three maps $X \to X'$, $Y \to Y'$, and $Z \to Z'$ commuting with f, g, and h.

Remark 2.4.7. There are many other equivalent ways to define cofiber and fiber sequences. For instance, we could ask for any stable equivalence $Cf \to Z$, instead of asking for one that comes from a homotopy. See exercise 22 for another equivalent definition.

One benefit of our definition is that both notions have the same data, only the condition is different. This makes it easier to prove that cofiber and fiber sequences coincide (Proposition 2.4.12).

Lemma 2.4.8. *For each fiber sequence we may construct a long exact sequence*

$$\dots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \dots$$

so that maps of fiber sequences give maps of long exact sequences.

Proof. Without loss of generality, $X \to Fg$ is an isomorphism. Then for each n, the fiber sequence of spaces $X_n \to Y_n \to Z_n$ gives a three-term exact sequence

$$\pi_{k+n}(X_n) \xrightarrow{f_*} \pi_{k+n}(Y_n) \xrightarrow{g_*} \pi_{k+n}(Z_n).$$

Sequential colimits preserve exact sequences, so this passes to a three-term sequence of stable homotopy groups

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z)$$

that is exact at $\pi_k(Y)$. Finally, just as in Proposition 1.2.14, we can continue to take homotopy fibers to get two more fiber sequences

$$\Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y, \qquad \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{\partial} X.$$

Repeating the same argument with these two sequences and identifying $\pi_k(\Omega Z) \cong \pi_{k+1}(Z)$ shows that the desired sequence is exact. (Strictly speaking, we get a sequence in which some signs are negated, but this does not change exactness.) $\square$

Lemma 2.4.9. *For each cofiber sequence we may construct a long exact sequence*

$$\dots \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \dots$$

so that maps of cofiber sequences give maps of long exact sequences.

Remark 2.4.10. This lemma, unlike the previous one, is not true for spaces. It only holds for highly connected spaces in a range, by the homotopy excision theorem.

Proof. Again we may assume that $Cf \to Z$ is an isomorphism. As in the previous lemma, it suffices to prove that

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Cf)$$

is exact in the middle. It is clear that the composite is zero. Going the other way, suppose $\alpha \in \pi_k(Y)$ and $g_*(\alpha)=0$ in $\pi_k(Cf)$. An element that is zero in the colimit must attain zero at some finite stage, so there is a representative $\alpha_n \in \pi_{k+n}(Y_n)$ such that $(g_n)_*(\alpha_n)=0$ in $\pi_{k+n}(Cf_n)$. Concretely, this means the composite

$$S^{k+n} \xrightarrow{\alpha_n} Y_n \xrightarrow{g_n} Cf_n$$

is nullhomotopic. It therefore extends to a map $\overline{\alpha}_n\colon CS^{k+n} \to Cf_n$. Now define β_{n+1} to be the composite

$$S^{k+n+1} \xleftrightarrow{\cong} CS^{k+n} \cup_{S^{k+n}} CS^{k+n} \xrightarrow{C\alpha_n \cup \overline{\alpha}_n} CY_n \cup_{Y_n} Cf_n \xrightarrow{\sim} \Sigma X_n \xrightarrow{\xi_n} X_{n+1}$$

where the second-to-last map is the usual identification in the Puppe sequence that collapses CY_n to a point. Let $\beta \in \pi_k(X)$ be the corresponding element in the stable homotopy of X. The proof will be complete once we show that $f_*(\beta)=\pm\alpha$.

To prove this, consider the map $\varphi\colon Cf_n \to CY_n$ that applies f_n to the cone on X_n. This map fits into a commuting diagram

$$\begin{tikzcd}
CS^{k+n} \cup_{S^{k+n}} CS^{k+n} \arrow[r, "C\alpha_n \cup \overline{\alpha}_n"] & CY_n \cup_{Y_n} Cf_n \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\cup\varphi"] & \Sigma X_n \arrow[r, "\xi_n"] \arrow[d, "\Sigma f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\
& CY_n \cup_{Y_n} CY_n \arrow[r, "\sim"] & \Sigma Y_n \arrow[r, "\upsilon_n"] & Y_{n+1}
\end{tikzcd}$$

in which both $\sim$ maps collapse the first CY_n. The composite along the bottom route of the diagram is $\upsilon_n \circ \Sigma\alpha_n = \alpha_{n+1}$, possibly up to a flip of the suspension coordinate. Therefore $f_*(\beta)=\pm\alpha$. □

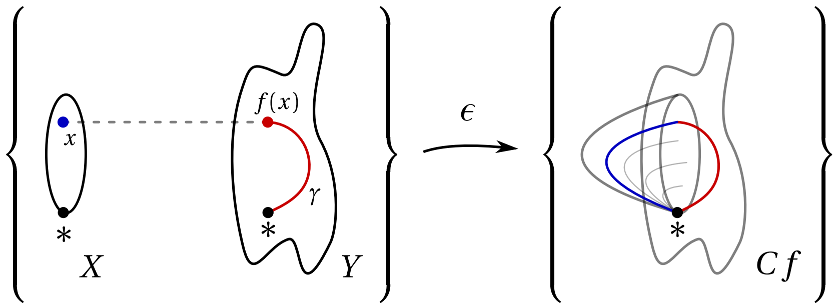
Lemma 2.4.11. *For each map of spectra $f\colon X \to Y$ there is a natural stable equivalence $\epsilon\colon \Sigma Ff \to Cf$, or equivalently $\widetilde{\epsilon}\colon Ff \to \Omega Cf$.*

Proof. Inside this proof, we use the definition $S^1 = I/\{0,1\}$, so that the reduced suspension ΣX is a quotient of $I \times X$. We define $\epsilon\colon \Sigma Ff \to Cf$ by the formula

$$\epsilon : \Sigma(X \times_Y PY) \longrightarrow Y \cup_X CX$$

$$\epsilon(t,x,\gamma) = \begin{cases} \gamma(2t) & t \le 1/2 \\ (x, 2-2t) & t \ge 1/2. \end{cases}$$

In other words, given a point $x \in X_n$ and a path $\gamma\colon I \to Y_n$ from $*$ to $f(x)$, we create a loop in Cf_n by composing the path γ in Y_n with the path that traverses $I \times \{x\}$ in the cone CX_n.



We check that the following squares of spectra commute up to homotopy, where the top row is the suspension of the fiber Puppe sequence for f, and the bottom row is the cofiber Puppe sequence for f, and the vertical maps on the left are the counit maps from Corollary 2.4.5.

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma(X \times_Y PY) \arrow[r] \arrow[d, "\epsilon"] & \Sigma X \arrow[r, "\Sigma f"] \arrow[d, "\mathrm{flip}"] & \Sigma Y \arrow[d, "\mathrm{flip}"] \\
X \arrow[r, "f"] & Y \arrow[r] & Y \cup_X CX \arrow[r] & \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y
\end{tikzcd}$$

The top row forms an exact sequence on stable homotopy groups, by Lemma 2.4.8 and Proposition 2.4.1. The bottom row forms an exact sequence on stable homotopy groups by Lemma 2.4.9. The two vertical maps to the left of ϵ are π_*-isomorphisms by Corollary 2.4.5, while the two maps to the right are π_*-isomorphisms by direct inspection. By the five-lemma, ϵ is a π_*-isomorphism as well. □

Proposition 2.4.12. *The data (X,Y,Z,f,g,h) describes a cofiber sequence iff it describes a fiber sequence.*

Proof. We check that the squares in the following diagram commute up to homotopy, and that the composite $\epsilon \circ \Sigma(f \times h)$ is equal to $h \cup Cf$.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & CX \cup_X Y \arrow[r] \arrow[d, "h\cup g"] & CX \cup_X CX \arrow[r, "\Sigma f"] \arrow[d, "h\cup Cf"] \arrow[dd, bend right=60, "\Sigma(f\times h)"'] & \Sigma Y \arrow[d, equal] \\
Y \arrow[r, "g"] & Z \arrow[r] & Z \cup_Y CY \arrow[r] & \Sigma Y \\
 & & \Sigma(Y \times_Z F(I,Z)) \arrow[u, "\epsilon"', "\simeq"] &
\end{tikzcd}$$

Each row describes a long exact sequence of homotopy groups, so we get the implications

$$\begin{aligned}
(X,Y,Z,f,g,h) \text{ is a cofiber sequence} &\Leftrightarrow h \cup g \text{ is a stable equivalence} \\
&\Leftrightarrow h \cup Cf \text{ is a stable equivalence} \\
&\Leftrightarrow \Sigma(f \times h) \text{ is a stable equivalence} \\
&\Leftrightarrow f \times h \text{ is a stable equivalence} \\
&\Leftrightarrow (X,Y,Z,f,g,h) \text{ is a fiber sequence.}
\end{aligned}$$

□

2.4.3 Pushout and pullback squares

Definition 2.4.13. Given a commuting square of spectra

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D,
\end{tikzcd}$$

we say it is a **homotopy pushout square** if the induced map $B \cup_A^h C \to D$ is a stable equivalence. It is a **homotopy pullback square** if the induced map $A \to B \times_D^h C$ is a stable equivalence.

The following is left as an exercise (exercise 20).

Lemma 2.4.14. *A square is homotopy pushout iff the induced map of homotopy cofibers $Cf \to Cg$ is a stable equivalence. It is a homotopy pullback iff the induced map of homotopy fibers $Ff \to Fg$ is a stable equivalence.*

Of couse, since the definition of homotopy pushout is symmetric, the homotopy cofibers and fibers could be taken along the vertical maps h and k instead. Combining Lemma 2.4.11 and Lemma 2.4.14 gives:

Corollary 2.4.15. *A commuting square of spectra is homotopy pushout iff it is homotopy pullback.*

This finishes the proof of the stability theorems. We conclude with some corollaries.

2.4.4 Wedge sums are equivalent to products

Let X and Y be spectra. The inclusions $X_n \vee Y_n \subseteq X_n \times Y_n$ give a map of spectra

$$X \vee Y \longrightarrow X \times Y. \tag{2.4.16}$$

Composing with the inclusions of X and Y into $X \vee Y$, and the projection of the product $X \times Y$ onto each of its factors, we get maps on homotopy groups

$$\pi_k(X) \oplus \pi_k(Y) \longrightarrow \pi_k(X \vee Y) \longrightarrow \pi_k(X \times Y) \longrightarrow \pi_k(X) \times \pi_k(Y). \tag{2.4.17}$$

Proposition 2.4.18. *All three maps in* (2.4.17) *are isomorphisms. Therefore the inclusion* $X \vee Y \to X \times Y$ *from* (2.4.16) *is a stable equivalence.*

Proof. The last map is left to exercise 25. It suffices to prove the first map is an isomorphism, since the composite of all three maps clearly is. Observe that

$$C(X \to X \vee Y) \to Y$$

is a level equivalence of spectra, and therefore there is a cofiber sequence of the form

$$X \longrightarrow X \vee Y \longrightarrow Y.$$

The second of these maps is split by the inclusion $Y \to X \vee Y$, and therefore the associated long exact sequence is split. We conclude that $\pi_k(X) \oplus \pi_k(Y) \to \pi_k(X \vee Y)$ is an isomorphism. □

2.4.5 Retracts are summands

Suppose that X is a spectrum, and A is a retract of X, in the rather weak sense that we can find maps

$$A \xrightarrow{i} X \xrightarrow{p} A'$$

whose composite is a stable equivalence. Then A is a summand of X,

$$X \simeq A \vee (?).$$

To make this precise, let Fp be the homotopy fiber of p, and $h\colon I \wedge Fp \to A'$ the canonical homotopy from the composite $Fp \xrightarrow{j} X \xrightarrow{p} A'$ to the zero map.

Proposition 2.4.19. *The map $j \vee i \colon Fp \vee A \longrightarrow X$ is a stable equivalence.*

Proof. We form the map of cofiber/fiber sequences

$$\begin{tikzcd}
Fp \arrow[r] \arrow[d, "="] & Fp \vee A \arrow[r] \arrow[d, "j\vee i"] & (I \wedge Fp) \vee A \arrow[d, "h\vee(p\circ i)"] \\
Fp \arrow[r, "j"] & X \arrow[r, "p"] & A'.
\end{tikzcd}$$

Both squares commute. In addition, the nullhomotopy h for the top composite is carried to the nullhomotopy for the bottom composite. We therefore get a commuting map of long exact sequences of homotopy groups. Since $I \wedge Fp$ is contractible, the two outside vertical maps are π_*-isomorphisms, therefore so is the map in the middle. □

The complementary summand could also be described as Ci, so that $X \simeq A \vee Ci$, see exercise 23. Using Proposition 2.4.18, we conclude:

Corollary 2.4.20. *If X contains A as a retract, then*

$$\pi_*(X) \cong \pi_*(A) \oplus \pi_*(Fp) \cong \pi_*(A) \oplus \pi_*(Ci).$$

2.4.6 Smash products and cofiber sequences

Finally we explain how the smash product interacts with cofiber sequences.

Lemma 2.4.21. *If $X \to Y \to Cf$ is a cofiber sequence of spaces or spectra, smashing with a space or spectrum W produces another cofiber sequence*

$$X \wedge W \to Y \wedge W \to Cf \wedge W.$$

Dually, the cotensors $F(-,-)$ also preserve cofiber and fiber sequences, see exercise 30.

Proof. For spectra, this quickly reduces to checking the statement for spaces. It suffices to check that the operation $- \wedge W$ commutes with the formation of the based homotopy cofiber Cf. This occurs because $- \wedge W$ is a left adjoint, so it preserves pushouts (Example 1.6.11), and it also commutes with smash products because the smash product is associative:

$$\begin{aligned}
C(X \wedge W \to Y \wedge W) &= (I \wedge X \wedge W) \cup_{(X \wedge W)} (Y \wedge W) \\
&\cong \big((I \wedge X) \cup_X Y\big) \wedge W \\
&= C(X \to Y) \wedge W.
\end{aligned}$$

See also Section 1.7, exercise 26. □

Remark 2.4.22. If $X \to Y \to Z$ is a cofiber sequence and Z is only equivalent to Cf, not isomorphic, then we might want

$$X \wedge W \to Y \wedge W \to Z \wedge W$$

to be a cofiber sequence too. This will work, but we have to be a little careful – we should make sure that $Z \wedge W$ is also equivalent to $Cf \wedge W$. This will happen so long as all the spaces or spectrum levels are well-based.

The statement of Lemma 2.4.21 doesn't require anything to be well-based to work. This has the following surprising consequence:

Corollary 2.4.23. *If $f: X \to Y$ is a stable equivalence of spectra, with no conditions on basepoints, and K is a based CW complex, then $K \wedge X \to K \wedge Y$ is also a stable equivalence.*

Proof. Let $K^{(n)}$ be the nth level of the skeletal filtration of K. Inductively, we show that

$$\mathrm{id} \wedge f: K^{(n)} \wedge X \to K^{(n)} \wedge Y$$

is a stable equivalence. For $n = 0$, this is a wedge of copies of f, so it follows from exercise 29. For larger n, we take the cofiber sequence

$$K^{(n-1)} \xrightarrow{i} K^{(n)} \longrightarrow Ci \cong \bigvee_\alpha S^n$$

and smash with X and Y to get two cofiber sequences, using Lemma 2.4.21:

$$\begin{tikzcd}
K^{(n-1)} \wedge X \arrow[r, "i\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge f", "\sim"'] & K^{(n)} \wedge X \arrow[r] \arrow[d, "\mathrm{id}\wedge f"] & \bigvee_\alpha \Sigma^n X \arrow[d, "\Sigma^n f", "\sim"'] \\
K^{(n-1)} \wedge Y \arrow[r, "i\wedge\mathrm{id}"] & K^{(n)} \wedge Y \arrow[r] & \bigvee_\alpha \Sigma^n Y
\end{tikzcd}$$

This induces a map of long exact sequences by Lemma 2.4.9. Since the outside maps are stable equivalences, so is the map in the middle.

Finally, by exercise 27 the stable homotopy groups of $K \wedge X$ are the colimit of the stable homotopy groups of $K^{(n)} \wedge X$. Since a colimit of isomorphisms is an isomorphism, the induced map $\pi_*(K \wedge X) \to \pi_*(K \wedge Y)$ is an isomorphism. □

2.5 Extraordinary homology and cohomology

2.5.1 Extraordinary homology

Recall that a CW pair (X, A) is a choice of CW complex X and subcomplex $A \subseteq X$. A map of pairs $(X, A) \to (Y, B)$ is a continuous map $X \to Y$ sending A into B.

Definition 2.5.1. An **extraordinary homology theory** is any sequence of functors $E_k(-,-)$ from CW pairs to abelian groups, along with a natural "boundary map"

$$E_k(X,A) \xrightarrow{\delta} E_{k-1}(A,\emptyset),$$

satisfying the first four Eilenberg-Steenrod axioms (Propositions 1.3.1–1.3.4), but not necessarily the fifth axiom (Proposition 1.3.5). The **coefficient group** is the graded abelian group $E_*(*,\emptyset)$.

For instance, bordism theory $\mathfrak{N}_k(X)$ from Example 0.1.1 is an extraordinary homology theory. Stable homotopy groups $\pi_k(\Sigma^\infty_+ X)$ are another.

Definition 2.5.2. For an unbased space X, the **unreduced E-homology** of X is

$$E_k(X) = E_k(X,\emptyset).$$

For a based space X, the **reduced E-homology** of X is

$$E_k(X) = E_k(X,*)$$

Reduced homology is sometimes denoted $\widetilde{E}_k(X)$ to distinguish it from unreduced homology, but it is more common to use the same notation for both, using the word "unreduced" or "reduced" to indicate which one we mean.

Lemma 2.5.3. *There is a canonical splitting for based spaces X*

$$E_k(X,\emptyset) \cong E_k(X,*) \oplus E_k(*,\emptyset),$$

therefore the unreduced and reduced groups always differ by $E_k(*,\emptyset)$.

Proof. This follows from the exactness axiom and the fact that the map $E_k(*,\emptyset) \to E_k(X,\emptyset)$ is split by the map from X back to a point. □

Lemma 2.5.4. *The reduced homology groups have suspension isomorphisms*

$$E_k(X,*) \cong E_{k+n}(\Sigma^n X,*)$$

for all $n \geq 0$ and all values of k.

Proof. Using Lemma 2.5.3, we get a long exact sequence on reduced homology for any cofiber sequence of based CW complexes $A \to X \to X/A$. We apply this to the cofiber sequence $X \to CX \to \Sigma X$ to get suspension isomorphisms $E_{k+1}(\Sigma X,*) \cong E_k(X,*)$. □

Remark 2.5.5. In Definition 2.5.1, it does not matter whether the functors E_k are defined only for $k \geq 0$ or for all $k \in \mathbb{Z}$. Any theory only defined for $k \geq 0$ can be canonically extended to negative k using the above two lemmas. See exercise 2.

Remark 2.5.6. The **direct limit axiom** for an extraordinary homology theory E_* states that if X is a CW complex, the natural map from the colimit over all the finite subcomplexes of X,

$$\operatorname*{colim}_{K\subseteq X \text{ finite}} E_n(K) \xrightarrow{\cong} E_n(X),$$

is an isomorphism. This holds whether we use reduced or unreduced E-homology. This axiom actually follows from the other axioms given in Propositions 1.3.1–1.3.4. See exercise 4.

Every spectrum E represents an extraordinary homology theory E_*. This is actually a straightforward corollary of the stability theorems from the previous section:

Proposition 2.5.7. *For any spectrum E, the E-homology groups*

$$E_k(X,A) := \pi_k((X/A)\wedge E)$$

form an extraordinary homology theory, whose coefficient group is $\pi_*(E)$.

Here the smash product is the tensor spectrum from Definition 2.3.6. Recalling that $X/\emptyset = X_+$, the unreduced homology groups in this theory are given by $\pi_k(X_+\wedge E)$, and the reduced groups are given by $\pi_k(X\wedge E)$. By Corollary 2.4.23, a stable equivalence $E\to E'$ gives an isomorphism on the extraordinary homology groups.

Proof. For simplicity we focus on the reduced homology groups of based CW complexes – the proof can be easily adapted to pairs. Let $A\to X\to X/A$ be a cofiber sequence of based CW complexes. Since $A\to X$ is a cofibration, X/A is homotopy equivalent to Cf, the homotopy cofiber of $A\to X$. Taking the smash product with E creates a sequence of spectra

$$A\wedge E \longrightarrow X\wedge E \longrightarrow Cf\wedge E. \tag{2.5.8}$$

By Lemma 2.4.21, this is a cofiber sequence. The long exact sequence and boundary map for E_* therefore come from Lemma 2.4.9. The additivity axiom follows from exercise 29 at the end of this section. The homotopy and excision axioms are immediate. □

Example 2.5.9. The sphere spectrum therefore represents an extraordinary homology theory, stable homotopy. The reduced version of this theory is

$$\pi_k^S(X) := \pi_k(\Sigma^\infty X),$$

while the unreduced version is $\pi_k(\Sigma^\infty_+ X)$. In particular, we get additivity and long exact sequences on stable homotopy groups, unlike the case of ordinary homotopy groups. The coefficient group is the stable homotopy groups of spheres,

$$\pi_*(\mathbb{S}) = \quad \cdots \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad \mathbb{Z}/24 \quad \cdots$$

Example 2.5.10. The Eilenberg-Maclane spectrum HG also represents a homology theory. Its coefficient group is

$$\pi_*(HG) = \quad \cdots \quad 0 \quad 0 \quad 0 \quad G \quad 0 \quad 0 \quad 0 \quad \cdots$$

Since these coefficients are concentrated in degree 0, the dimension axiom is satisfied, so we have defined an ordinary homology theory with G coefficients. By the uniqueness of ordinary homology (Theorem 1.3.7), we conclude:

Corollary 2.5.11. *There is a canonical natural isomorphism*

$$\pi_k((X/A) \wedge HG) \cong H_k(X, A; G)$$

for CW pairs (X, A) *and abelian groups* G.

Remark 2.5.12. The spectral definition of homology should remind you of the definition of homology with G coefficients:

$$\begin{array}{cccccc} \pi_k(& X_+ & \wedge & E &) \\ H_k(& C_*(X) & \otimes & G &) \end{array}$$

The smash product with E plays the role of tensoring with G. We have gone from taking homology with coefficients in an ordinary abelian group G to homology with exotic coefficients in a spectrum E. You might even want to write "$X_+ \otimes E$" to strengthen this analogy.

We highlight a theorem that we will prove later in the book, due to G.W. Whitehead [Whi62]. It states that every extraordinary homology theory arises by the recipe given in Proposition 2.5.7.

Theorem 2.5.13 (Whitehead representability)**.** *If* h_* *is any extraordinary homology theory, there is a spectrum* E *and a natural isomorphism of homology theories* $E_* \cong h_*$*, where* E_* *is defined as in Proposition 2.5.7. Furthermore* E *is unique up to stable equivalence.*

For example, bordism theory $\mathfrak{N}_k(X)$ is isomorphic to $MO_k(X)$ where MO is the Thom spectrum from Example 2.1.20, see also Example 2.5.36. We will give the proof of Theorem 2.5.13 in Theorem 4.2.23.

2.5.2 Extraordinary cohomology

We next turn to extraordinary cohomology, by considering contravariant functors (i.e. ones that reverse the direction of maps).

Definition 2.5.14. An **extraordinary cohomology theory** is any sequence of *contravariant* functors $E^k(-,-)$ from CW pairs to abelian groups, along with natural boundary maps

$$E^k(X,A) \xleftarrow{\delta} E^{k-1}(A,\emptyset),$$

satisfying the first four Eilenberg-Steenrod axioms (Propositions 1.3.10–1.3.13), but not necessarily the fifth axiom (Proposition 1.3.14). The coefficient group is the graded abelian group $E^*(*,\emptyset)$.

As before, this has both a reduced and an unreduced version, but they are equivalent. Without loss of generality, the groups are defined for all $k \in \mathbb{Z}$.

Remark 2.5.15. There is no direct limit axiom for extraordinary cohomology. Instead, there is a $\lim^1$ exact sequence for countable sequential limits, as in (1.5.6). See Section 3.5, exercise 16.

Proposition 2.5.16. *For any Ω-spectrum E, the E-cohomology groups*

$$\begin{aligned} E^k(X,A) &:= \pi_{-k}(F(X/A,E)) \\ &\cong [X/A, E_k]_* \quad \text{for } k \geq 0 \end{aligned}$$

form an extraordinary cohomology theory, whose coefficient group is $\pi_{-*}(E)$.

Here $F(X/A,E)$ is the cotensor or function spectrum from Definition 2.3.8. The proof of Proposition 2.5.16 is left to exercise 31 at the end of the chapter.

Note that, compared to Proposition 2.5.7, the condition on E was made stronger, but we also get a description in terms of the levels E_k of the spectrum E that wasn't possible for homology. This version of the theorem is really the more natural one.

Example 2.5.17. Taking the fibrant sphere spectrum $f\mathbb{S}$ fro Example 2.2.6, we get a cohomology theory called **stable cohomotopy**, whose reduced version is

$$\pi_S^k(X) := \operatorname*{colim}_{n\to\infty}[\Sigma^n X, S^{k+n}]_*.$$

Its coefficient group is $\pi_{-*}(\mathbb{S})$. So it has the stable homotopy groups of spheres in negative degrees, and is zero in positive degrees.

Example 2.5.18. The Eilenberg-Maclane spectrum HG also represents a cohomology theory, whose coefficients are simply G in degree 0. Again by uniqueness of ordinary cohomology, we conclude:

Corollary 2.5.19. *There is a canonical natural isomorphism*

$$\pi_{-k}(F(X/A,HG)) \cong H^k(X,A;G)$$

for CW pairs (X,A) and abelian groups G.

Remark 2.5.20. The spectral definition of cohomology should remind you of the definition of cohomology with G coefficients:

$$\begin{array}{cccccccc} \pi_{-k}(& F & (& X_+ & , & E &) &) \\ H_{-k}(& \mathrm{Hom} & (& C_*(X) & , & G &) &) \end{array}$$

The function spectrum into E plays the role of taking cochains with coefficients in G. You might even want to write "Hom(X_+, E)" to strengthen this analogy.

When E is an Ω-spectrum, the reduced homology and cohomology groups it represents can be drawn as follows. Note that for cohomology each functor E^k corresponds to a single spectrum level, whereas for homology each spectrum level corresponds to a single sphere S^k.

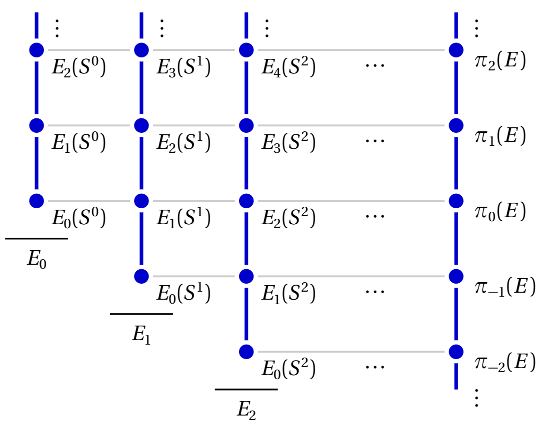


Figure 2.5.21: An Ω-spectrum representing a homology theory. All homology groups are reduced.

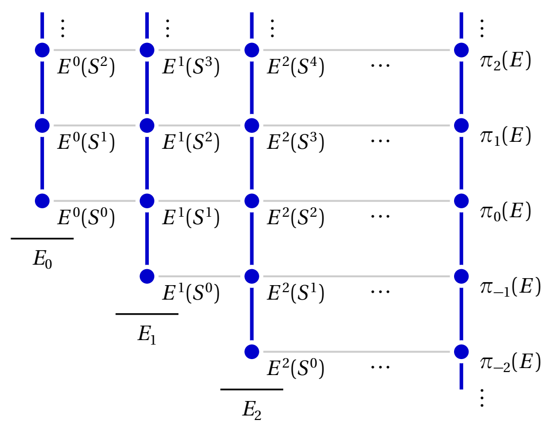


Figure 2.5.22: The same Ω-spectrum representing a cohomology theory. All cohomology groups are reduced.

Remark 2.5.23. The minus signs on the homotopy groups in Proposition 2.5.16 arise because cohomology is traditionally graded with the boundary maps going up in degree. This convention is called *cohomological grading*. If one is willing to make the opposite convention of *homological grading*, the minus sign goes away, and the figure on the right changes to become the figure on the left. But then one must define the functors for all $k \in \mathbb{Z}$, not just $k \geq 0$, because the functors that determine the theory are now the ones with negative k.

The converse of Proposition 2.5.16 is the following classical theorem of Brown [Bro62, Bro65], that every extraordinary cohomology theory can be constructed this way.

Theorem 2.5.24 (Brown representability)**.** *If h^* is any extraordinary cohomology theory, then there is an Ω-spectrum E and a natural isomorphism of cohomology theories $E^* \cong$*

h^*, where E^* is defined as in Proposition 2.5.16. Furthermore E is unique up to stable equivalence.

Proof. We give a moderately detailed sketch – a version with much more detail can be found in e.g. [Hat02, Thm 4E.1], [May99]. By the excision axiom, it is enough to focus on the reduced groups $h^k(X,*)$. It suffices to build for each $k \geq 0$ a CW complex E_k and an isomorphism $h^k(X,*) \cong [X,E_k]_*$. By the Yoneda Lemma (Lemma 1.4.20), the space E_k is unique up to homotopy equivalence, and the suspension isomorphisms in the cohomology theory are represented by weak equivalences $E_{k-1} \simeq \Omega E_k$, making these spaces into an Ω-spectrum.

The main part of the argument is therefore actually building the spaces E_k. We know E_k will be a loopspace, and we have the adjunction $[X,\Omega E_{k+1}]_* \cong [\Sigma X, E_{k+1}]$, where ΣX is always connected. Therefore it suffices to build the connected component of the basepoint of E_{k+1}, and to prove it represents $h^{k+1}(-,*)$ on the category of *connected* based CW complexes.

Simplifying notation, we are given a "set-valued cohomology theory," in other words a functor F from connected based CW complexes to sets, such that

- [Homotopy axiom] F takes homotopic maps to the same map of sets,
- [Wedge axiom] F takes wedge sums to products, in particular $F(*) = *$, and
- [Mayer-Vietoris axiom] for each homotopy pushout $X \cup_A Y$, any pair of elements in $F(X)$ and $F(Y)$ that agree when restricted to $F(A)$, must come from some element of $F(X \cup_A Y)$.

Our goal is to build a connected based CW complex C and a cohomology class $\alpha \in F(C)$ such that the map

$$\begin{aligned} [X,C]_* &\longrightarrow F(X) \\ f &\longmapsto f^*(\alpha) \end{aligned} \tag{2.5.25}$$

is an isomorphism for all connected based CW complexes X.

Inductively, we can build a C and $\alpha \in F(C)$ such that (2.5.25) is an isomorphism for $X = S^i$, $1 \leq i < n$, and surjective when $X = S^n$. In the inductive step, we kill the kernel of (2.5.25) for $X = S^{n-1}$ by attaching n-cells, using the Mayer-Vietoris axiom to extend α to the result. When X is a sphere, the pinch maps $S^i \to S^i \vee S^i$ make the maps (2.5.25) into group homomorphisms, so killing this kernel has the effect of making the map injective. We then make (2.5.25) surjective for $X = S^n$ by taking a wedge sum with n-spheres, using the wedge axiom to extend α to the result. These steps don't affect (2.5.25) for lower-dimensional spheres because a map that attaches n-cells is $(n-1)$-connected (Proposition 1.4.9).

At the end, we have produced a space C and $\alpha \in F(C)$ such that (2.5.25) is an isomorphism whenever X is a sphere. For arbitrary X, if $x \in F(X)$ is any cohomology class, we can apply the above induction again, starting with $X \vee C$, and attaching cells. This creates a new space C_x, and a cohomology class $\alpha_x \in F(C_x)$ restricting to $x \in F(X)$ and $\alpha \in F(C)$, such that pulling back α_x gives isomorphisms $[S^n, C_x]_* \cong F(S^n)$ for every $n \geq 1$. The inclusion $C \to C_x$ is therefore a weak equivalence. By Theorem 1.4.12, C_x deformation retracts to C. Composing the inclusion of X with this retraction produces a map $f: X \to C$ such that $f^*(\alpha) = x$. This shows that (2.5.25) is surjective for any connected based CW complex X.

For any pair of maps $f, g: X \rightrightarrows C$ giving the same class $x \in F(X)$, we consider the based double mapping cylinder $(X \wedge I_+) \vee_{X \wedge \{0,1\}_+} C$. Since $f^*(\alpha) = g^*(\alpha) = x$, the Mayer-Vietoris axiom produces a cohomology class on this cylinder that restricts to $\alpha \in F(C)$. We once again apply our inductive procedure, starting with this double mapping cylinder and producing a space $C_{f,g}$ that deformation retracts to C. The inclusion of $X \wedge I_+ \to C_{f,g}$ composed with this retraction produces a homotopy between f and g. Therefore (2.5.25) is injective for any connected based CW complex X. □

The above proof can be improved, at the expense of additional headaches involving inverse limits, to a statement that only requires the cohomology theory h^* to be defined on *finite* complexes [Ada71, Thm 1.6].

Theorem 2.5.26 (Brown representability for finite complexes)**.** *If h^* is any extraordinary cohomology theory, defined only on finite complexes, then there is an Ω-spectrum E and a natural isomorphism of cohomology theories $E^* \cong h^*$. Furthermore E is unique up to stable equivalence.*

2.5.3 More examples

We finish this section with several more interesting examples of extraordinary homology and cohomology theories.

Example 2.5.27. Complex K-theory from Example 0.1.2 is an unreduced cohomology theory, represented by the spectrum KU defined in Example 2.2.3. In particular, for unbased spaces X the theory gives

$$KU^0(X) := KU^0(X, \emptyset) \cong \pi_0(F(X_+, KU)) \cong [X, \mathbb{Z} \times BU].$$

There is also a connective version $ku^*(X)$, represented by a spectrum ku in which $\pi_{2k} = \mathbb{Z}$ for all $k \geq 0$, and all other homotopy groups are zero.

Example 2.5.28. Similarly, we may take **real K-theory**, the K-theory of real vector bundles. This is represented by an Ω-spectrum KO whose infinite loop space is $\mathbb{Z}\times BO$, the classifying space for stable real vector bundles:

$$KO^0(X) := KO^0(X,\emptyset) \cong \pi_0(F(X_+, KO)) \cong [X, \mathbb{Z}\times BO].$$

There is also a connective version $ko^*(X)$, represented by ko, with the same homotopy groups in the nonnegative degrees only.

Example 2.5.29. Let p be a prime. We define **sphere spectrum mod p** by

$$\mathbb{S}/p = F_n M(\mathbb{Z}/p, n)$$

where the Moore space $M(\mathbb{Z}/p, n)$ is the homotopy cofiber of the degree p map

$$S^n \xrightarrow{p} S^n \longrightarrow M(\mathbb{Z}/p, n).$$

The value of n is irrelevant – it does not affect $\mathbb{S}/p$ up to stable equivalence.

We define "stable homotopy mod p" to be the reduced homology theory represented by this spectrum:

$$\pi_k^S(X;\mathbb{Z}/p) := \pi_k(X\wedge \mathbb{S}/p).$$

The cofiber sequence defining $M(\mathbb{Z}/p, n)$ gives a long exact sequence

$$\cdots \longrightarrow \pi_k^S(X) \xrightarrow{\cdot p} \pi_k^S(X) \longrightarrow \pi_k^S(X;\mathbb{Z}/p) \longrightarrow \pi_{k-1}^S(X) \longrightarrow \cdots$$

just like the long exact sequence for homology with mod p coefficients. Alternatively, this can be presented by a short exact sequence

$$0 \longrightarrow \pi_k^S(X)\otimes \mathbb{Z}/p \longrightarrow \pi_k^S(X;\mathbb{Z}/p) \longrightarrow \mathrm{Tor}(\pi_{k-1}^S(X), \mathbb{Z}/p) \longrightarrow 0.$$

Example 2.5.30. We define the **rational sphere spectrum** by

$$\mathbb{S}_{\mathbb{Q}} = F_n M(\mathbb{Q}, n)$$

where the Moore space $M(\mathbb{Q}, n)$ is the homotopy colimit of the maps

$$S^n \xrightarrow{1} S^n \xrightarrow{2} S^n \xrightarrow{3} S^n \cdots \longrightarrow M(\mathbb{Q}, n).$$

Rational stable homotopy is the reduced homology theory represented by this spectrum:

$$\pi_k^S(X;\mathbb{Q}) := \pi_k(X\wedge \mathbb{S}_{\mathbb{Q}}).$$

The homotopy colimit defining $M(\mathbb{Q}, n)$, together with exercise 27, can be used to show that $\pi_k^S(X;\mathbb{Q}) \cong \pi_k^S(X)\otimes\mathbb{Q}$. It turns out that $\mathbb{S}_Q \simeq H\mathbb{Q}$ (see exercise 39), so rational stable homotopy is isomorphic to rational homology,

$$\pi_k^S(X)\otimes\mathbb{Q} \cong H_k(X, *;\mathbb{Q}).$$

Example 2.5.31. We define the **sphere spectrum localized at** p by

$$\mathbb{S}_{(p)} = F_n M(\mathbb{Z}_{(p)}, n)$$

where the Moore space $M(\mathbb{Z}_{(p)}, n)$ is the homotopy colimit of the maps in Example 2.5.30, except we only take those maps whose degrees are not multiples of p. We define stable homotopy localized at p by taking the associated homology theory, giving $\pi_k^S(X;\mathbb{Z}_{(p)}) \cong \pi_k^S(X) \otimes \mathbb{Z}_{(p)}$.

Example 2.5.32. We can similarly define the **sphere spectrum completed at** p to be the homotopy inverse limit of $\mathbb{S}/(p^k)$ as $k \to \infty$. This can be defined by

$$\mathbb{S}_p^\wedge = \mathrm{sh}^n\, F(M(\mathbb{Z}/(p^\infty), n), f\mathbb{S}),$$

where $f\mathbb{S}$ is the Ω-sphere spectrum of Example 2.2.6, F is the cotensor or function spectrum of Definition 2.3.8, and $M(\mathbb{Z}/(p^\infty), n)$ is the Moore space whose homology is the Prüfer group $\mathbb{Z}/(p^\infty) = \operatorname*{colim}_{k\to\infty} \mathbb{Z}/(p^k)$ in degree n.

For finite CW complexes X, the associated homology theory is $\pi_k^S(X)_p^\wedge \cong \pi_k^S(X) \otimes \mathbb{Z}_p^\wedge$, where $\mathbb{Z}_p^\wedge = \lim_{k\to\infty} \mathbb{Z}/(p^k)$ is the p-adic integers. (For infinite CW complexes X, it's a little more natural to take the p-completion of the suspension spectrum $(\Sigma^\infty X)_p^\wedge$, rather than taking the smash product $X \wedge (\mathbb{S}_p^\wedge)$.)

Example 2.5.33. The previous four examples can be applied to any spectrum E. We define

$$\begin{aligned}
E/p &= E \wedge \mathbb{S}/p \simeq \mathrm{sh}^{-n}\, M(\mathbb{Z}/p, n) \wedge E, \\
E_{(p)} &= E \wedge \mathbb{S}_{(p)} \simeq \mathrm{sh}^{-n}\, M(\mathbb{Z}_{(p)}, n) \wedge E, \\
E_{\mathbb{Q}} &= E \wedge \mathbb{S}_{\mathbb{Q}} \simeq \mathrm{sh}^{-n}\, M(\mathbb{Q}, n) \wedge E, \\
E_p^\wedge &= F(\mathbb{S}/(p^\infty), RE) \simeq \mathrm{sh}^n\, F(M(\mathbb{Z}/(p^\infty), n), RE).
\end{aligned}$$

We call homology theory associated to E/p the "E-homology mod p," $E_*(X;\mathbb{Z}/p)$. The cofiber sequence defining $M(\mathbb{Z}/p, n)$ gives an exact sequence

$$0 \longrightarrow E_k(X) \otimes \mathbb{Z}/p \longrightarrow E_k(X;\mathbb{Z}/p) \longrightarrow \mathrm{Tor}(E_{k-1}(X), \mathbb{Z}/p) \longrightarrow 0.$$

The next two spectra define the p-local E-homology $E_*(X;\mathbb{Z}_{(p)})$ and rational E-homology $E_*(X;\mathbb{Q})$. Using exercise 27 we get isomorphisms

$$E_k(X;\mathbb{Z}_{(p)}) \cong E_k(X) \otimes \mathbb{Z}_{(p)}, \qquad E_k(X;\mathbb{Q}) \cong E_k(X) \otimes \mathbb{Q}.$$

Applying these constructions to ordinary homology recovers the usual notion of homology with coefficients in $\mathbb{Z}/p$, $\mathbb{Z}_{(p)}$, or $\mathbb{Q}$ (exercise 40).

The last construction defines (on finite CW complexes) p-complete E-homology, which fits into a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/(p^\infty), E_k(X)) \longrightarrow E_k(X)^\wedge_p \longrightarrow \mathrm{Hom}(\mathbb{Z}/(p^\infty), E_{k-1}(X)) \longrightarrow 0.$$

When the groups $E_*(X)$ are finitely generated, this simplifies to

$$E_k(X)^\wedge_p \cong E_k(X) \otimes \mathbb{Z}^\wedge_p.$$

On infinite complexes, it is a little better to complete the smash product $(X \wedge E)^\wedge_p$, rather than taking the smash product $X \wedge (E^\wedge_p)$, though the two agree if X is finite CW.

Example 2.5.34. The complex K-theory spectrum KU from Example 2.2.3 and Example 2.5.27 becomes simpler after rationalizing – it becomes a wedge sum of shifted copies of $H\mathbb{Q}$:

$$KU_{\mathbb{Q}} \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} H\mathbb{Q}.$$

In fact, this is true for every spectrum E: the rationalization $E_{\mathbb{Q}}$ is a wedge sum of shifted copies of $H\mathbb{Q}$. In particular, this means that the topological K-theory of CW complexes is rationally nothing more than shifted copies of ordinary homology:

$$KU^0(X) \otimes \mathbb{Q} \cong \bigoplus_{n \geq 0} H^{2n}(X; \mathbb{Q}).$$

This isomorphism is called the **Chern character**.

Example 2.5.35. Suppose we only localize the spectrum KU at a prime p, rather than rationalizing. The resulting spectrum $KU_{(p)}$ decomposes into a wedge sum of shifted copies of a single spectrum L called the **Adams summand**:

$$KU_{(p)} \simeq L \vee \Sigma^2 L \vee \ldots \vee \Sigma^{2p-4} L.$$

Its homotopy groups are:

$$\pi_n(L) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } n \equiv 0 \mod 2p-2, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, when $p \neq 2$ the p-localized real K-theory spectrum $KO_{(p)}$ splits into half as many summands

$$KO_{(p)} \simeq L \vee \Sigma^4 L \vee \ldots \vee \Sigma^{2p-6} L.$$

More generally, as soon as 2 is inverted, KU is equivalent to $KO \vee \Sigma^2 KO$.

Example 2.5.36. **Bordism theory** $\mathfrak{N}_k(X)$ from Example 0.1.1 is an unreduced homology theory, represented by the Thom spectrum MO from Example 2.1.20:

$$\mathfrak{N}_k(X) \cong \pi_k(X_+ \wedge MO).$$

The proof of this uses the Pontryagin-Thom isomorphism, see [Wes96, Kup17, Mil94, Sto68, Coh20]. The spectrum MO has homotopy groups the underlying abelian group of the graded $\mathbb{Z}/2$-algebra

$$\pi_*(MO) \cong (\mathbb{Z}/2)[\ x_n : n \neq 2^k - 1\ ] = (\mathbb{Z}/2)[x_2, x_4, x_5, x_6, x_8, \ldots], \qquad |x_n| = n.$$

In particular, the first few homotopy groups are $\mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2^2, \ldots$. As a spectrum, it is a theorem that MO is equivalent to a wedge of Eilenberg-Maclane spectra

$$MO \simeq \bigvee_\alpha \Sigma^{n_\alpha} H(\mathbb{Z}/2),$$

one for each $\mathbb{Z}/2$ summand in the graded abelian group $\pi_*(MO)$. It follows that the bordism groups of a space X are calculated as

$$\mathfrak{N}_k(X) \cong \pi_*(MO) \otimes_{\mathbb{Z}/2} H_*(X; \mathbb{Z}/2).$$

In particular, bordism is determined by mod 2 homology.

Example 2.5.37. There is similarly a theory of **complex cobordism** $MU_*(X)$, see [Rav86, Mil94]. It is represented by a spectrum MU, that is built just as MO is built in Example 2.1.20, but with complex vector bundles. The universal classifying space is $BU(n)$, with tautological bundle $\gamma^n \to BU(n)$. Since the the compactification of $\mathbb{C}$ is a two-sphere, the structure maps have the form

$$\Sigma^2 \mathrm{Th}(\gamma^n) \to \mathrm{Th}(\gamma^{n+1}). \tag{2.5.38}$$

We therefore let the Thom space for γ^n be spectrum level $2n$, rather than spectrum level n. We interpolate between these using suspensions

$$MU_{2n} = \mathrm{Th}(\gamma^n), \qquad MU_{2n+1} = \Sigma \mathrm{Th}(\gamma^n),, \qquad MU_{2n+2} = \mathrm{Th}(\gamma^{n+1}), \qquad \cdots$$

and use the map (2.5.38) to define the bonding map $\Sigma MU_{2n+1} \to MU_{2n+2}$. Again, intuitively, MU is a twisted suspension spectrum of the classifying space $BU = \operatorname*{colim}_{n\to\infty} BU(n)$.

The coefficient group is the underlying abelian group of the graded $\mathbb{Z}$-algebra

$$\pi_*(MU) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots], \qquad |x_n| = 2n.$$

Example 2.5.39. The spectrum MU does not split quite as nicely as MO, but after localizing at a prime p, it splits into shifted copies of a single spectrum BP called the **Brown-Peterson spectrum**. Its homotopy groups are the underlying abelian group of the graded ring

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \qquad |v_n| = 2(p^n - 1).$$

There are a number of other interesting spectra, distilled out of MU, that play an important role in computations in stable homotopy theory. They are derived from BP by killing some elements of π_* and inverting others:

$$\begin{aligned} \pi_*(BP\langle n\rangle) &= \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n] \\ \pi_*(E(n)) &= \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}] \\ \pi_*(K(n)) &= (\mathbb{Z}/p)[v_n, v_n^{-1}] \end{aligned}$$

The last one in particular, **Morava K-theory**, is very computable – it acts like homology with field coefficients, for every value of n. As n varies, it interpolates between rational homology $K(0) = H\mathbb{Q}$ and mod p homology $K(\infty) = H\mathbb{Z}/p$. The first theory $K(1)$ is also equivalent to L/p, where L is the Adams summand of Example 2.5.35.

These spectra allow us to pick out phenomena in $\pi_*(\mathbb{S})$ that occur at one prime p and one "frequency" $2(p^n - 1)$, like separating light into colors using a prism. For this reason, the study of stable homotopy category from this point of view is called **chromatic homotopy theory**. See e.g. [Rav86, Rav92, Lur10] for references in this direction.

Remark 2.5.40. The analog of the universal coefficient theorem for extraordinary homology is called the Atiyah-Hirzebruch spectral sequence, which for unreduced homology is written

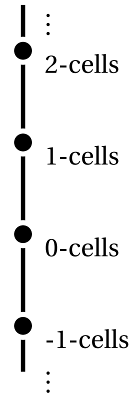
$$H_p(X; E_q(*)) \Rightarrow E_{p+q}(X),$$

and for cohomology

$$H^p(X; E^q(*)) \Rightarrow E^{p+q}(X).$$

See **??**. The much more impressive Adams spectral sequence makes it possible to go from cohomology of a spectrum to homotopy groups, at least after localizing or completing at a prime p – see **??**.

2.6 Cellular and CW spectra

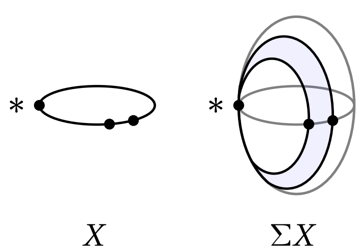


CW spectra are essentially CW complexes with negative-dimensional cells. Every spectrum is stably equivalent to a CW spectrum, so we can think of spectra as complexes with negative-dimensional cells. When thinking this way, we draw the cells as black dots and connect them with lines as shown.

This is a satisfying intuition, and it's useful on a technical level too. We can use it to prove versions of the Whitehead theorem, cellular approximation, Postnikov towers, and the Hurewicz theorem for spectra.

2.6.1 Definition

Recall from Definition 1.1.8 that a cell complex X is a space that is built by repeatedly attaching discs, of varying dimensions. If X is a based space, we ask that the basepoint be one of the cells. A CW complex is a cell complex in which we only attach n-cells to cells of lower dimension.

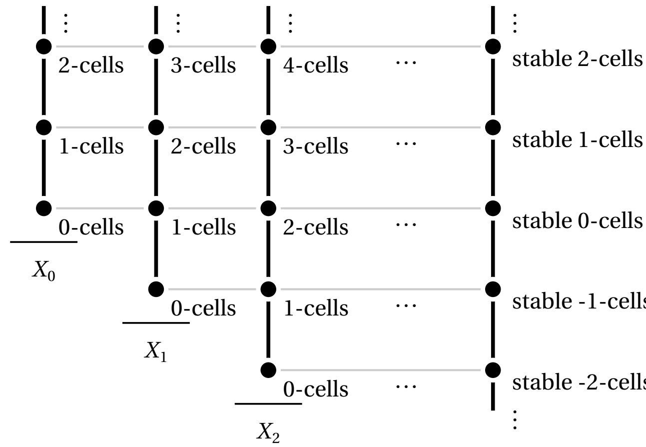


Lemma 2.6.1. *If X is a based cell complex, then ΣX is a based cell complex in a natural way, with one $(n+1)$-cell for each n-cell of X other than the basepoint.*

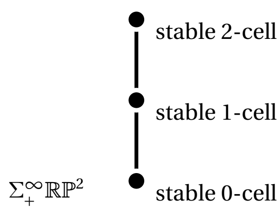
Proof. For each n-cell $D^n \to X$, we write the map as a map of based spaces $D^n_+ \to X$ and then take the reduced suspension, giving $\Sigma D^n_+ \to \Sigma X$. The space ΣD^n_+ is a quotient of $I \times D^n \cong D^{n+1}$ by an equivalence relation along its boundary. We therefore get a map $D^{n+1} \to \Sigma X$, in other words an $(n+1)$-cell in ΣX. It is now straightforward to check that ΣX is homeomorphic to the disjoint union of these cells modulo their attaching maps, making ΣX into a cell complex. □

Definition 2.6.2. The spectrum X is a **cellular spectrum** if each space X_n can be given the structure of a based cell complex, so that each bonding map $\Sigma X_n \to X_{n+1}$ is the inclusion of a subcomplex. A **CW spectrum** is the same except that each X_n is a CW complex.

If X is a cellular spectrum and $k \in \mathbb{Z}$, a **stable** k**-cell** in X is a non-basepoint $(k+n)$-cell in X_n for any $n \geq 0$. This is identified with the corresponding cell of dimension $(k+n+1)$ in $\Sigma X_n \subseteq X_{n+1}$, and so on as $n \to \infty$.



As we did with the stable homotopy groups, we can depict these stable cells by drawing the levels of the spectrum in a staggered formation. Then each cell creates a cell immediately to its right, and the limit of this process at the far right is the set of stable cells of X.



Example 2.6.3. If K is a based cell or CW complex then its suspension spectrum $\Sigma^\infty K$ is a cellular or CW spectrum, respectively. It has a stable k-cell for every k-cell of K, other than the basepoint. More generally, the free spectrum $F_n K$ is cellular, with a stable $(k-n)$-cell for every k-cell of K, other than the basepoint.

Example 2.6.4. As a special case of the previous example, for any $d \in \mathbb{Z}$, the d-sphere spectrum $\mathbb{S}^d$ from Example 2.1.9 is characterized by the fact that it has a single stable d-cell.

Example 2.6.5. If B is an unbased cell or CW complex and ζ is a virtual bundle over B then the Thom spectrum $\mathrm{Th}(\zeta)$ constructed in Example 2.1.17 is a cellular or CW spectrum, respectively. The Thom spectrum MO from Example 2.1.20 and MU from Example 2.5.37 are both CW spectra.

Example 2.6.6. When building the Eilenberg-Maclane spectrum HG from Example 2.2.2, we may choose each level $K(G,n)$ to be a CW complex, and iteratively replace $K(G,n+1)$ with the mapping cylinder of the map $\Sigma K(G,n) \to K(G,n+1)$, so as to make the entire spectrum into a CW spectrum.

You should picture cellular spectra as built by an iterative process. At the first step, we start with the zero spectrum, and attach cells (of all dimensions) to spectrum level 0 to make X_0. As we do this, we're also attaching the suspensions of these cells to all the higher spectrum levels. At the end, we've made the suspension spectrum $\Sigma^\infty X_0$.

$$\begin{array}{r|c}
X_0 & \text{cells attached at level 0} \\
\Sigma X_0 & \Sigma(\text{cells attached at level 0}) \\
\Sigma^2 X_0 & \Sigma^2(\text{cells attached at level 0}) \\
\vdots & \vdots
\end{array}$$

Then, we attach cells to spectrum level 1, and all of their suspensions to the higher levels. At level 1, this makes a space X_1 that looks like ΣX_0 with more cells attached, and we get the suspensions of this space at all the higher levels:

$$\begin{array}{r|cc}
X_0 & \text{cells attached at level 0} & \\
X_1 & \Sigma(\text{cells attached at level 0}) & \text{cells attached at level 1} \\
\Sigma X_1 & \Sigma^2(\text{cells attached at level 0}) & \Sigma(\text{cells attached at level 1}) \\
\Sigma^2 X_1 & \Sigma^3(\text{cells attached at level 0}) & \Sigma^2(\text{cells attached at level 1}) \\
\vdots & \vdots & \vdots
\end{array}$$

At level 2, attach more cells to ΣX_1 to make X_2, and their suspensions, and so on.

$$\begin{array}{r|ccc}
X_0 & \text{cells attached at level 0} & & \\
X_1 & \Sigma(\text{cells attached at level 0}) & \text{cells attached at level 1} & \\
X_2 & \Sigma^2(\text{cells attached at level 0}) & \Sigma(\text{cells attached at level 1}) & \text{cells attached at level 2} \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Clearly, we can do this whole process relative to some fixed starting spectrum A. We attach cells to A_0 to form X_0, and also attach their suspensions to the higher levels of A, giving the spectrum in the second column just below. Then we attach cells to $A_1 \cup_{\Sigma A_0} \Sigma X_0$ to form X_1, and their suspensions to the higher levels, giving the third column below, and so on:

to start	after one step	after two steps	after three steps
A_0	X_0	X_0	X_0
A_1	$A_1 \cup_{\Sigma A_0} \Sigma X_0$	X_1	X_1
A_2	$A_2 \cup_{\Sigma^2 A_0} \Sigma^2 X_0$	$A_2 \cup_{\Sigma A_1} \Sigma X_1$	X_2
A_3	$A_3 \cup_{\Sigma^3 A_0} \Sigma^3 X_0$	$A_3 \cup_{\Sigma^2 A_1} \Sigma^2 X_1$	$A_3 \cup_{\Sigma A_2} \Sigma X_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Definition 2.6.7. For a map of spectra $f\colon A \to X$ the nth **relative bonding map** is the map

$$A_n \cup_{\Sigma A_{n-1}} \Sigma X_{n-1} \longrightarrow X_n \tag{2.6.8}$$

given by f_n and ξ_{n-1}. For $n = 0$, it is just the map $f_0\colon A_0 \to X_0$.

The map f is a **relative cellular spectrum** if each of the relative bonding maps is a relative cell complex of spaces. A relative CW spectrum is defined the same way.

Of course, $* \to X$ is a relative cellular spectrum when X is a cellular spectrum.

2.6.2 Stable cell attachments

This iterative process can be described more holistically. When we attach a cell to one spectrum level of A, and all of its suspensions to the higher levels, we are really attaching a free spectrum $F_n D_+^{k+n}$ to A.

To explain this further, recall that for a unbased space K, the shift desuspension spectrum $F_n K_+$ is the one-point space at every level until n, then K_+ at level n, then suspensions of K_+ after that:

$$F_n K_+ = \{* \quad * \quad \dots \quad * \quad K_+ \quad \Sigma(K_+) \quad \Sigma^2(K_+) \quad \dots\}$$

We call this the **free spectrum** on the space K at level n, for the following reason:

Lemma 2.6.9. *A map of spectra $F_n K_+ \to X$ is the same data as a map of unbased spaces $K \to X_n$.*

In other words, $F_n(-)_+$ is the left adjoint of the operation ev_n that takes every spectrum X to its nth space X_n and forgets the basepoint of X_n (see exercise 17).

Definition 2.6.10. For any spectrum A and any map $\varphi\colon S^{(k-1)+n} \to A_n$, we **attach a k-cell** to A along φ by taking the pushout spectrum

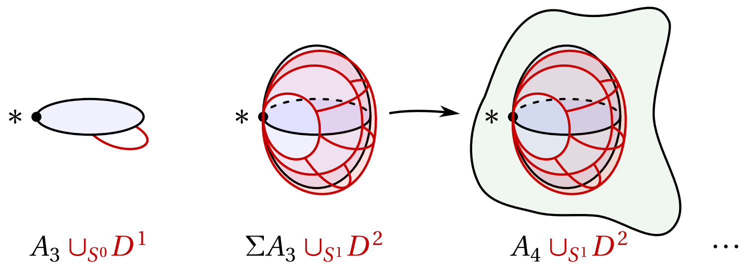
$$A' = (F_n D_+^{k+n}) \cup_{F_n(S_+^{(k-1)+n})} A.$$

At each spectrum level $m \geq n$, using Lemma 2.6.1, this gives the pushout

$$A'_m = (\Sigma^{m-n} D_+^{k+n}) \cup_{\Sigma^{m-n}(S_+^{(k-1)+n})} A_m \cong D^{k+m} \cup_{S^{(k-1)+m}} A_m.$$

In other words, it is A_m with a $(k+m)$-cell attached. So, this operation attaches a single stable k-cell (in the sense of Definition 2.6.2) to the spectrum A.

We illustrate just below the process of attaching a stable (−2)-cell to A, by attaching a 1-cell to spectrum level 3, and all of its suspensions to the higher spectrum levels.



Proposition 2.6.11. *The map $f: A \to X$ is a relative cellular spectrum if and only if it is a countable composition*

$$A = X^{(-1)} \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \ldots \longrightarrow X^{(j-1)} \longrightarrow X^{(j)} \longrightarrow \ldots \longrightarrow X$$

in which each map $X^{(j-1)} \to X^{(j)}$ attaches an arbitrary number of cells of varying dimensions:

$$\begin{tikzcd}
\bigvee_i F_{n_i} S_+^{(k_i-1)+n_i} \arrow[r] \arrow[d] & \bigvee_i F_{n_i} D_+^{k_i+n_i} \arrow[d] \\
X^{(j-1)} \arrow[r] & X^{(j)}.
\end{tikzcd}$$

Proof. This is mainly an exercise in bookkeeping. It is clear that any f of this form attaches cells and their suspensions as described earlier in this section, though it may attach cells to the levels in different orders. Still, it is enough to see that the relative bonding maps are all cell complexes, so f is a relative cellular spectrum. Conversely, if f is a relative cellular spectrum, the procedure described just before Definition 2.6.7 expresses f as a countable composition of stable cell attachments. Strictly speaking, it is a countable composition *of countable compositions* of cell attachments, but this can be re-indexed by attaching each cell earlier in the process, to make a single countable composition. See exercise 28. □

Recall from Theorem 2.6.12 that any map of topological spaces $A \to X$ factors into a relative CW complex $A \to B$, followed by a weak equivalence $B \to X$. As a consequence, every topological space X is weakly equivalent to a CW complex QX. We now prove the same fact for spectra.

Theorem 2.6.12. *If $f : A \to X$ is any map of spectra, it can be factored into a relative CW spectrum $A \to B$ and a level equivalence $B \to X$.*

Corollary 2.6.13. *Every spectrum X is level equivalent to a CW spectrum QX, and stably equivalent to a CW Ω-spectrum QRX. (That is, QRX is a CW spectrum and also an Ω-spectrum.)*

Proof of Theorem 2.6.12. By Corollary 1.4.11, the map $A_0 \to X_0$ can be factored into a CW complex followed by an equivalence,

$$A_0 \longrightarrow B_0 \xrightarrow{\sim} X_0. \tag{2.6.14}$$

The reduced suspension of (2.6.14) fits in with the structure maps of A and X in the following way:

$$\begin{tikzcd}
\Sigma A_0 \arrow[r] \arrow[d] & \Sigma B_0 \arrow[r] \arrow[d] & \Sigma X_0 \arrow[d] \\
A_1 \arrow[r] & A_1 \cup_{\Sigma A_0} \Sigma B_0 \arrow[r, dashed] & X_1
\end{tikzcd}$$

The dotted-arrow map is induced from our original map $f_1 : A_1 \to X_1$ and the suspension of the map $B_0 \to X_0$ that we constructed in the previous step. We may factor it into a relative cell complex followed by a weak equivalence, giving a new space we call B_1:

$$A_1 \cup_{\Sigma A_0} \Sigma B_0 \longrightarrow B_1 \xrightarrow{\sim} X_1$$

Notice that B_1 is equipped with compatible maps coming in from A_1 and ΣB_0, and a map going out to X_1. By construction, the relative bonding map $A_1 \cup_{\Sigma A_0} \Sigma B_0 \to B_1$ is a cell complex and $B_1 \to X_1$ is a weak equivalence.

Repeating this procedure for each spectrum level gives a spectrum B and the maps $A \to B$ and $B \to X$ with the desired properties. □

Remark 2.6.15. The factorization in Theorem 2.6.12, just as the one in Corollary 1.4.11, can be defined naturally, so that it is a functor. In particular, we get a functor $Q : \mathbf{Sp} \to \mathbf{Sp}$ that replaces each spectrum X by a CW spectrum QX, and a natural level equivalence $QX \to X$. See also Section 5.1.

Next we prove the Whitehead theorem for spectra.

Proposition 2.6.16. *If $f : X \to Y$ is a level equivalence of cellular spectra, then it is a homotopy equivalence, in the sense that there is a map $g : Y \to X$ and homotopies from $f \circ g$ and $g \circ f$ to the identity.*

Corollary 2.6.17 (Whitehead theorem for spectra)**.** *If X and Y are cellular Ω-spectra then $f : X \to Y$ is a stable equivalence iff it is a homotopy equivalence.*

Proof of Proposition 2.6.16. Replace Y with the mapping cylinder $M = (X \wedge I_+) \cup_X Y$. It suffices to show that M deformation retracts onto X as a spectrum. Since f is a level equivalence and both X_n and Y_n are cell complexes, M_n is a cell complex that deformation retracts onto the subcomplex X_n.

The complex M_n also contains ΣM_{n-1} as a subcomplex, and the intersection with X_n is exactly ΣX_{n-1}. We therefore have inclusions of cell complexes

$$X_n \longrightarrow X_n \cup_{\Sigma X_{n-1}} \Sigma M_{n-1} \longrightarrow M_n. \tag{2.6.18}$$

The second map here is the relative bonding map for $X \to M$ from Definition 2.6.7.

The first map of (2.6.18) is a homotopy equivalence, since ΣM_{n-1} deformation retracts onto ΣX_{n-1}. The composite is also a homotopy equivalence. Therefore the second inclusion is a homotopy equivalence as well. We conclude that M_n deformation retracts onto the pushout $X_n \cup_{\Sigma X_{n-1}} \Sigma M_{n-1}$.

Composing $(n+1)$ of these deformation retractions together gives a deformation retraction

$$M_n \to X_n \cup_{\Sigma X_{n-1}} \Sigma M_{n-1} \to X_n \cup_{\Sigma^2 X_{n-2}} \Sigma^2 M_{n-2} \to \cdots \to X_n \cup_{\Sigma^n X_0} \Sigma^n M_0 \to X_n.$$

We make these into a single homotopy by having the last deformation happen from time $1/2$ to 1, the next to last happen from time $1/4$ to $1/2$, and so on, until the first deformation which happens from time $1/2^{n+1}$ to $1/2^n$. We take the constant homotopy at the identity map of M_n from time 0 to $1/2^{n+1}$.

These formulas give a continuous homotopy of maps $M \to M$ at each spectrum level n, and our choices of parametrization ensure that they agree along the bonding maps. This gives a homotopy from the identity of M to a map that retracts onto X. □

Remark 2.6.19. We will later give a second proof of this using the abstract theory of model categories. See also Section 3.5, exercise 8.

Remark 2.6.20. The Whitehead theorem is named after J.H.C. Whitehead, while Whitehead representability is named after G.W. Whitehead.

2.6.3 Relative homotopy groups

To get Postnikov towers and the Hurewicz theorem, we will have to make cellular approximations of spectra in a different way, that does not proceed one level at a time. As before, we attach cells to kill the relative homotopy groups. But, we're going to think of the relative homotopy groups of the spectrum as a whole, not on each level separately.

Recall from Chapter 1 that for a map $f: A \to X$ of topological spaces, the relative homotopy group $\pi_n(X,A)$ describes commuting squares of based maps

$$\begin{tikzcd} S^{n-1} \arrow[r, "\subseteq"] \arrow[d] & D^n \arrow[d] \\ A \arrow[r, "f"] & X \end{tikzcd}$$

up to homotopies of the vertical maps. Each element $\alpha \in \pi_n(X,A)$ therefore gives a way to attach an n-cell to A, and to extend the map $A \to X$ to the new, larger version of A that has this cell attached. By Proposition 1.4.9, this has the effect of killing $\alpha \in \pi_n(X,A)$ without changing $\pi_{<n}(X,A)$.

Remark 2.6.21. By the long exact sequence on homotopy groups, this means that attaching an n-cell changes $\pi_*(A)$ in a way that makes

$$\begin{aligned} \pi_n(A) \to \pi_n(X) &\quad \text{more surjective,} \\ \pi_{n-1}(A) \to \pi_{n-1}(X) &\quad \text{more injective, and} \\ \pi_k(A) \to \pi_k(X) &\quad \text{unchanged for } k < n-1. \end{aligned}$$

Applying this process iteratively, we can make A larger and larger, and agree up to π_n with X for progressively larger values of n.

We want to do the same process in spectra. It is surprisingly straightforward; we just have to define relative homotopy groups of spectra as the colimit of relative homotopy of the levels, and attach cells to spectra using Definition 2.6.10.

Definition 2.6.22. If $f: A \to X$ is a map of spectra, and $k \in \mathbb{Z}$, we define the kth **relative stable homotopy group** $\pi_k(X,A)$ as the colimit of the system

$$\dots \longrightarrow \pi_{k+n}(X_n, A_n) \xrightarrow{\sigma} \pi_{k+n+1}(\Sigma X_n, \Sigma A_n) \xrightarrow{\xi_n} \pi_{k+n+1}(X_{n+1}, A_{n+1}) \longrightarrow \dots$$

The map σ is defined as in Definition 2.1.2. To make the definitions agree as nicely as possible, we define $\pi_m(X_n, A_n)$ to mean maps of pairs $(D^m, S^{m-1}) \to (X_n, A_n)$, where S^{m-1} is the one-point compactification of $\mathbb{R}^{m-1}$ and D^m is the one-point compactification of $[0,\infty) \times \mathbb{R}^{m-1}$.

Proposition 2.6.23. *For any map of spectra $f: A \to X$ there is a long exact sequence*

$$\dots \longrightarrow \pi_k(A) \xrightarrow{f_*} \pi_k(X) \longrightarrow \pi_k(X,A) \xrightarrow{\partial} \pi_{k-1}(A) \longrightarrow \dots$$

and a natural isomorphism $\pi_k(X,A) \cong \pi_k(Cf)$.

Proof. This follows by taking the colimit of the corresponding exact sequences at each spectrum level (Theorem 1.4.4). See exercise 14. □

Each element of $\pi_k(X,A)$ can be represented by a commuting diagram of based spaces as shown, for any sufficiently large n.

$$\begin{tikzcd} S^{(k-1)+n} \arrow[r] \arrow[d, "\partial\alpha_n"'] & D^{k+n} \arrow[d, "\alpha_n"] \\ A_n \arrow[r, "f_n"] & X_n \end{tikzcd} \tag{2.6.24}$$

Thinking of this as a diagram of unbased spaces, we can therefore attach a k-cell to A along $\partial\alpha_n$, as in Definition 2.6.10. It is easy to see that up to stable equivalence, the result only depends on $\partial\alpha \in \pi_{k-1}(A)$, and not on n.

Proposition 2.6.25. *For any map of spectra $f: A \to X$, any integer $k \in \mathbb{Z}$, and any element $\alpha \in \pi_k(X,A)$, this produces a factorization $A \to A' \to X$ such that*

$$\pi_i(X,A) \to \pi_i(X,A')$$

is an isomorphism for $i < k$, and surjective for $i = k$, with kernel containing α.

Proof. By Proposition 1.4.9, this happens at each spectrum level n for sufficiently large n. In other words, the map $\pi_{i+n}(X_n, A_n) \to \pi_{i+n}(X_n, A'_n)$ is an isomorphism for $i < k$, and surjective for $i = k$, with kernel containing α_n. Taking the colimit over n gives the result. □

Remark 2.6.26. We can do the same with any collection of elements $\alpha \in \pi_k(X,A)$, or even the entire group. This has the effect of killing $\pi_k(X,A)$ while preserving all of the lower homotopy groups.

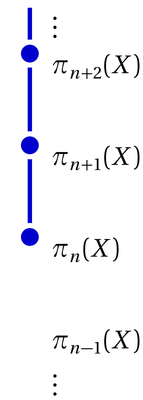
As discussed in Remark 2.6.21, attaching stable k-cells to A makes

$$\begin{aligned} \pi_k(A) \to \pi_k(X) &\quad \text{more surjective,} \\ \pi_{k-1}(A) \to \pi_{k-1}(X) &\quad \text{more injective,} \end{aligned}$$

and doesn't affect the lower homotopy groups. Proceeding up one value of k at a time, we can therefore make the map $A \to X$ an isomorphism on homotopy groups, one group at a time.

But we have to start somewhere. In spectra, there could be homotopy groups in infinitely many negative degrees, which would make it impossible to start.

Definition 2.6.27. A spectrum X is $(n-1)$-**connected** or n-**connective** if $\pi_i(X) = 0$ for $i < n$. Similarly a map of spectra $A \to X$ is $(n-1)$-connected if $\pi_i(X, A) = 0$ for $i < n$.



A (-1)-connected spectrum X is also called **connective**. A spectrum is **bounded below** if it is $(n-1)$-connected for some $n \in \mathbb{Z}$.

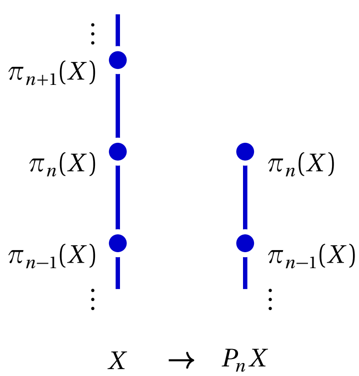
Example 2.6.28. Every suspension spectrum $\Sigma^\infty K$ and Eilenberg-Maclane spectrum HG is connective. Every Thom spectrum $\mathrm{Th}(\zeta)$ is bounded below by the dimension of the virtual bundle ζ. In particular, MO and MU are connective. The complex K-theory spectra KU and KO are not bounded below.

Repeated application of Proposition 2.6.25 and Proposition 2.6.23 gives:

Proposition 2.6.29. *The spectrum X is $(n-1)$-connected iff it is stably equivalent to a CW spectrum with stable cells in dimension n and above.*

The map $A \to X$ is $(n-1)$-connected iff it is stably equivalent to a relative CW spectrum with stable cells in dimension n and above.

This gives a second way to replace spectra by CW spectra, different than Theorem 2.6.12.



Definition 2.6.30. For any spectrum X and any integer n, we construct the n**th Postnikov stage** $X \to P_n X$ as follows. We attach all possible stable $(n+1)$-cells to the map $X \to *$, then all possible $(n+2)$-cells, and so on. In other words, we use a cell for every diagram of spectra of the form (2.6.24), with the appropriate stable dimension. This produces a relative CW spectrum $X \to P_n X$.

By Proposition 2.6.25, the map $X \to P_n X$ is an isomorphism on homotopy up to degree n, while above degree n the homotopy of $P_n X$ is zero.

Example 2.6.31. The 0th Postnikov stage of the sphere spectrum $P_0\mathbb{S}$ has only a $\mathbb{Z}$ in degree zero. By uniqueness of Eilenberg-Maclane spectra (exercise 38), it is therefore stably equivalent to $H\mathbb{Z}$. We therefore have constructed a map (up to stable equivalences) $\mathbb{S} \to H\mathbb{Z}$, giving an isomorphism on π_0. We could also construct such a map more simply by taking $S^0 \to K(\mathbb{Z}, 0)$ by taking the non-basepoint to the point $1 \in \mathbb{Z}$, and then taking its suspensions.

Example 2.6.32. More generally, if X is any connective spectrum, the 0th Postnikov stage is a map of spectra $X \to H(\pi_0 X)$ that is an isomorphism on π_0. So for instance we get a map $MO \to H\mathbb{Z}/2$, and $MU \to H\mathbb{Z}$.

Lemma 2.6.33. *For each $n \in \mathbb{Z}$ there is a canonical map $P_{n+1}X \to P_nX$ giving an isomorphism on homotopy groups up to degree n.*

Proof. Each of the cells that we attach to form $P_{n+1}X$ is also a cell that we used for P_nX, since we took *all possible* cells when killing π_{n+2} and higher. This gives the map, and since it commutes with the map in from X, it gives an isomorphism on the lower homotopy groups. □

Definition 2.6.34. The above construction and lemma define the **Postnikov tower** of X, a bi-infinite tower of the following form.

$$X \longrightarrow \ldots \longrightarrow P_2X \longrightarrow P_1X \longrightarrow P_0X \longrightarrow P_{-1}X \longrightarrow P_{-2}X \longrightarrow \ldots$$

The cofiber or fiber of each map $P_nX \to P_{n-1}X$ has homotopy concentrated in a single degree. By exercise 38, it is therefore a shift of an Eilenberg-Maclane spectrum. We therefore get a fiber Puppe sequence

$$\Sigma^n H\pi_n(X) \longrightarrow P_nX \longrightarrow P_{n-1}X \longrightarrow \Sigma^{n+1}H\pi_n(X).$$

In contrast to the case with spaces, sometimes this tower is bottomless. There is a lowest stage precisely when X is bounded below.

Example 2.6.35. The last three stages of the Postnikov tower for the sphere spectrum have the following homotopy groups:

$\vdots$	$\vdots$	$\vdots$	$\vdots$
π_3	0	0	0
π_2	$\mathbb{Z}/2$	0	0
π_1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
π_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
	$P_2\mathbb{S}$	$P_1\mathbb{S}$	$P_0\mathbb{S}$

Since $P_0\mathbb{S}$ has only one homotopy group $\mathbb{Z}$ in degree 0, it is stably equivalent to $H\mathbb{Z}$. (See exercise 38.)

We also form the **tower of connective covers** or **Whitehead tower** of X by defining $X\langle n\rangle$ to be the homotopy fiber of $X \to P_{n-1}X$. This gives a bi-infinite tower mapping to X

$$\ldots \longrightarrow X\langle 2\rangle \longrightarrow X\langle 1\rangle \longrightarrow X\langle 0\rangle \longrightarrow X\langle -1\rangle \longrightarrow X\langle -2\rangle \longrightarrow \ldots \longrightarrow X,$$

where $X\langle n\rangle$ is n-connective and the map $X\langle n\rangle \to X$ is an isomorphism on $\pi_{\geq n}$. If X is bounded below, then this tower is eventually the constant tower at X.

2.6.4 Homology of spectra

Now that cellular spectra are in place, we can take their homology groups, and prove the Hurewicz theorem. There are actually two equivalent ways to define the kth homology group $H_k(X;G)$ of the spectrum X:

Definition 2.6.36.

- As in Definition 2.1.2, define $H_k(X;G)$ as the colimit of the system

$$\dots \longrightarrow H_{k+n}(X_n,*;G) \xrightarrow{\sigma} H_{k+n+1}(\Sigma X_n,*;G) \xrightarrow{\xi_n} H_{k+n+1}(X_{n+1},*;G) \longrightarrow \dots$$

The map σ is the composite

$$\begin{tikzcd} H_{k+n}(X_n,*;G) \arrow[rr, bend right, "\sigma"'] & H_{k+n+1}(CX_n,X_n;G) \arrow[l, "\delta", "\cong"'] \arrow[r] & H_{k+n+1}(\Sigma X_n,*;G). \end{tikzcd}$$

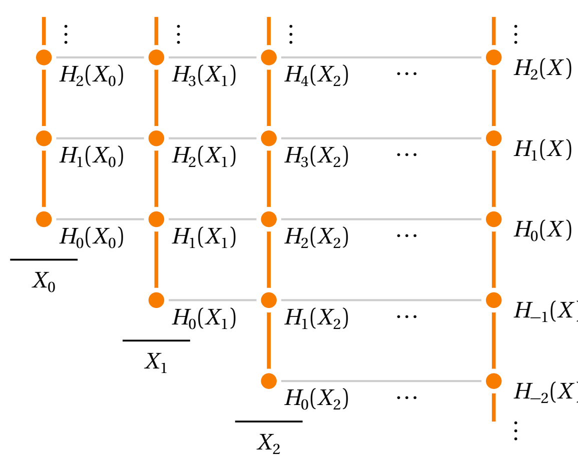
It is an isomorphism if X_n is well-based, so that ΣX_n has the right homotopy type. But it is defined even if X_n is not well-based.

- As in Proposition 2.5.7, define

$$H_k(X;G) = \pi_k(X \wedge HG)$$

where the $\wedge$ is the handicrafted smash product of Definition 2.3.23. We assume that the levels of X and HG are CW complexes, so that $X_p \wedge (HG)_q$ has the correct homotopy type.

It is an easy exercise to generalize these definitions to the E-homology of a spectrum X, for any extraordinary homology theory E. See also Example 3.2.16 and Example 3.2.21 for the definition of cohomology of spectra.



The first definition of homology is illustrated to the right. As for homotopy groups, what we really care about is the limiting behavior, so we focus on the single line of homology groups at the far right. The rest of the figure is just a presentation of those groups.

The following result is true for E-homology, but we state and prove it just for ordinary homology.

Proposition 2.6.37. *The above two definitions give isomorphic homology groups $H_k(X;G)$. Furthermore, any stable equivalence $X \to Y$ induces an isomorphism on homology $H_k(X;G) \cong H_k(Y;G)$ for all $k \in \mathbb{Z}$.*

Proof. The homotopy groups of the handicrafted smash product are given by the colimit of the following grid.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\pi_{k+2}(X_0 \wedge (HG)_2) \arrow[u, "+"] \arrow[r, "+"] & \pi_{k+3}(X_1 \wedge (HG)_2) \arrow[u, "+"] \arrow[r, "+"] & \pi_{k+4}(X_2 \wedge (HG)_2) \arrow[u, "+"] \arrow[r, "+"] & \dots \\
\pi_{k+1}(X_0 \wedge (HG)_1) \arrow[u, "+"] \arrow[r, "-"] & \pi_{k+2}(X_1 \wedge (HG)_1) \arrow[u, "+"] \arrow[r, "-"] & \pi_{k+3}(X_2 \wedge (HG)_1) \arrow[u, "+"] \arrow[r, "-"] & \dots \\
\pi_{k}(X_0 \wedge (HG)_0) \arrow[u, "+"] \arrow[r, "+"] & \pi_{k+1}(X_1 \wedge (HG)_0) \arrow[u, "+"] \arrow[r, "+"] & \pi_{k+2}(X_2 \wedge (HG)_0) \arrow[u, "+"] \arrow[r, "+"] & \dots
\end{tikzcd}$$

The colimit along the ith column is $\pi_{k+i}(X_i \wedge HG) \cong H_{k+i}(X_i,*)$. Passing to the $(i+1)$st row induces the map $H_{k+i}(X_i,*) \to H_{k+i+1}(X_{i+1},*)$ from Definition 2.6.36. Taking the colimit of these gives the first definition of homology from Definition 2.6.36. But their colimit is equal to the colimit of the entire grid of abelian groups, which is the second definition of homology from Definition 2.6.36. Therefore the two definitions are isomorphic.

To prove that homology respects stable equivalences, we take colimits along the rows instead. The colimit along the jth row is isomorphic to $\pi_{k+j}((HG)_j \wedge X)$. Any stable equivalence $X \to Y$ induces isomorphisms on these groups by Corollary 2.4.23, and therefore an isomorphism on the colimit of the entire grid. □

Example 2.6.38. The homology of a suspension spectrum is just the homology of the underlying space:

$$H_n(\Sigma^\infty_+ A) \cong H_n(A).$$

More generally, if $\zeta \to B$ is an oriented virtual bundle of dimension d then

$$H_n(\mathrm{Th}(\zeta)) \cong H_{n-d}(B).$$

See exercise 7.

Warning 2.6.39. The homology of a spectrum X is *not* the same as the homology of its infinite loopspace $\Omega^\infty X$, even though they have the same homotopy:

$$\pi_n(\Omega^\infty X) \cong \pi_n(X), \qquad H_n(\Omega^\infty X) \ncong H_n(X) \text{ or } H_n(X,*).$$

For instance, if $X = \mathrm{sh}^{2n} H\mathbb{Q}$ is a shifted rational Eilenberg-Maclane spectrum, its homology is just a $\mathbb{Q}$ in degree 2n, but the homology of its infinite loop space $K(\mathbb{Q}, 2n)$ has a $\mathbb{Q}$ in every degree that is a multiple of $2n$:

$$\begin{array}{rcccccccccccc} & (0) & \cdots & (2n-1) & (2n) & (2n+1) & \cdots & (4n-1) & (4n) & (4n+1) & \cdots & (6n) & \cdots \\ H_*(X) = & 0 & \cdots & 0 & \mathbb{Q} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ H_*(\Omega^\infty X) = & \mathbb{Z} & \cdots & 0 & \mathbb{Q} & 0 & \cdots & 0 & \mathbb{Q} & 0 & \cdots & \mathbb{Q} & \cdots \end{array}$$

If we shifted $H\mathbb{Q}$ to an odd degree, the infinite loop space $K(\mathbb{Q}, 2n+1)$ would have the same reduced homology as the spectrum $\mathrm{sh}^{2n+1} H\mathbb{Q}$, just a $\mathbb{Q}$ in degree $2n+1$:

$$\begin{array}{rcccccc} & (0) & \cdots & (2n) & (2n+1) & (2n+2) & \cdots \\ H_*(X) = & 0 & \cdots & 0 & \mathbb{Q} & 0 & \cdots \\ H_*(\Omega^\infty X) = & \mathbb{Z} & \cdots & 0 & \mathbb{Q} & 0 & \cdots \end{array}$$

In general, $H_*(X;\mathbb{Q})$ is the *indecomposables* of $H_*(\Omega^\infty X;\mathbb{Q})$ as an algebra, or the *primitives* of $H_*(\Omega^\infty X;\mathbb{Q})$ as a coalgebra. See **??** for more details.

Remark 2.6.40. If X is a CW spectrum, there is a third way to define the homology of X. Define the cellular chain complex $C_*(X)$ by taking $C_k(X)$ to be the free abelian group on the stable k-cells of X, for all $k \in \mathbb{Z}$. This is a colimit of the cellular chain complexes of the levels X_n:

$$C_k(X) = \operatorname*{colim}_{n\to\infty} C_{k+n}(X_n, *).$$

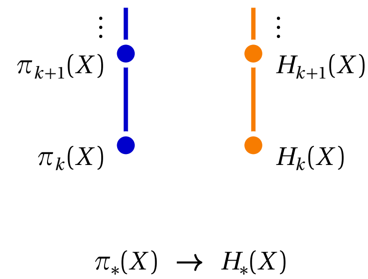
The cellular boundary maps of the spaces X_n give well-defined boundary maps on $C_*(X)$, and we define $H_*(X;G)$ to be the homology of $C_*(X) \otimes G$. Since this is defined using the colimit of the chain complexes $C_*(X_n)$, on homology we get the colimit of the homology of the levels, so this is isomorphic to the first definition in Definition 2.6.36.

Definition 2.6.41. For any spectrum X and $k \in \mathbb{Z}$, the Hurewicz map $\pi_k(X) \to H_k(X)$ is defined by applying the space-level Hurewicz maps to the colimit system in Definition 2.6.36. Equivalently, take any map of spectra $\mathbb{S} \to H\mathbb{Z}$ that is an isomorphism on π_0, and smash with X to get

$$\pi_k(X \wedge \mathbb{S}) \longrightarrow \pi_k(X \wedge H\mathbb{Z}).$$

Theorem 2.6.42 (Hurewicz theorem for spectra)**.** *If the spectrum X is $(k-1)$-connected then $\pi_k(X) \to H_k(X)$ is an isomorphism.*

Proof. By Proposition 2.6.29, X admits a stable equivalence from a spectrum Y that has stable cells only in degree k and above. By Proposition 2.6.37, the map $Y \to X$ induces an isomorphism on homology, so we just have to show that $\pi_k(Y) \to H_k(Y)$ is an isomorphism. Since Y has only stable cells of degree k and higher, Y_n only has cells (other than the basepoint) of dimension $(k+n)$ and higher. By the Hurewicz theorem for spaces, Proposition 1.4.14, $\pi_{k+n}(Y_n) \to H_{k+n}(Y_n, *)$ is an isomorphism. Taking the colimit of these isomorphisms gives the desired isomorphism. □



Corollary 2.6.43. *If X and Y are bounded below then the map $f : X \to Y$ is a stable equivalence iff it induces an isomorphism on homology groups.*

As a result, we frequently use ordinary homology to tell whether a map is a stable equivalence. This is great because homology is often easier to compute than stable homotopy.

2.7 Exercises

1. Modify the definition of spectra by allowing levels X_n for all $n \in \mathbb{Z}$. Explain why the theory of such spectra is equivalent to the theory of spectra presented in this chapter.

2. Similarly to the last problem, prove that extraordinary homology theories E_n defined with $n \in \mathbb{Z}$ can be recovered up to isomorphism from their restrictions to $n \geq 0$.

3. In this chapter we defined extraordinary homology theories as functors on CW pairs. One can similarly define reduced homology theories as functors on based CW complexes, or unreduced theories on unbased CW complexes. Write out the variant of the Eilenberg-Steenrod axioms in each of these cases. Show how to go back and forth between these versions without changing the homology theory, up to isomorphism.

4. (*) Suppose h_* is an extraordinary homology theory on based CW complexes, defined as in exercise 3. In this exercise we show that h_* also satisfies the reduced version of the direct limit axiom from Remark 2.5.6.

   (a) Define a new homology theory c_* on each CW complex X by the formula

$$c_n(X) = \operatorname*{colim}_{K \subseteq X \text{ finite}} h_n(K).$$

Prove that c_* is a functor on the category of based CW complexes and continuous based maps, and that it satisfies the axioms for an extraordinary homology theory. (You'll need to use the fact that filtered colimits preserve exact sequences.)

(b) There is a canonical natural transformation $c_* \to h_*$. Observe that this is an isomorphism on all finite complexes. Prove that it is also an isomorphism on arbitrary n-dimensional CW complexes, for each $n \geq 0$, by induction on n. You might find the usual cofiber sequence helpful:

$$X^{(n-1)} \longrightarrow X^{(n)} \longrightarrow \bigvee S^n.$$

(c) Given an arbitrary CW complex X, write it as a mapping telescope of n-dimensional CW complexes $X^{(n)}$, and use the cofiber sequence

$$\bigvee_{n\geq 0} X^{(n)} \longrightarrow \operatorname*{hocolim}_{n\to\infty} X^{(n)} \longrightarrow \bigvee_{n\geq 0} \Sigma X^{(n)}$$

to show that $c_* \to h_*$ is an isomorphism on X as well. Conclude that h_* satisfies the direct limit axiom.

5. (a) Prove that adding a trivial line bundle to a vector bundle E has the effect of suspending its Thom spaces, as in (2.1.16).

(b) More generally, given two vector bundles $E_1 \to B_1$ and $E_2 \to B_2$, form the external product bundle by taking their Cartesian product

$$E_1 \times E_2 \to B_1 \times B_2.$$

Prove that this goes to the smash of the Thom spaces,

$$\mathrm{Th}(E_1 \times E_2) \cong \mathrm{Th}(E_1) \wedge \mathrm{Th}(E_2).$$

6. A virtual bundle ζ can be presented in several equivalent ways. If

$$\zeta = E_1 - \epsilon^m = E_2 - \epsilon^n$$

are two presentations of the same virtual bundle, then for some k we have an isomorphism of bundles

$$E_1 \oplus \epsilon^{n+k} \cong E_2 \oplus \epsilon^{m+k}.$$

(Here $\epsilon^n = B \times \mathbb{R}^n$ is a trivial bundle.) Verify that for any two such presentations, the resulting constructions of the Thom spectrum $\mathrm{Th}(\zeta)$ are stably equivalent. Therefore $\mathrm{Th}(\zeta)$ is a well-defined spectrum up to stable equivalence.

7. The Thom isomorphism says that if $E \to B$ is an orientable vector bundle of dimension d, then
$$H_{n+d}(\mathrm{Th}(E), *) \cong H_n(B).$$
See e.g. [Coh98, Hat03]. Use this to prove a similar result for the Thom *spectrum* of a virtual bundle $\zeta = E - \epsilon^n$, if E is orientable. (See Example 2.6.38.)

8. Combine the constructions of Example 2.1.17 and Example 2.1.20 to define the Thom spectrum of a virtual bundle ζ over a non-compact base space B. In other words, start with the data of a virtual bundle $\zeta|_A$ over each finite subcomplex $A \subseteq B$, agreeing along intersections.

9. (a) Prove that a product of Ω-spectra $\prod_\alpha X_\alpha$ is again an Ω-spectrum. More generally, show that any limit of Ω-spectra is an Ω-spectrum.

   (b) Prove that if E is an Ω-spectrum and A is a CW complex then $F(A, E)$ is also an Ω-spectrum. Conclude that the homotopy fiber of any map of Ω-spectra is an Ω-spectrum.

10. Prove that $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$. Here X could be a based space or a spectrum, and Y and Z could likewise both be based spaces, or both be spectra.

11. Consider a system of abelian groups
$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$
Define a new system by taking $B_1 = A_{i_1}$ for some i_1, $B_2 = A_{i_2}$ for some $i_2 > i_1$, and so on. Define the maps $B_k \to B_{k+1}$ by composing the maps f_i. This is usually called a **cofinal subsystem**. Prove that
$$\operatorname*{colim}_i A_i \cong \operatorname*{colim}_k B_k.$$
Explain how this is being implicitly used in the proof of Proposition 2.4.1.

12. Consider the functor ev_0 that sends a spectrum X to its 0th space X_0, in the category of based spaces. Prove that the suspension spectrum functor Σ^∞, from based spaces to spectra, is the left adjoint. How does this change if we think of ev_0 as a functor to unbased spaces instead?

13. (a) Prove that the coproduct $X \vee Y$ satisfies the universal property of the coproduct in the category of spectra. More generally, prove that the colimit of spectra satisfies the universal property of the colimit.

    (b) Prove that the two bonding maps we defined for the product spectrum $X \times Y$ actually agree.

(c) Prove that the product $X \times Y$ satisfies the universal property of the product in the category of spectra. More generally, prove that the limit of spectra satisfies the universal property of the colimit.

14. Prove that the relative stable homotopy groups $\pi_k(X,A)$ from Definition 2.6.22 fit into a natural long exact sequence

$$\dots \longrightarrow \pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X,A) \longrightarrow \pi_{k-1}(A) \longrightarrow \dots$$

and more generally for two maps $A \to B \to X$ a long exact sequence

$$\dots \longrightarrow \pi_k(B,A) \longrightarrow \pi_k(X,A) \longrightarrow \pi_k(X,B) \longrightarrow \pi_{k-1}(B,A) \longrightarrow \dots$$

Define a natural map $\pi_k(X,A) \to \pi_k(Cf)$ and prove it is an isomorphism. (You may find Lemma 2.4.9 and its proof helpful.)

15. Recall the tensor $K \wedge -$ from Definition 2.3.6, the cotensor $F(K,-)$ from Definition 2.3.8, and the mapping space $\mathrm{Map}_*(-,-)$ from Definition 2.3.12.

(a) Prove that for any based space K, the functors $K \wedge -$ and $F(K,-)$ on spectra form an adjoint pair. (The idea is to deduce this from the same statement for based spaces.)

(b) Prove that for any spectrum X, the functors $- \wedge X$ and $\mathrm{Map}(X,-)$ form an adjoint pair between spaces and spectra. In summary, for spaces K and spectra X and Y, we get bijections

$$K \to \mathrm{Map}_*(X,Y) \quad \longleftrightarrow \quad K \wedge X \to Y \quad \longleftrightarrow \quad X \to F(K,Y).$$

(c) Use the Yoneda Lemma (Lemma 1.4.20) to upgrade these to homeomorphisms of topological spaces

$$\mathrm{Map}_*(K, \mathrm{Map}_*(X,Y)) \cong \mathrm{Map}_*(K \wedge X, Y) \cong \mathrm{Map}_*(X, F(K,Y)).$$

16. Prove that the un-shift functor $\mathrm{sh}^{-1}(-)$ and the shift functor $\mathrm{sh}(-)$ form an adjoint pair. It is easy to see that, like Σ and Ω, they are inverses up to stable equivalence.

17. (a) Show that the free spectrum functor $F_n \colon \mathbf{Top}_* \to \mathbf{Sp}$ is the left adjoint of the forgetful map $\mathrm{ev}_n \colon \mathbf{Sp} \to \mathbf{Top}_*$ that sends each spectrum X to its nth level X_n. As a special case, Σ^∞ is the left adjoint of ev_0.

(b) Show the same for the functor $F_n(-)_+ \colon \mathbf{Top} \to \mathbf{Sp}$ and the forgetful map to *unbased* spaces $\mathbf{Sp} \to \mathbf{Top}$.

18. Suppose X and Y are spectra. A **weak map** $f: X \rightsquigarrow Y$ consists of based maps $f_n: X_n \to Y_n$ and *homotopies* $\Sigma X_n \wedge I_+ \to Y_{n+1}$, from $f_{n+1} \circ \xi_n$ to $\upsilon_n \circ \Sigma f_n$, for each $n \geq 0$.

    Show that a weak map gives a zig-zag $X \leftarrow X' \to Y$ of actual maps of spectra, where $X' \to X$ is a level equivalence and the maps $X'_n \to Y_n$ are in the homotopy class of f_n. (Hint: Take X'_n to be the mapping telescope of $\Sigma^n X_0 \to \Sigma^{n-1} X_1 \to \ldots \to X_n$.)

19. Let X be any spectrum. By Proposition 2.4.1, the spectra ΣX and $\operatorname{sh} X$ (see Definition 2.3.6 and Definition 2.3.4) have isomorphic homotopy groups.

    (a) Find a zig-zag of stable equivalences between ΣX and $\operatorname{sh} X$. You may find exercise 18 helpful.

    (b) Use this to conclude that ΩX and $\operatorname{sh}^{-1} X$ are also stably equivalent.

    (c) Take the homotopy colimit of the stable equivalences $X \to \Omega \operatorname{sh} X$, to get a second construction of RX from Proposition 2.2.9. It is not homeomorphic to the first construction, only equivalent, but along the bottom of the last grid in the proof, we now get the groups $\pi_{k+m}(\Omega^m X)$ instead of A_m.

20. Prove Lemma 2.4.14.

21. Prove that for any homotopy pushout/pullback square of spectra as in Definition 2.4.13, there is a homotopy Mayer-Vietoris sequence as in Section 1.7, exercise 18:

$$\ldots \longrightarrow \pi_k(A) \xrightarrow{(f_*,h_*)} \pi_k(Y) \oplus \pi_k(Z) \xrightarrow{k_*-g_*} \pi_k(W) \xrightarrow{\partial} \pi_{k-1}(X) \longrightarrow \ldots$$

22. Consider the following alternate definition of cofiber sequence of spectra. A cofiber sequence of spectra $X \to Y \to Z$ is a homotopy pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ C \arrow[r] & Z \end{tikzcd}$$

    where C is weakly contractible, i.e. $C \to *$ is a stable equivalence. Prove that every cofiber sequence in our sense is a cofiber sequence in this sense. Conversely, prove that every cofiber sequence in this sense can be changed up to stable equivalence to be a cofiber sequence in our sense. (You may find it helpful to make the spectra into CW Ω-spectra and to use the Whitehead theorem.)

23. Dualize the proof of Proposition 2.4.19 to get the following: when the composite $A \xrightarrow{i} X \xrightarrow{p} A'$ is a stable equivalence, the canonical map $X \to Ci \times A'$ is also a stable equivalence. Conclude that the composite $Fp \to X \to Ci$ is a stable equivalence.

So when we have a retract in spectra, the complementary piece is both the cofiber of the inclusion, and the fiber of the projection.

24. Define the free loop spectrum LX of a spectrum X by the cotensor

$$LX = F(S^1_+, X).$$

- Check that on each spectrum level this takes the space of unbased loops.
- Show that LX is stably equivalent to the product $X \times \Omega X$.
- Show that $L(\Sigma^\infty_+ A)$ is not equivalent to $\Sigma^\infty_+ LA$ for any nonempty unbased space A.

25. (a) Fill in the missing step of Proposition 2.4.18 by proving that homotopy groups preserve finite products:

$$\pi_k\left(\prod_{i=1}^n X(i)\right) \cong \prod_{i=1}^n \pi_k(X(i)),$$

for any spectra $X(1)$, ..., $X(n)$ and $k \in \mathbb{Z}$.

(b) Prove that stable homotopy groups preserve infinite products of Ω-spectra:

$$\pi_k\left(\prod_\alpha X(\alpha)\right) \cong \prod_\alpha \pi_k(X(\alpha)),$$

for any collection of Ω-spectra $\{X(\alpha)\}_{\alpha \in A}$ and $k \in \mathbb{Z}$.

(c) Show that stable homotopy groups do not preserve infinite products in general. (This is what we mean by infinite products having the wrong homotopy type.) (You might consider taking a product of infinitely many copies of a colimit system that does not stabilize, such as the one in exercise 39.)

26. Dualize the proof of Corollary 2.4.23 and prove that if K is a *finite* CW complex, then the cotensor $F(K, -)$ preserves all stable equivalences. What goes wrong if K is infinite? (Consider exercise 25.)

27. Use Section 1.7, exercise 23 to prove that stable homotopy groups commute with sequential homotopy colimits of spectra:

$$\pi_k\left(\operatorname*{hocolim}_{n\to\infty} X(n)\right) \cong \operatorname*{colim}_{n\to\infty} \pi_k(X(n)).$$

They also commute with the strict colimit, provided the maps $X(n) \to X(n+1)$ are closed inclusions at each spectrum level.

28. Recall from Section 1.7, exercise 23 that if Y is compact and

$$X^{(-1)} \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \dots \longrightarrow X$$

is a sequence of closed inclusions with colimit X, then any map $Y \to X$ factors through some $X^{(j)}$.

   (a) Prove that the composite of two relative cell complexes is a relative cell complex. (Either in spaces or in spectra; the argument is the same.)

   (b) Prove that the composite of countably many relative cell complexes is a relative cell complex. (This is also true for larger transfinite compositions, we would just need more set theory to define and work with them.)

29. (a) Suppose S is a compact (CGWH) space, and $\bigvee_\alpha K(\alpha)$ is a wedge sum of based (CGWH) spaces. Prove that any map

$$S \longrightarrow \bigvee_{\alpha \in A} K(\alpha)$$

must factor through a finite wedge, i.e. $\bigvee_{\alpha \in F} K(\alpha)$ for some finite subset $F \subseteq A$. (This is similar to Section 1.7 exercise 23.)

   (b) Prove that stable homotopy groups preserve infinite coproducts:

$$\pi_k\left(\bigvee_\alpha X(\alpha)\right) \cong \bigoplus_\alpha \pi_k(X(\alpha)),$$

for any collection of spectra $\{X(\alpha)\}_{\alpha \in A}$ and $k \in \mathbb{Z}$. This finishes the proof of Proposition 2.5.7, that spectra define extraordinary homology theories.

30. (a) Prove that for any spectrum E, the cotensor $F(-,E)$ sends coproducts of based spaces $\bigvee_\alpha X_\alpha$ to products of spectra. More generally, it sends colimits to limits.

   (b) Prove also that the cotensor $F(-,E)$ sends cofiber sequences of CW complexes $A \xrightarrow{f} X \to Cf$ to fiber sequences of spectra.

   (c) Switching to the other slot, prove that the cotensor $F(A,-)$ sends fiber sequences of spectra $Ff \to E \to B$ to fiber sequences of spectra.

We have to be a little careful in this exercise – if we have a cofiber sequence of spaces $A \to X \to C$, where C is only *equivalent* to the cofiber, not isomorphic, then $F(-,E)$ may not respect that equivalence, so applying $F(-,E)$ may not give a fiber sequence. This can be corrected by taking the right-derived functor of $F(A,E)$, in other words, by making A a CW complex and making E an Ω-spectrum before feeding them into F. See Definition 3.3.15.

31. Let E be any Ω-spectrum. Prove that $\pi_{-*}(F(X/A,E))$ satisfies the Eilenberg-Steenrod axioms for a cohomology theory (Proposition 2.5.16). Explain how the proof fails if E is not an Ω-spectrum. (You may find it helpful to use exercises 9, 25, and 30.)

32. Prove that the homology theory defined in Proposition 2.5.7 extends to all pairs (X,A) in which $A \to X$ is a cofibration, and preserves all weak equivalences of such pairs.

    On the other hand, the cohomology theory in Proposition 2.5.7 does not extend this way, because weak equivalences $X \to X'$ do not always induce weak equivalences on $\mathrm{Map}(-,Y)$. Instead, we extend it by replacing an arbitrary pair (X,A) by an equivalent CW pair (QX,QA) and then feeding it into the earlier definition. In other words, we have to right-derive cohomology to extend it to all spaces (see Definition 3.3.15).

33. Suppose X and Y are based CW complexes.

    (a) Prove that $\Sigma^\infty X_+$ is stably equivalent to the wedge sum $\mathbb{S} \vee \Sigma^\infty X$.

    (b) Prove that $\Sigma^\infty(X \times Y)$ is stably equivalent to the wedge sum

    $$\Sigma^\infty X \vee \Sigma^\infty Y \vee \Sigma^\infty (X \wedge Y).$$

    (c) How does this generalize to $\Sigma^\infty(X_1 \times \cdots \times X_n)$?

34. (a) Prove that the suspension spectrum of a cofiber sequence of based spaces is a cofiber sequence of spectra.

    (b) Show by counterexample that the suspension spectrum of a fiber sequence of spaces may not be a fiber sequence of spectra. (Hint: Use exercise 33.)

    (c) Check that the right adjoint ev_0 preserves fiber sequences of Ω-spectra. Does it also preserve cofiber sequences?

35. Check that $H_*(\Sigma^\infty A) \cong H_*(A,*)$ when A is well-based.

36. Explain why a cofiber/fiber sequence of spectra $X \to Y \to Z$ gives a long exact sequences on homology:

    $$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

    Does this work for extraordinary homology as well?

37. A spectrum X is n-connected if $\pi_k(X) = 0$ for $k \le n$; it is n-connective if $\pi_k(X) = 0$ for $k < n$. Suppose we have a cofiber/fiber sequence $X \to Y \to Z$.

    (a) If X and Z are n-connected, prove that Y is n-connected.

(b) If X and Y are n-connected, what does this imply about Z? What if Y and Z are n-connected?

38. (Uniqueness of Eilenberg-Maclane spectra) Fix an abelian group G. Suppose X is a spectrum such that $\pi_0(X) \cong G$ and all other homotopy groups are zero. Prove there is a stable equivalence $HG \xrightarrow{\sim} X$. This shows that Eilenberg-Maclane spectra are unique up to stable equivalence. (Hint: Make sure you already know the corresponding argument for topological spaces. Then think carefully about how to build HG as a cellular spectrum.)

39. (a) Let A be any abelian group and let $A \xrightarrow{n} A$ be the map that multiplies by $n \in \mathbb{Z}$. Prove that the colimit of the maps

$$A \xrightarrow{1} A \xrightarrow{2} A \xrightarrow{3} \cdots$$

is the rationalization $A \otimes \mathbb{Q}$. (This gets easy if you know that $- \otimes \mathbb{Q}$ commutes with colimits.)

(b) Recall that the rational homotopy groups of spheres $\pi_k(S^n) \otimes \mathbb{Q}$ are

$$\pi_n(S^n) \otimes \mathbb{Q} = \mathbb{Q}, \qquad \pi_{4n-1}(S^{2n}) \otimes \mathbb{Q} = \mathbb{Q},$$

and zero otherwise. Use this to prove that the Moore space $M(\mathbb{Q}, n)$ from Example 2.5.30 is an Eilenberg-Maclane space $K(\mathbb{Q}, n)$ when n is odd.

(c) Using exercise 38, prove that the rational sphere $\mathbb{S}_{\mathbb{Q}} := F_n M(\mathbb{Q}, n)$ is equivalent to the Eilenberg-Maclane spectrum $H\mathbb{Q}$.

40. (a) Prove that the mod p version of ordinary homology, in the sense of Example 2.5.33, agrees with homology with $\mathbb{Z}/p$ coefficients. In other words, show that $H\mathbb{Z} \wedge \mathbb{S}/p \simeq H(\mathbb{Z}/p)$.

(b) Similarly, prove that the rationalization of $H\mathbb{Z}$ agrees with $H\mathbb{Q}$.

(c) Argue that the Hurewicz map from stable homotopy to homology comes from a map of spectra $\mathbb{S} \to H\mathbb{Z}$. Show that this map is an equivalence after rationalization, but not before.

41. Prove that the following conditions are equivalent. We say the spectrum X is **finite** if these conditions hold.

(1) X is stably equivalent to a cellular spectrum with finitely many stable cells.

(2) X is stably equivalent to a free spectrum $F_k A$ on a finite cell complex A.

(3) the direct sum of the homology groups $\bigoplus_k H_k(X; \mathbb{Z})$ is finitely generated.

(For (3) ⇒ (1), you'll have to use the long exact sequence on homology, and the fact that any subgroup of $\mathbb{Z}^{\oplus n}$ is isomorphic to $\mathbb{Z}^{\oplus k}$ for some k.)

42. Define spectra in simplicial sets, using $\Delta[1]/\partial\Delta[1]$ as the circle. Define a geometric realization functor to spectra in topological spaces, and a singular complex functor coming back. Prove these are inverses up to stable equivalence.

43. Use the Hurewicz theorem and exercise 33 to verify the calculations of π_0 and π_1 of suspension spectra from Example 2.1.7:

$$\pi_0(\Sigma^\infty A) \cong H_0(A,*;\mathbb{Z}),$$
$$\pi_1(\Sigma^\infty A) \cong H_0(A,*;\mathbb{Z}/2) \oplus H_1(A,*;\mathbb{Z}).$$

Chapter 3

The stable homotopy category

Let X and Y be spectra. In Proposition 2.4.18 we showed that the map

$$X \vee Y \to X \times Y$$

is a stable equivalence. We therefore bravely declare that $X \vee Y$ and $X \times Y$ are "the same spectrum."

It is easy enough to say this, but not so easy to put those words into practice. For example, suppose we define a map from a third spectrum Z into $X \times Y$. Then it should be possible to lift it to a map to the wedge, $Z \to X \vee Y$. After all, $X \vee Y$ is "the same" as $X \times Y$. But how would we actually construct such a lift? The inclusion $X \vee Y \to X \times Y$ isn't actually an isomorphism, so we can't compose with its inverse.

The simplest way to solve this problem is to take the category of spectra **Sp** and formally turn the stable equivalences into isomorphisms. This gives a new category called the **stable homotopy category**, or Ho **Sp** for short. It has the same objects as **Sp**, but different morphisms. In particular, the map $X \vee Y \to X \times Y$ turns into an isomorphism.

In this chapter we give all of the fundamental properties of the stable homotopy category, save for those that involve the smash product $\wedge$, which we put off to Section 4.1. With these properties, we can now effectively work with spectra up to stable equivalence.

3.1 Three equivalent definitions

3.1.1 First definition: zig-zags

Our first definition of Ho **Sp** will have the same objects as **Sp**, but the maps will be zig-zags of maps of spectra.

Definition 3.1.1. The stable homotopy category $\operatorname{Ho}\mathbf{Sp} = \mathbf{Sp}[s^{-1}]$ has an object for every spectrum X. The morphisms from X to Y, denoted $[X,Y]_s$, are equivalence classes of zig-zags of maps from X to Y. We require that the backwards-pointing maps are all stable equivalences. For instance:

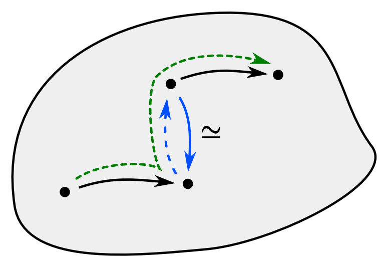
$$X \longrightarrow X_1 \xleftarrow{\sim} X_2 \xleftarrow{\sim} X_3 \longrightarrow X_4 \xleftarrow{\sim} X_5 \longrightarrow X_6 \xleftarrow{\sim} Y$$

Two zig-zags give the same morphism if they can be related by the following moves:

- compose two maps pointing in the same direction,
- cancel any identity map, or
- cancel out any pair of the form $\xrightarrow{f}\xleftarrow{f}$ or $\xleftarrow{f}\xrightarrow{f}$.

We compose two zig-zags by concatenating them. The identity is the empty zig-zag of maps from X to X, equivalently the one-term zig-zag $\left(\xrightarrow{\mathrm{id}_X}\right)$. We let $\delta\colon \mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$ denote the functor that takes each map f to the one-term zig-zag $\left(\xrightarrow{f}\right)$.

Intuitively, $\operatorname{Ho}\mathbf{Sp}$ is the category that turns the stable equivalences into isomorphisms, and does as little else as possible. To see why this gives zig-zags, consider the figure to the right. The blob represents the category of spectra. The blue morphism in the center is a stable equivalence, so we give it a formal inverse, drawn as a blue dashed arrow.



Sp

However, once this morphism exists, we can compose it with existing morphisms. So for instance we can get the zig-zag that follows the green dashed arrow along the top of the figure.

Example 3.1.2. We can define a "degree n" map $\mathbb{S} \to \mathbb{S}$ in the stable homotopy category by taking the following zig-zag.

$$\begin{array}{ccccc}
\mathbb{S} & \xleftarrow{\sim} & F_1 S^1 & \longrightarrow & \mathbb{S} \\
\hline\hline
S^0 & \longleftarrow & * & \longrightarrow & S^0 \\
S^1 & \xleftarrow{\cong} & S^1 & \xrightarrow{n} & S^1 \\
S^2 & \xleftarrow{\cong} & S^2 & \xrightarrow{n} & S^2 \\
S^3 & \xleftarrow{\cong} & S^3 & \xrightarrow{n} & S^3 \\
\vdots & & \vdots & & \vdots
\end{array}$$

Note that this does not define a morphism in **Sp**, because there is no map of spaces $S^0 \to S^0$ that suspends to give the degree n map $S^1 \to S^1$, unless $n = 0$ or 1. It only defines a morphism in $\operatorname{Ho}\mathbf{Sp}$.

In fact, every map $\mathbb{S} \to \mathbb{S}$ in the stable homotopy category is of this form, for some $n \in \mathbb{Z}$, by Example 3.1.35. This is hard to prove right now though. You would have to show that any zig-zag from $\mathbb{S}$ to $\mathbb{S}$ can be simplified to the one above.

Example 3.1.3. The Hopf map $\eta\colon S^3 \to S^2$ passes to a map in the stable homotopy category $\mathbb{S}^1 \to \mathbb{S}^0$. Informally, this is because suspension is invertible. Formally, we take the following zig-zag.

$$\begin{array}{ccccccc}
\mathbb{S}^1 & \xleftarrow{\quad\sim\quad} & F_2S^3 & \xrightarrow{\quad F_2\eta\quad} & F_2S^2 & \xrightarrow{\quad\sim\quad} & \mathbb{S}^0 \\
\hline\hline
S^1 & \longleftarrow & * & \xrightarrow{\quad 0\quad} & * & \longrightarrow & S^0 \\
S^2 & \longleftarrow & * & \xrightarrow{\quad 0\quad} & * & \longrightarrow & S^1 \\
S^3 & \xleftarrow{\quad\cong\quad} & S^3 & \xrightarrow{\quad\eta\quad} & S^2 & \xrightarrow{\quad\cong\quad} & S^2 \\
S^4 & \xleftarrow{\quad\cong\quad} & S^4 & \xrightarrow{\quad\Sigma\eta\quad} & S^3 & \xrightarrow{\quad\cong\quad} & S^3 \\
S^5 & \xleftarrow{\quad\cong\quad} & S^5 & \xrightarrow{\quad\Sigma^2\eta\quad} & S^4 & \xrightarrow{\quad\cong\quad} & S^4 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Again, η does not define a morphism in the category of spectra **Sp**, only the homotopy category $\operatorname{Ho}\mathbf{Sp}$. We had to pass to the homotopy category, to be able to cut off a few levels from $\mathbb{S}^1$ and then define the map on the rest.

Example 3.1.4. For any spectrum X, we can define a degree n map $X \xrightarrow{n} X$ in many ways, including the following:

$$\begin{array}{ccccccc}
X & \xrightarrow{\quad(\mathrm{id},\ldots,\mathrm{id})\quad} & \prod_{i=1}^n X & \xleftarrow{\quad\sim\quad} & \bigvee_{i=1}^n X & \xrightarrow{\quad(\mathrm{id},\ldots,\mathrm{id})\quad} & X \\
X & \xleftarrow{\quad\sim\quad} & \mathrm{sh}^{-1}\Sigma X & \xrightarrow{\quad S^1 \xrightarrow{n} S^1\quad} & \mathrm{sh}^{-1}\Sigma X & \xleftarrow{\quad\sim\quad} & X \\
X & \xleftarrow{\quad\sim\quad} & \mathrm{sh}^1\Omega X & \xrightarrow{\quad S^1 \xrightarrow{n} S^1\quad} & \mathrm{sh}^1\Omega X & \xleftarrow{\quad\sim\quad} & X.
\end{array}$$

These all define the same map in the stable homotopy category by exercise 11.

Example 3.1.5. If $f: X \to Y$ is a stable equivalence, then $\delta(f) = \left(\xrightarrow{f}\right)$ is an isomorphism in $\mathrm{Ho}\,\mathbf{Sp}$. Its inverse is the zig-zag $\left(\xleftarrow{f}\right)$:

$$\left(\xrightarrow{f}\right)\left(\xleftarrow{f}\right) = \left(\xrightarrow{f}\xleftarrow{f}\right) = (\mathrm{id}_X), \qquad \left(\xleftarrow{f}\right)\left(\xrightarrow{f}\right) = \left(\xleftarrow{f}\xrightarrow{f}\right) = (\mathrm{id}_Y).$$

In fact, $\mathrm{Ho}\,\mathbf{Sp}$ is the universal category in which the stable equivalences in $\mathbf{Sp}$ become isomorphisms.

Proposition 3.1.6. *Suppose $F: \mathbf{Sp} \to \mathbf{D}$ is any functor taking stable equivalences to isomorphisms. Then there is a unique functor making this triangle commute:*

$$\begin{tikzcd} & \mathrm{Ho}\,\mathbf{Sp} \arrow[d, dashed, "\mathrm{Ho}\,F\ \text{(or just } F\text{)}"] \\ \mathbf{Sp} \arrow[ur, "\delta"] \arrow[r, "F"'] & \mathbf{D} \end{tikzcd} \tag{3.1.7}$$

Proof. We define the functor $\mathrm{Ho}\,F$ by sending X to $F(X)$, and sending each zig-zag to the composite of F of the morphisms (inverted when they point backwards). This gives the desired commuting triangle. Any other functor making the triangle commute would have to agree with $\mathrm{Ho}\,F$ on objects and one-term zig-zags. But then it follows that it agrees on every zig-zag. □

Example 3.1.8. By definition, each stable equivalence $X \to Y$ gives an isomorphism on the stable homotopy groups $\pi_k(-)$. By Proposition 3.1.6, π_k therefore defines a functor on the stable homotopy category, $\mathrm{Ho}\,\mathbf{Sp} \to \mathbf{Ab}$. By Proposition 2.6.37, homology $H_k(-;G)$ also defines a functor $\mathrm{Ho}\,\mathbf{Sp} \to \mathbf{Ab}$.

It is clear that the proof of Proposition 3.1.6 has nothing to do with spectra.

Definition 3.1.9. Given any category $\mathbf{C}$, and any collection of morphisms W that we call the "weak equivalences" in $\mathbf{C}$, we form the category $\mathrm{Ho}\,\mathbf{C} = \mathbf{C}[W^{-1}]$ by taking the objects of $\mathbf{C}$, and defining the morphisms to be equivalence classes of zig-zags with backwards maps in W, as in Definition 3.1.1. We may as well assume that W is closed under composition, so that we can compose arrows pointing to the left. Considering one-term zig-zags, we see there is a functor

$$\delta: \mathbf{C} \longrightarrow \mathbf{C}[W^{-1}].$$

Theorem 3.1.10. *[GZ67, 1.1] Any functor $F: \mathbf{C} \to \mathbf{D}$ that sends weak equivalences to isomorphisms, must factor uniquely through $\mathbf{C}[W^{-1}]$.*

Definition 3.1.11. The homotopy category of unbased spaces $\mathrm{Ho}\,\mathbf{Top} = \mathbf{Top}[w^{-1}]$ is formed from $\mathbf{Top}$ by inverting the weak homotopy equivalences. A morphism in $\mathrm{Ho}\,\mathbf{Top}$ is a zig-zag of maps of unbased spaces, where the maps pointing to the left are weak homotopy equivalences.

Definition 3.1.12. The homotopy category of based spaces $\operatorname{Ho}\mathbf{Top}_* = \mathbf{Top}_*[w^{-1}]$ is formed from $\mathbf{Top}_*$ by inverting the weak homotopy equivalences. A morphism in $\operatorname{Ho}\mathbf{Top}_*$ is a zig-zag of based maps of based spaces, where the maps pointing to the left are weak homotopy equivalences.

Example 3.1.13. Singular homology $H_k(-;G)$ preserves weak equivalences. By Theorem 3.1.10, it therefore defines a functor on the homotopy category $\operatorname{Ho}\mathbf{Top} \to \mathbf{Ab}$.

Remark 3.1.14. There is a size issue in these constructions: for any two objects X and Y in $\mathbf{C}$, the collection of morphisms from X to Y in $\operatorname{Ho}\mathbf{C}$ may be so large that it is not a set, but rather a proper class. This is addressed by allowing the homotopy category to have sets lying in a larger universe. In the examples we actually encounter, such as $\operatorname{Ho}\mathbf{Sp}$, the morphisms do in fact form a set, so this set-theory workaround isn't necessary. We won't worry so much about this issue here.

We end with a lemma about the homotopy category that will be useful when comparing it to other models. Recall that two maps of spectra $f, g\colon X \rightrightarrows Y$ are homotopic if they extend to a map of spectra $h\colon I_+ \wedge X \to Y$.

Lemma 3.1.15. *If f and g are homotopic, then $\delta(f) = \delta(g)$ in* $\operatorname{Ho}\mathbf{Sp}$*. The same is true for spaces: homotopic maps of spaces give the same morphism in* $\operatorname{Ho}\mathbf{Top}$.

Proof. Let $i_0, i_1\colon X \rightrightarrows I_+ \wedge X$ be the inclusion of the top and bottom of the cylinder, and let $p\colon I_+ \wedge X \to S^0 \wedge X \cong X$ be the projection. Then in $\operatorname{Ho}\mathbf{Sp}$,

$$\left(\xrightarrow{f}\right) = \left(\xrightarrow{i_0}\xrightarrow{h}\right) = \left(\xrightarrow{i_0}\xrightarrow{p}\xleftarrow{p}\xrightarrow{h}\right) = \left(\xleftarrow{p}\xrightarrow{h}\right)$$

$$= \left(\xrightarrow{i_1}\xrightarrow{p}\xleftarrow{p}\xrightarrow{h}\right) = \left(\xrightarrow{i_1}\xrightarrow{h}\right) = \left(\xrightarrow{g}\right).$$

□

Definition 3.1.16. For spectra X and Y, we let

- $[X, Y]_s$ denote the set of maps from X to Y in $\operatorname{Ho}\mathbf{Sp}$, i.e. zig-zags up to the equivalence relation of Definition 3.1.1, and
- $[X, Y]_h$ denote maps of spectra $X \to Y$ up to homotopy.

Remark 3.1.17. By Lemma 3.1.15, there is a well-defined map $[X, Y]_h \to [X, Y]_s$. But this is often not a bijection. For instance, $[\mathbb{S}, \mathbb{S}]_h = \{0, 1\}$, but we will prove later that $[\mathbb{S}, \mathbb{S}]_s \cong \mathbb{Z}$.

3.1.2 Equivalences of categories and a second definition

In this section we give several definitions of $\mathrm{Ho}\,\mathbf{Sp}$ that are equivalent to the one in Definition 3.1.1. We first recall what an equivalence of categories is.

Definition 3.1.18. A functor $F : \mathbf{C} \to \mathbf{D}$ is an **isomorphism of categories** if it is a bijection on objects and on morphisms. The functor F is an **equivalence of categories** if it is a bijection on isomorphism classes of objects

$$\mathrm{ob}\,\mathbf{C}/\mathrm{isom} \xrightarrow{\cong} \mathrm{ob}\,\mathbf{D}/\mathrm{isom} \tag{3.1.19}$$

and a bijection on morphisms between any pair of objects,

$$\mathbf{C}(X, Y) \xrightarrow{\cong} \mathbf{D}(F(X), F(Y)). \tag{3.1.20}$$

Implicit in this definition is the fact that any functor F induces a function (3.1.19) on isomorphism classes of objects. This is because every functor preserves isomorphisms,

$$X \cong Y \;\Rightarrow\; F(X) \cong F(Y).$$

We say $F : \mathbf{C} \to \mathbf{D}$ is **fully faithful** if (3.1.20) is a bijection. Informally, this means $\mathbf{C}$ and $\mathbf{D}$ have “the same morphisms.” By exercise 1, this implies that (3.1.19) is injective.

We say that F is **essentially surjective** if every $Z \in \mathrm{ob}\,\mathbf{D}$ is isomorphic to $F(X)$ for some $X \in \mathrm{ob}\,\mathbf{C}$. Of course, this is the same thing as saying that (3.1.19) is surjective. Putting this all together:

Lemma 3.1.21. *F is an equivalence of categories iff it is both fully faithful and essentially surjective.*

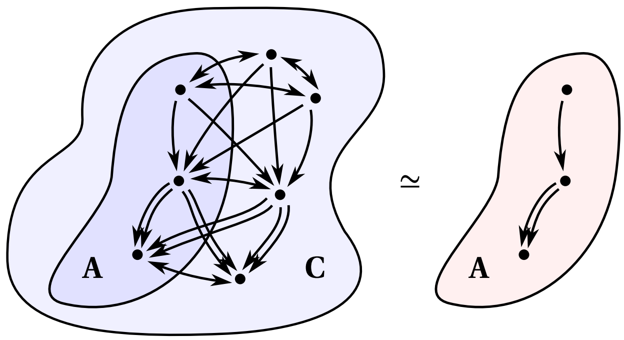
The distinction between “isomorphism” and “equivalence” of categories is very much like the difference between a homeomorphism and a homotopy equivalence of topological spaces. A homeomorphism $X \to Y$ gives a bijection on the underlying set. A homotopy equivalence only gives a bijection on the path components and the homotopy groups. The following standard lemma takes this analogy further.

Lemma 3.1.22. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. If there exists a second functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $GFX \cong X$ and $FGY \cong Y$ for all $X \in \mathrm{ob}\,\mathbf{C}$ and $Y \in \mathrm{ob}\,\mathbf{D}$, then F is an equivalence of categories. The converse holds as well, modulo issues of size and the axiom of choice.*

Example 3.1.23. Let $\mathbf{A} \subseteq \mathbf{C}$ be a full subcategory. That is, we select some objects from $\mathbf{C}$, and take all morphisms between them:

$$\mathbf{A}(X, Y) := \mathbf{C}(X, Y).$$

The inclusion $\mathbf{A} \subseteq \mathbf{C}$ is obviously fully faithful. So, it is an equivalence of categories if $\mathbf{A}$ contains at least one object in each isomorphism class. In this case the inverse functor $G \colon \mathbf{C} \to \mathbf{A}$ from Lemma 3.1.22 can be chosen to be the identity on $\mathbf{A}$, so that it is a "deformation retract" of categories.



Example 3.1.24. If $\mathbf{A}$ is a **skeleton** of $\mathbf{C}$, containing exactly one object in each isomorphism class, then $\mathbf{A} \subseteq \mathbf{C}$ is an equivalence of categories.

To give a common example from linear algebra, let $\mathbf{C}$ be the category of finite-dimensional real vector spaces, and $\mathbf{A} \subseteq \mathbf{C}$ be the full subcategory consisting of the vector space $\mathbb{R}^n$ for each $n \geq 0$. Then $\mathbf{A}$ is a skeleton of $\mathbf{C}$, and is therefore equivalent to $\mathbf{C}$. Choosing a deformation retract of $\mathbf{C}$ onto $\mathbf{A}$ amounts to choosing a basis for each finite-dimensional vector space.

Definition 3.1.25. Define $\operatorname{Ho}\mathbf{CW} = \mathbf{CW}[w^{-1}]$ to have objects the CW complexes, and maps the zig-zags of CW complexes as in Definition 3.1.1, where the backwards maps are weak homotopy equivalences. Define $h\mathbf{CW}$ to have objects the CW complexes, and morphisms the homotopy classes of maps.

By Lemma 3.1.15, homotopic maps give the same zig-zag, so we have functors

$$h\mathbf{CW} \longrightarrow \operatorname{Ho}\mathbf{CW} = \mathbf{CW}[w^{-1}] \longrightarrow \operatorname{Ho}\mathbf{Top} = \mathbf{Top}[w^{-1}].$$

Proposition 3.1.26. *These functors are equivalences of categories.*

Proof. $h\mathbf{CW}$ and $\operatorname{Ho}\mathbf{CW}$ have the same objects, so we just have to show they have the same morphisms. The easiest way to do this is to show that $h\mathbf{CW}$ satisfies the universal property of $\operatorname{Ho}\mathbf{CW}$: given any map $F \colon \mathbf{CW} \to \mathbf{D}$ taking weak homotopy equivalences to isomorphisms, it factors uniquely through $h\mathbf{CW}$.

By the Whitehead theorem (Theorem 1.4.12), weak homotopy equivalences between CW complexes are homotopy equivalences. Therefore $\mathbf{CW} \to h\mathbf{CW}$ does in fact take weak homotopy equivalences to isomorphisms. Furthermore, if $F: \mathbf{CW} \to \mathbf{D}$ takes weak homotopy equivalences to isomorphisms, then for homotopic maps f, g we have

$$F(f) = F(h) \circ F(i_0) = F(h) \circ F(p)^{-1} = F(h) \circ F(i_1) = F(g)$$

where p, i_0, and i_1 are defined as in Lemma 3.1.15. Therefore F factors through $h\mathbf{CW}$ in a unique way. This verifies the universal property, so $h\mathbf{CW} \cong \operatorname{Ho}\mathbf{CW}$.

For the second part of the proof, we know that $\operatorname{Ho}\mathbf{CW} \to \operatorname{Ho}\mathbf{Top}$ is essentially surjective, because every space is weakly equivalent to a CW complex. So we just have to show these categories have the same morphisms. In other words, for two CW complexes X and Y, we have to show we get a bijection between zig-zags of CW complexes, and zig-zags of all spaces.

Let $Q(-)$ be a functor that replaces spaces by equivalent CW complexes. To each zig-zag of arbitrary spaces between X and Y, we apply Q and get a commuting diagram

$$\begin{tikzcd}
X \arrow[r] & X_1 & X_2 \arrow[l, "\sim"'] \arrow[r] & X_3 & X_4 \arrow[l, "\sim"'] \arrow[r] & X_5 \arrow[r] & Y \\
QX \arrow[r] \arrow[u, "\sim"'] & QX_1 \arrow[u, "\sim"'] & QX_2 \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"'] & QX_3 \arrow[u, "\sim"'] & QX_4 \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"'] & QX_5 \arrow[r] \arrow[u, "\sim"'] & QY. \arrow[u, "\sim"']
\end{tikzcd}$$

We define the inverse function $\operatorname{Ho}\mathbf{Top}(X, Y) \to \operatorname{Ho}\mathbf{CW}(X, Y)$ by taking the zig-zag along the top to the zig-zag along the bottom and sides. This is an inverse to the map $\operatorname{Ho}\mathbf{CW}(X, Y) \to \operatorname{Ho}\mathbf{Top}(X, Y)$ by exercise 6, so we have a bijection on the morphism sets.

□

Definition 3.1.27. Let

- $\mathbf{Sp}^{CW} \subseteq \mathbf{Sp}$ be the full subcategory of CW spectra,
- $\mathbf{Sp}^{\Omega} \subseteq \mathbf{Sp}$ be the full subcategory of Ω-spectra, and
- $\mathbf{Sp}^{CW,\Omega} \subseteq \mathbf{Sp}$ be the full subcategory of CW Ω-spectra.

We take the homotopy category of each of these by inverting the stable equivalences. We also let $h\mathbf{Sp}^{CW,\Omega}$ have the same objects as $\mathbf{Sp}^{CW,\Omega}$, but the morphisms are maps of spectra up to homotopy, $[X, Y]_h$.

Proposition 3.1.28. *The inclusions of categories*

$$\begin{tikzcd}
\mathbf{Sp}^{CW,\Omega} \arrow[r] \arrow[d] & \mathbf{Sp}^{\Omega} \arrow[d] \\
\mathbf{Sp}^{CW} \arrow[r] & \mathbf{Sp}
\end{tikzcd}$$

induce equivalences of homotopy categories

$$\begin{tikzcd} h\mathbf{Sp}^{CW,\Omega} \arrow[r, leftrightarrow, "\cong"] & \mathrm{Ho}\,\mathbf{Sp}^{CW,\Omega} \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{Ho}\,\mathbf{Sp}^{\Omega} \arrow[d, "\sim"] \\ & \mathrm{Ho}\,\mathbf{Sp}^{CW} \arrow[r, "\sim"] & \mathrm{Ho}\,\mathbf{Sp} \end{tikzcd}$$

Proof. The proof is the same as in Proposition 3.1.26. For the first part, we use the Whitehead theorem (Corollary 2.6.17) to show that stable equivalences go to isomorphisms in $h\mathbf{Sp}^{CW,\Omega}$. For the second part, we use the CW replacement functor Q constructed in Theorem 2.6.12 and the Ω-spectrum replacement functor R from Proposition 2.2.9. □

Proposition 3.1.28 tells us that if X and Y are CW Ω-spectra, then the map

$$[X,Y]_h \to [X,Y]_s$$

is a bijection. This is not true for arbitrary X and Y, as mentioned in Remark 3.1.17. See exercise 5.

Proposition 3.1.28 gives us four more models for the stable homotopy category. The most interesting of these is the one on the far left – we restrict our attention to CW Ω-spectra, but now we just take single maps (not zig-zags) up to homotopy. This is our second definition of the stable homotopy category.

Remark 3.1.29. We can give an explicit inverse to the inclusion $\mathrm{Ho}\,\mathbf{Sp}^{CW} \subseteq \mathrm{Ho}\,\mathbf{Sp}$. Let $Q\colon \mathbf{Sp} \to \mathbf{Sp}^{CW}$ be the functor that replaces each spectrum X by a stably equivalent CW spectrum. Since Q is stably equivalent to the identity, composing Q with the inclusion $\mathrm{Ho}\,\mathbf{Sp}^{CW} \subseteq \mathrm{Ho}\,\mathbf{Sp}$ (in either order) is isomorphic to the identity. Therefore Q gives the inverse equivalence of categories $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}^{CW}$.

We can similarly use R to go back to Ω-spectra. So we have a diagram of equivalences of categories

$$\begin{tikzcd} h\mathbf{Sp}^{CW,\Omega} \arrow[r, leftrightarrow, "\cong"] & \mathrm{Ho}\,\mathbf{Sp}^{CW,\Omega} \arrow[r, "\sim"'] \arrow[d, "\sim"] & \mathrm{Ho}\,\mathbf{Sp}^{\Omega} \arrow[l, bend right, "Q"'] \arrow[d, "\sim"'] \\ & \mathrm{Ho}\,\mathbf{Sp}^{CW} \arrow[r, "\sim"] & \mathrm{Ho}\,\mathbf{Sp} \arrow[l, bend left, "Q"] \arrow[u, bend right, "R"'] \end{tikzcd}$$

We conclude that there is a bijection

$$[X,Y]_s \cong [QRX, QRY]_h.$$

Example 3.1.30. The degree n map of the sphere from Example 3.1.2 can be described in $h\mathbf{Sp}^{CW,\Omega}$ by taking the map $\Omega^\infty\Sigma^\infty S^0 \to \Omega^\infty\Sigma^\infty S^0$ that applies the degree n map to every sphere in the colimit system after S^0. With a fair amount of effort, this can be extended to the rest of the fibrant sphere spectrum $f\mathbb{S}$ from Example 2.2.6. We then have to apply the CW replacement functor Q to both sides, and we arrive at a map in $h\mathbf{Sp}^{CW,\Omega}$.

Phew! Hopefully this example demonstrates that it is usually not practical to define maps of CW Ω-spectra directly.

3.1.3 Third definition: cells now, maps later

We have now seen two definitions of the stable homotopy category. The first definition takes zig-zags of maps up to an equivalence relation. This is convenient because it is easy to name a single morphism explicitly. But it is inconvient because it is very hard to count how many morphisms there are from X to Y. Literally any object in the entire category could show up in the zig-zag!

The second definition takes CW Ω-spectra and homotopy classes of maps. In principle, this makes it easier to count how many morphisms there are. In practice, though, this is not actually very useful. CW Ω-spectra are too complicated.

We can now give a third definition that captures the best of both worlds.

Definition 3.1.31. Let X be a CW spectrum. A **cofinal subspectrum** $X' \subseteq X$ consists of a subcomplex $X'_n \subseteq X_n$ on each level, such that $\xi_n(\Sigma X'_n) \subseteq X'_{n+1}$, and such that every stable cell of X has a representative in X'.

Example 3.1.32. The sphere spectrum $\mathbb{S}$ has the shift desuspension $F_n S^n$ as a cofinal subspectrum, for each $n \geq 0$. The suspension spectrum $\Sigma^\infty \mathbb{RP}^2$ has a stable 1-cell e_1 and a stable 2-cell e_2. We could form a cofinal spectrum by taking e_1 at spectrum level 15, where it is a 16-cell, and e_2 at spectrum level 18, where it is a 20-cell attached to the 19-cell representing e_1.

Definition 3.1.33. Suppose X is a CW spectrum and Y is a spectrum. An **eventually-defined map** $X \to Y$ is a cofinal subspectrum $X' \subseteq X$ and a map of spectra $X' \to Y$. In other words, the map is defined on each stable cell, but not right away – we may wait several levels before the map becomes defined.

Given two eventually-defined maps $(X', f : X' \to Y)$ and $(X'', g : X'' \to Y)$, an eventually-defined homotopy is a third cofinal subspectrum $X''' \subseteq X' \cap X''$ and a homotopy from f to g on X'''.

Let $[X, Y]_e$ denote the set of eventually-defined maps up to eventually-defined homotopy.

The above definition was famously summarized by Adams as "cells now, maps later." See [Ada74].

Definition 3.1.34. The category $e\mathbf{Sp}^{CW}$ has an object for each CW spectrum. The morphisms from X to Y are $[X,Y]_e$, the eventually-defined maps up to eventually-defined homotopy. In particular, we can always restrict f to a smaller cofinal subspectrum $X'' \subseteq X'$ without changing the resulting morphism in $[X,Y]_e$.

We compose two morphisms

$$(X' \subseteq X, f\colon X' \to Y), \quad (Y' \subseteq Y, g\colon Y' \to Z)$$

by taking any cofinal $X'' \subseteq X'$ whose image is contained in Y', and taking the composite $g \circ f$ on X''. Such an X'' exists because each stable cell of X has image in Y contained in finitely many stable cells, and hence is eventually contained in Y'. It is straightforward to check that this rule is well-defined on homotopy classes of maps.

Example 3.1.35. The maps $[\mathbb{S},\mathbb{S}]_e$ are easily computed. The sphere has just one stable 0-cell, so a cofinal subspectrum must be of the form $F_n S^n$ for some n. The set of maps $[S^n,S^n]_*$ is $\mathbb{Z}$ so long as $n \geq 1$. It follows that $[\mathbb{S},\mathbb{S}]_e \cong \mathbb{Z}$.

Similarly, using the fact that $[S^{n+1},S^n]_* \cong \mathbb{Z}/2$ for $n \geq 3$, we can compute that $[\mathbb{S}^1,\mathbb{S}^0]_e \cong \mathbb{Z}/2$. The nonzero element is the Hopf map η, defined as in Example 3.1.3 by waiting until spectrum level two and then taking the Hopf map $S^3 \to S^2$.

We will prove that $e\mathbf{Sp}^{CW}$ is equivalent to the other models of the homotopy category, extending the diagram from Remark 3.1.29 to the following.

$$\begin{tikzcd}
h\mathbf{Sp}^{CW,\Omega} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\sim"] & \mathrm{Ho}\,\mathbf{Sp}^{CW,\Omega} \arrow[r, "\sim"'] \arrow[d, "\sim"] & \mathrm{Ho}\,\mathbf{Sp}^{\Omega} \arrow[l, bend right, "Q"'] \arrow[d, "\sim"'] \\
e\mathbf{Sp}^{CW} \arrow[r, leftrightarrow, "\cong"] & \mathrm{Ho}\,\mathbf{Sp}^{CW} \arrow[r, "\sim"] & \mathrm{Ho}\,\mathbf{Sp} \arrow[l, bend left, "Q"] \arrow[u, bend right, "R"']
\end{tikzcd} \tag{3.1.36}$$

Lemma 3.1.37. *The inclusion of a cofinal subspectrum $X' \to X$ is a stable equivalence.*

Proof. To show that $\pi_k(X') \to \pi_k(X)$ is surjective, we take any element and represent it by a map $S^{k+n} \to X_n$. Since S^{k+n} is compact, the image of this map is contained in a finite subcomplex. If n is sufficiently large then all the cells of this complex are contained in X'_n, so the map lifts to X'. The proof of injectivity is the same argument with homotopies $S^{k+n} \wedge I_+ \to X_n$. □

Proposition 3.1.38. *Every stable equivalence of CW spectra $f\colon X \to Y$ is an isomorphism in $e\mathbf{Sp}^{CW}$; it has an eventually-defined homotopy inverse.*

Proof. We factor f into the mapping cylinder $X \to M$ and then the homotopy equivalence of spectra $M \to Y$. The fully-defined homotopy equivalence $M \simeq Y$ is clearly an isomorphism in $e\mathbf{Sp}^{CW}$. So, we just need to show that $X \to M$ is an isomorphism as well.

We do this by defining an eventually-defined deformation retract of M onto X. We define this on a cofinal subspectrum of $I_+ \wedge M$, working one stable cell of M at a time. For each k-cell of M, represented by a disc $D^{k+n} \to M_n$ for all n sufficiently large, we are required to define a map of pairs

$$(D^{k+n} \times I, D^{k+n} \times 0) \to (M_n, X_n)$$

that is already specified on the top $D^{k+n} \times \{1\}$ (as the identity of M_n) and on the sides $\partial D^{k+n} \times I$ (by the previously defined cells), when n is sufficiently large. Since the top and sides of $D^{k+n} \times I$ are homeomorphic to D^{k+n}, we get a class $\alpha_n \in \pi_{k+n}(M_n, X_n)$, and the extension exists if this class is zero. Since $\pi_k(M, X) = 0$, the class α_n is zero for sufficiently large n. Therefore, after possibly waiting longer to define the deformation retract on this cell, the required extension exists. Repeating for each cell, we get the desired eventually-defined deformation retract. $\square$

Theorem 3.1.39. *There is an isomorphism of categories $e\mathbf{Sp}^{CW} \cong \mathrm{Ho}\,\mathbf{Sp}^{CW}$.*

Proof. We define a functor $F\colon e\mathbf{Sp}^{CW} \longrightarrow \mathrm{Ho}\,\mathbf{Sp}^{CW}$, in other words maps

$$[X, Y]_e \to [X, Y]_s,$$

by taking the eventually-defined map $(X' \subseteq X, f\colon X' \to Y)$ to the zig-zag

$$X \xleftarrow{\subseteq} X' \xrightarrow{f} Y.$$

Passing to a further cofinal subspectrum $X'' \subseteq X'$ gives the same morphism in $[X, Y]_s$ by examining the commuting diagram

$$\begin{tikzcd}
 & X' \arrow[dl, hook', "\subseteq"'] \arrow[dr, "f"] \arrow[d, "\subseteq"] & \\
X & & Y. \\
 & X' \arrow[ul, hook, "\subseteq"] \arrow[ur, "f|_{X''}"'] &
\end{tikzcd}$$

If we change f by a homotopy, we also get the same morphism in $[X, Y]_s$ by Lemma 3.1.15.

We check that this respects compositions of maps, so that it defines a functor. Given a second morphism $Y \longrightarrow Z$ defined on $Y' \subset Y$, $F(g \circ f)$ becomes the top branch of the diagram

$$\begin{tikzcd}
 & X'' \arrow[r] \arrow[d, "\sim"] & Y' \arrow[r] \arrow[d, "\sim"] & Z \\
X & X' \arrow[l, hook', "\sim"] \arrow[r] & Y &
\end{tikzcd}$$

while $F(g) \circ F(f)$ is the bottom branch. Since the diagram commutes in $\mathbf{Sp}^{CW}$, it also commutes as a diagram in $\mathrm{Ho}\,\mathbf{Sp}^{CW}$, so these two routes give the same morphism of $\mathrm{Ho}\,\mathbf{Sp}^{CW}$.

To show that F is an isomorphism, we define its inverse $G\colon \operatorname{Ho}\mathbf{Sp}^{CW} \to e\mathbf{Sp}^{CW}$. There is an obvious functor $\mathbf{Sp}^{CW} \to e\mathbf{Sp}^{CW}$. By Proposition 3.1.38, this functor takes stable equivalences to isomorphisms. Therefore it factors through the homotopy category $\operatorname{Ho}\mathbf{Sp}^{CW}$, giving our desired functor $G\colon \operatorname{Ho}\mathbf{Sp}^{CW} \to e\mathbf{Sp}^{CW}$.

The composite $G \circ F$ takes each eventually-defined map $(X' \subseteq X, f\colon X' \to Y)$ to the inverse of $X' \subseteq X$, composed with $f\colon X' \to Y$. The inverse of the inclusion is the eventually-defined map $X \to X'$ that is just the identity map on X'. Composing this with f gives the original map back, $(X' \subseteq X, f\colon X' \to Y)$, so $G \circ F$ is the identity functor.

The composite $F \circ G$ only has to be calculated on $\mathbf{Sp}^{CW}$, by the universal property of $\operatorname{Ho}\mathbf{Sp}^{CW}$. For each map of spectra $f\colon X \to Y$, it is sent to the eventually-defined map $(X = X, f\colon X \to Y)$, and then back to $f\colon X \to Y$ again. So $F \circ G$ is the identity functor. $\square$

In summary, all of the models of the stable homotopy category from the last three sections are equivalent to each other. We are free to use whichever one is the most convenient at the moment:

$$[X,Y]_s \cong [X,Y]_h \text{ when } X \text{ and } Y \text{ are both CW } \Omega\text{-spectra, and}$$
$$[X,Y]_s \cong [X,Y]_e \text{ when } X \text{ and } Y \text{ are both CW spectra.}$$

It will be helpful to weaken these hypotheses a little more. Notice that eventually-defined maps $[X,Y]_e$ make sense if X is CW, but Y is arbitrary. In this setting we have three maps

$$[X,Y]_h \to [X,Y]_e \to [X,Y]_s,$$

which are not necessarily bijections.

Proposition 3.1.40. *If X is CW and Y is an Ω-spectrum, both of the maps $[X,Y]_h \to [X,Y]_e \to [X,Y]_s$ are bijections.*

Proof. Let QY be the CW replacement of Y. The level equivalence $QY \to Y$ gives a commuting diagram

$$\begin{tikzcd}
{[X,QY]_h} \arrow[r] \arrow[d] & {[X,QY]_e} \arrow[r, "\cong"] \arrow[d, "(2)"] & {[X,QY]_s} \arrow[d, "\cong"] \\
{[X,Y]_h} \arrow[r, "(1)"] & {[X,Y]_e} \arrow[r] & {[X,Y]_s}.
\end{tikzcd}$$

The isomorphism in the top row is by Theorem 3.1.39, while the isomorphism on the right is because $QY \to Y$ is an isomorphism in $\operatorname{Ho}\mathbf{Sp}$. To finish proving the claim, it is enough to show that the maps (1) and (2) are isomorphisms.

To show that (1) is surjective, we show each eventually-defined map $f: X' \to Y$, defined on a cofinal subspectrum $X' \subseteq X$, is homotopic to an actual map $g: X \to Y$. In other words, we seek to define a map of spectra

$$X \wedge \{0\}_+ \cup_{X' \wedge \{0\}_+} X' \wedge I_+ \longrightarrow Y$$

that agrees with f along $X' \wedge \{1\}_+$. As in the proof of Proposition 3.1.38, we define this extension one cell of X at a time. Suppose we have a stable k-cell in X, first appearing in X as as $D^{k+m} \to X_m$. For each $n \geq m$, this gives a cell

$$D^{k+n} \to \Sigma^{n-m} D^{k+m}_+ \to X_n.$$

Since X' is cofinal in X, there is a first value of n on which f is defined on this cell,

$$\Sigma^{n-m} D^{k+m}_+ \to Y_n.$$

This rearranges to a map $D^{k+m} \to \Omega^{n-m} Y_n$, and our goal is to modify this map up to homotopy to a map that lifts to Y_m. Furthermore, this homotopy has already been defined on the boundary ∂D^{k+m}, because we have already defined the homotopy on the lower-dimensional cells. All together, this defines a class in $\pi_{k+m}(\Omega^{n-m} Y_n, Y_m)$. Since Y is an Ω-spectrum, this relative homotopy group is zero, so the desired extension exists.

This proves that (1) is surjective. The proof of injectivity is similar: we define a map

$$X \wedge (I \times \{0\})_+ \cup_{X' \wedge (I \times \{0\})_+} X' \wedge (I \times I)_+ \longrightarrow Y$$

that is already defined on the top and sides of $I \times I$. Again, we do this one cell of X at a time, and the required extensions exist because Y is an Ω-spectrum. The proof that (2) is an isomorphism is similar but somewhat easier than the proof of (1), and is left as an exercise (exercise 9). □

Proposition 3.1.41. *If X is CW, then $[X, Y]_e \to [X, Y]_s$ is a bijection.*

Proof. This follows from the fact that the map (2) in the previous proof is an isomorphism. The proof (exercise 9) does not require us to assume that Y is an Ω-spectrum. □

Remark 3.1.42. Throughout this section and the last, we have used CW spectra for simplicity. In fact, all of the definitions and results also work for the larger category of cellular spectra. In particular, $[X, Y]_s \cong [X, Y]_h$ if X is cellular and Y is an Ω-spectrum. We will give another proof of this using model categories in Theorem 5.2.11 and Theorem 5.2.26.

3.2 Fundamental properties

We have now defined three equivalent models for the stable homotopy category:

- All spectra, with maps $[X, Y]_s$ defined as zig-zags,
- CW spectra, with homotopy classes of eventually-defined maps $[X, Y]_e$, and
- CW Ω-spectra, with homotopy classes of actual maps $[X, Y]_h$.

These are equivalent categories – they have the same objects up to isomorphism, and the same morphisms between those objects. In fact, we have bijections

$$\begin{aligned} &[X, Y]_s \cong [X, Y]_h \text{ when } X \text{ is CW and } Y \text{ is an } \Omega\text{-spectrum, and} \\ &[X, Y]_s \cong [X, Y]_e \text{ when } X \text{ is CW.} \end{aligned}$$

In this section, we begin moving past the *definition* of the stable homotopy category and start proving its fundamental properties. These properties are somewhat parallel to the stability theorems and their corollaries from Section 2.4.

3.2.1 Suspension, coproducts, and additive structure

Proposition 3.2.1. *Σ and Ω define inverse equivalences of categories*

$$\begin{tikzcd} \mathrm{Ho}\,\mathbf{Sp} \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\sim"] & \mathrm{Ho}\,\mathbf{Sp}. \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

In particular,

$$[\Sigma X, \Sigma Y]_s \cong [X, Y]_s \cong [\Omega X, \Omega Y]_s.$$

Proof. Both suspension and loopspace preserve all stable equivalences by Corollary 2.4.3. Therefore by the universal property of the homotopy category (Proposition 3.1.6), they descend to functors on the homotopy category. The natural isomorphisms from Corollary 2.4.5 relate each composite back to the identity in the homotopy category. □

For the next proposition, let $\{X(\alpha)\}_{\alpha \in A}$ be any collection of spectra and let Y be any spectrum.

Proposition 3.2.2. *The canonical map that restricts the sum to each summand*

$$\left[\bigvee_\alpha X(\alpha), Y\right]_s \to \prod_\alpha [X(\alpha), Y]_s$$

is a bijection. Furthermore the canonical map that projects the product to each factor

$$\left[Y, \prod_\alpha X(\alpha)\right]_s \to \prod_\alpha [Y, X(\alpha)]_s$$

is a bijection, if either the product is finite or every $X(\alpha)$ is an Ω-spectrum.

Proof. For the first claim, we replace each $X(\alpha)$ by the CW spectrum $QX(\alpha)$. Since $QX(\alpha)$ is stably equivalent to α, and wedge sums preserve stable equivalences (Section 2.7, exercise 29), this gives an isomorphism on $[-,-]_s$. (See also exercise 4.) Similarly we replace Y by the Ω-spectrum RY.

Once we have done this, the wedge sum $\bigvee_\alpha QX(\alpha)$ is a CW spectrum and RY is an Ω-spectrum, so by Proposition 3.1.40, the maps in the stable category agree with homotopy classes of maps $[-,-]_h$. We therefore have the following isomorphisms.

$$\begin{tikzcd}
{[\bigvee_\alpha QX(\alpha), RY]_h} \arrow[r] \arrow[d, "\cong"] & \prod_\alpha [QX(\alpha), RY]_h \arrow[d, "\cong"] \\
{[\bigvee_\alpha QX(\alpha), RY]_s} \arrow[r] \arrow[d, "\cong"] & \prod_\alpha [QX(\alpha), RY]_s \arrow[d, "\cong"] \\
{[\bigvee_\alpha X(\alpha), Y]_s} \arrow[r] & \prod_\alpha [X(\alpha), Y]_s.
\end{tikzcd}$$

The top horizontal map in this diagram is an isomorphism, by the universal property of the coproduct applied to maps and to homotopies. Therefore all of the horizontal maps are isomorphisms, which proves the first claim.

The same argument applies to products, except that we use Section 2.7, exercise 25 to show that replacing the $X(\alpha)$ by Ω-spectra induces an equivalence on the product. □

Remark 3.2.3. Proposition 3.2.2 tells us that the wedge sum $\bigvee$ is the coproduct *both* in the category of spectra **Sp** and in the homotopy category Ho **Sp**. On the other hand, the product $\prod$ in spectra is only the product in the homotopy category if it is a finite product, or if the inputs are Ω-spectra. To get the product in Ho **Sp** in general, we replace the $X(\alpha)$ by Ω-spectra first, and then take their product.

Corollary 3.2.4. *The zero spectrum $*$ is a* ***zero object*** *in* Ho **Sp***: for any spectrum X, there is a unique map $X \to *$ and a unique map $* \to X$ in* Ho **Sp**.

Proof. The proof of Proposition 3.2.2 works perfectly well for empty coproducts and empty products, both of which give $*$. □

Theorem 3.2.5. *The stable homotopy category* Ho **Sp** *is* ***additive****:*

- *the sets $[X,Y]_s$ are abelian groups,*
- *the composition maps $[X,Y]_s \times [Y,Z]_s \to [X,Z]_s$ are bilinear,*
- Ho **Sp** *has all finite coproducts and products, and a zero object $*$, and*
- *the canonical map $X \vee Y \to X \times Y$ is an isomorphism.*

Proof. The statements about coproducts, products, and the zero object are already handled by the above results and Proposition 2.4.18.

To define the abelian group structure, for each spectrum X we pick another spectrum X' and a stable equivalence $X \simeq \Sigma X'$. For instance, X' could be ΩX, or $\mathrm{sh}^{-1} X$.

We also pick a map of spaces $p\colon S^1 \to S^1 \vee S^1$ that pinches the circle in the middle, or any other map whose degree is $(1,1)$. By Section 2.7, exercise 10, this gives a "pinch map" in the stable homotopy category

$$\begin{aligned} S^1 \wedge X' &\longrightarrow (S^1 \vee S^1) \wedge X' \\ &\cong (S^1 \wedge X') \vee (S^1 \wedge X') \\ \Sigma X' &\longrightarrow \Sigma X' \vee \Sigma X' \\ X &\longrightarrow X \vee X. \end{aligned}$$

By abuse of notation, we also let $p\colon X \to X \vee X$ denote this pinch map. This is a map in the *homotopy* category $\mathrm{Ho}\,\mathbf{Sp}$, not an actual map of spectra.

Suppose $f, g \in [X, Y]_s$ are two morphisms in the homotopy category, in other words, zig-zags of morphisms of spectra. We can form a map $(f,g)\colon X \vee X \to Y$ in the homotopy category, because $\vee$ is the coproduct, by Proposition 3.2.2. Concretely, this map applies the zig-zags to each summand separately to end at $Y \vee Y$, then maps each of these to Y by id_Y.

We define the sum $(f+g) \in [X, Y]_s$ by composing (f,g) with p:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[rr, bend right, "f+g"'] & X \vee X \arrow[r, "{(f,g)}"] & Y. \end{tikzcd}$$

The usual proof that $\pi_1(X)$ is a group applies here and shows that the zero map $X \to * \to Y$ is the identity for $+$, and that the inverse of a map f is formed by flipping the suspension coordinate of $\Sigma X' \simeq X$. See exercise 10.

To show this rule is well-defined and abelian, it will be helpful to write it a different way. The diagram of spaces

$$\begin{tikzcd} & S^1 \vee S^1 \arrow[d, "\subseteq"] \\ S^1 \arrow[ur, "p"] \arrow[r, "{(\mathrm{id}_{S^1}, \mathrm{id}_{S^1})}"'] & S^1 \times S^1 \end{tikzcd}$$

commutes up to homotopy. It follows using Lemma 3.1.15 that the diagram of zig-zags of maps of spectra

$$\begin{tikzcd} & X \vee X \arrow[d, "\subseteq"] \\ X \arrow[ur, "p"] \arrow[r, "{(\mathrm{id}_X, \mathrm{id}_X)}"'] & X \times X \end{tikzcd}$$

commutes in Ho **Sp**. Therefore our addition rule $f+g$ is equal to either route through the following commuting diagram in Ho **Sp**.

$$\begin{tikzcd}
 & X \vee X \arrow[r, "f \vee g"] \arrow[d, "\cong"] & Y \vee Y \arrow[r, "(\mathrm{id}_Y, \mathrm{id}_Y)"] \arrow[d, "\cong"] & Y \\
X \arrow[r, "(\mathrm{id}_X, \mathrm{id}_X)"] & X \times X \arrow[r, "f \times g"] & Y \times Y &
\end{tikzcd} \tag{3.2.6}$$

It is a straightforward diagram-chase to check this is bilinear, in other words $h\circ(f+g) = (h\circ f)+(h\circ g)$, and similarly $(f+g)\circ h = (f\circ h)+(g\circ h)$; see exercise 10.

To show that $+$ is commutative, we let $f \triangle g$ denote any other rule for adding two maps together in Ho **Sp**. Given two more maps $f', g' \colon X \rightrightarrows Y$, the map $(f\triangle f')+(g\triangle g')$ is given by the composite

$$\begin{tikzcd}
 & X \vee X \arrow[r, "(f\triangle f')\vee(g\triangle g')"] \arrow[d, "\sim"'] & Y \vee Y \arrow[r, "(\mathrm{id}_Y, \mathrm{id}_Y)"] \arrow[d, "\sim"] & Y \\
X \arrow[r, "(\mathrm{id}_X, \mathrm{id}_X)"] & X \times X \arrow[r, "(f\triangle f')\times(g\triangle g')"] & Y \times Y &
\end{tikzcd}$$

We have the equality $(f\triangle f')\vee(g\triangle g') = (f\vee g)\triangle(f'\vee g')$, because both give $f\triangle f'$ when restricted to the first copy of X, and $g\triangle g'$ when restricted to the second copy of X. By the bilinearity of $\triangle$, we can compose this with the remaining maps of the diagram and see that the result is $(f+g)\triangle(f'+g')$. All together, this proves the interchange law

$$(f\triangle f')+(g\triangle g') = (f+g)\triangle(f'+g').$$

We now apply the Eckmann-Hilton argument [], and conclude that the two operations $+$ and $\triangle$ are equal to each other, and are abelian. $\square$

The last step of the proof of Theorem 3.2.5 shows:

Proposition 3.2.7. *The abelian group structure in Theorem 3.2.5 is unique.*

Example 3.2.8. The isomorphism $[\mathbb{S},\mathbb{S}]_s \cong \mathbb{Z}$ is an isomorphism of abelian groups, because the rule from Theorem 3.2.5 for adding two maps $\mathbb{S}\to\mathbb{S}$ agrees with the usual rule for addition in $\pi_n(S^n)\cong\mathbb{Z}$.

Example 3.2.9. We can describe the addition $f+g$ in yet another way. For each spectrum Y, we pick another spectrum Y' and a stable equivalence $Y \simeq \Omega Y'$. Then the composition map $\Omega Y' \times \Omega Y' \to \Omega Y'$ gives a map in the stable homotopy category $Y\times Y \to Y$ that we call c. We add f and g by the rule

$$X \xrightarrow{(f,g)} Y\times Y \xrightarrow{c} Y.$$

To prove this agrees with the operation of Theorem 3.2.5, we could prove it is bilinear and then use Proposition 3.2.7, but it is a little faster to argue that the following square of spectra commutes:

$$\begin{tikzcd} Y \vee Y \arrow[r, "(\mathrm{id}_Y, \mathrm{id}_Y)"] \arrow[d, "\cong"'] & Y \\ Y \times Y \arrow[ur, "c"'] & \end{tikzcd}$$

Sticking this onto the right-hand side of (3.2.6), we see that this rule agrees with the previous one.

Example 3.2.10. The category of of abelian groups **Ab** is also additive, as is the category $_R$**Mod** of left modules over a ring R. In both cases, the morphisms form abelian groups, and the finite coproducts and finite products are isomorphic to each other.

It is common to write finite coproducts in an additive category with the notation

$$X_1 \oplus \ldots \oplus X_n,$$

also so we will sometimes use this notation when working in the stable homotopy category.[1] We can also denote maps between direct sums

$$X_1 \oplus \ldots \oplus X_n \longrightarrow Y_1 \oplus \ldots \oplus Y_m$$

using matrix notation, every entry of the matrix being an element of one of the abelian groups $[X_j, Y_i]_s$. The composition of maps is given by matrix multiplication.

Example 3.2.11. $[\mathbb{S}\oplus\mathbb{S}, \mathbb{S}\oplus\mathbb{S}]_s$ is isomorphic to the abelian group $M_{2\times 2}(\mathbb{Z})$ of 2×2 matrices over $\mathbb{Z}$. The composition is by matrix multiplication. See also exercise 12.

3.2.2 Computing the morphism sets $[X, Y]_s$

Recall from the beginning of Section 3.2 that we have bijections

$[X, Y]_s \cong [X, Y]_h$ when X is CW and Y is an Ω-spectrum, and
$[X, Y]_s \cong [X, Y]_e$ when X is CW.

These are bijections of abelian groups, if we write X as a suspension and use the usual formula for addition in $[\Sigma X', Y]_*$ to add two maps together. The description as eventually-defined maps $[X, Y]_e$ is useful for actually computing these sets, at least when X only has a few cells.

[1]Note that the point-set category of spectra, **Sp**, is not additive. So we have to continue to use the notation $\vee$ or $\times$ if we are not working in the homotopy category.

To move to larger examples, recall that for a based space A, the free spectrum $F_k A$ has the property that maps of spectra $F_k A \to Y$ are the same as maps of based spaces $A \to Y_k$, see Lemma 2.6.9. The same applies to homotopies, so we get

Lemma 3.2.12. *There is a canonical bijection* $[F_k A, Y]_h \cong [A, Y_k]_*$.

Corollary 3.2.13. *We get the following bijections:*

$$\begin{aligned}
[F_k A, Y]_s &\cong [A, Y_k]_* && \text{when } A \text{ is CW and } Y \text{ is an } \Omega\text{-spectrum,}\\
[F_k A, Y]_s &\cong \operatorname*{colim}_{n\to\infty}[\Sigma^n A, Y_{k+n}]_* && \text{when } A \text{ is finite CW and } Y \text{ is any spectrum, and}\\
[F_k A, F_\ell B]_s &\cong \operatorname*{colim}_{n\to\infty}[\Sigma^{n-k} A, \Sigma^{n-\ell} B]_* && \text{when } A \text{ is finite CW and } B \text{ is any space.}
\end{aligned}$$

Proof. The first bijection is because $[F_k A, Y]_s \cong [F_k A, Y]_h$ by Proposition 3.1.40. For the second bijection, we replace Y by the Ω-spectrum RY. By Section 1.7, exercise 24, we have

$$[A, RY_k]_* \cong \operatorname*{colim}_{n\to\infty}[A, \Omega^n Y_{k+n}]_* \cong \operatorname*{colim}_{n\to\infty}[\Sigma^n A, Y_{k+n}]_*.$$

Alternatively, we could use Proposition 3.1.40 and argue that because A has only finitely many cells, any eventually-defined map becomes actually defined at level n for sufficiently large n. The same argument gives the third bijection as well. □

Remark 3.2.14. Recall that a spectrum is **finite** if it is stably equivalent to $F_k A$ for a finite CW complex A. The subcategory of finite spectra $\operatorname{Ho}\mathbf{Sp}^{\text{fin}} \subseteq \operatorname{Ho}\mathbf{Sp}$ is called the **Spanier-Whitehead category**. By Corollary 3.2.13, the morphisms of this category are stabilized maps of spaces $\operatorname*{colim}_{n\to\infty}[\Sigma^{n-k} A, \Sigma^{n-\ell} B]_*$. See Proposition 4.2.12 for more details.

Recall from Example 2.1.9 that the d-dimensional sphere in spectra is

$$\mathbb{S}^d = \operatorname{sh}^d \mathbb{S} = \begin{cases} F_0 S^d \cong \Sigma^d \mathbb{S} & \text{when } d \geq 0, \\ F_{|d|} S^0 \simeq \Omega^{|d|}\mathbb{S} & \text{when } d \leq 0. \end{cases}$$

These are equivalent by Section 2.7, exercise 19.

Corollary 3.2.15. *For any spectrum X, there is a natural isomorphism*

$$[\mathbb{S}^k, X]_s \cong \operatorname*{colim}_{n\to\infty} \pi_{k+n}(X_n) = \pi_k(X).$$

This is an isomorphism of abelian groups, using the pinching definition of addition from Theorem 3.2.5. A category theorist would describe this result by saying that $\pi_k \colon \operatorname{Ho}\mathbf{Sp} \to \mathbf{Ab}$ is "co-represented" by $\mathbb{S}^k$.

Example 3.2.16. If X is any spectrum and G is an abelian group, then

$$[X, \Sigma^k HG]_s \cong H^k(X;G).$$

This is really a definition rather than a theorem – it generalizes Proposition 2.5.16 from the cohomology of spaces to the cohomology of spectra. However when X is a suspension spectrum of a well-based space A, we do recover the cohomology of A, by Corollary 2.5.19:

$$[\Sigma^\infty A, \Sigma^k HG]_s \cong H^k(A,*;G).$$

We similarly get for any spectrum E an isomorphism

$$[X, \Sigma^k E]_s \cong E^k(X).$$

Example 3.2.17. Note that for any X we have $[\mathbb{S}, X]_h = \pi_0(X_0)$. This is isomorphic to $\pi_0(X)$ if X is an Ω-spectrum. This is consistent with the claim that $[\mathbb{S}, X]_s \cong [\mathbb{S}, X]_h$ if $\mathbb{S}$ is CW and X is an Ω-spectrum.

Example 3.2.18. Since maps between finite wedges/products are given by products of mapping groups, we get

$$[\mathbb{S}^2 \vee \mathbb{S}^3, Y]_s \cong [\mathbb{S}^2, Y]_s \times [\mathbb{S}^3, Y]_s \cong \pi_2(Y) \times \pi_3(Y),$$
$$[X, Y \vee Z]_s \cong [X, Y \times Z]_s \cong [X, Y]_s \times [X, Z]_s.$$

Every spectrum is equivalent to a cellular spectrum, and every cellular spectrum is built by piecing together free spectra. So to go from Corollary 3.2.13 to any spectrum X, we only have to know how to glue these descriptions together. The following two results are the essential pieces to this process.

Proposition 3.2.19. *If (X, Y, Z, f, g, h) is a cofiber/fiber sequence of spectra and W is any fourth spectrum then there are long exact sequences*

$$\ldots \longrightarrow [W, X]_s \xrightarrow{f\circ -} [W, Y]_s \xrightarrow{g\circ -} [W, Z]_s \xrightarrow{\partial\circ -} [W, \Sigma X]_s \longrightarrow \ldots$$

$$\ldots \longrightarrow [\Sigma X, W]_s \xrightarrow{-\circ\partial} [Z, W]_s \xrightarrow{-\circ g} [Y, W]_s \xrightarrow{-\circ f} [X, W]_s \longrightarrow \ldots$$

Proof. For the first case, since X, Y, and Z form a fiber sequence, we can make Y and Z into Ω-spectra and replace X by the homotopy fiber, which is also an Ω-spectrum. Then we can make W a CW spectrum, so that $[-,-]_s$ is the same as $[-,-]_h$.

After these replacements, the proof is identical to the proof of the long exact sequence from Proposition 1.2.14: the sequence is exact at $[W, Y]_s$ because a nullhomotopy of the composite $W \to Y \to Z$ is exactly the data you need to extend the map to the homotopy fiber X.

For the second case, we instead make X and Y CW, replace Z by the homotopy cofiber, which is also CW, and make W an Ω-spectrum, so that $[-,-]_s$ is the same as $[-,-]_h$. Then we apply the proof of Proposition 1.2.9: the sequence is exact at $[Y,W]_s$ because a nullhomotopy of the composite $X \to Y \to W$ is exactly the data you need to extend the map to the homotopy cofiber Z. □

Proposition 3.2.20. *If X is the colimit of a sequence of relative cellular or CW spectra*

$$* \longrightarrow X^{(0)} \longrightarrow X^{(1)} \longrightarrow \dots \longrightarrow \operatorname*{colim}_{n\to\infty} X^{(n)} = X,$$

and Y is another spectrum, then there is a short exact sequence

$$0 \longrightarrow \lim{}^1[\Sigma X^{(n)}, Y]_s \longrightarrow [X,Y]_s \longrightarrow \lim [X^{(n)}, Y]_s \longrightarrow 0.$$

In particular, if the maps $[X^{(n)}, Y]_s \to [X^{(n-1)}, Y]_s$ are surjective for sufficiently large n, then $[X,Y]_s \cong \lim [X^{(n)}, Y]_s$.

Proof. This is deferred to exercise 15. □

Example 3.2.21. If X is a spectrum and $E^*(-)$ is any reduced extraordinary cohomology theory, the cohomology groups $E^k(X) = [X, \Sigma^k E]_s$ as defined in Example 3.2.16 fit into a short exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1 E^{k+n-1}(X_n) \longrightarrow E^k(X) \longrightarrow \lim_{n\to\infty} E^{k+n}(X_n) \longrightarrow 0.$$

This follows by applying Proposition 3.2.20 to the filtration of X in which we attach the cells one spectrum level at a time. Note the similarity to (1.5.6), except that the degrees of the cohomology groups are increasing as we increase the spectrum level, because we are using the suspension isomorphisms to go from each level to the next.

We also recall (Definition 2.6.36) that E-homology is given by the simpler formula

$$E_k(X) \cong \operatorname*{colim}_{n\to\infty} E_{k+n}(X_n).$$

Example 3.2.22. The previous example lets us compute maps between Eilenberg-Maclane spectra $[HG_1, \Sigma^k HG_2]_s$ as the stable cohomology of Eilenberg-Maclane spaces,

$$[HG_1, \Sigma^k HG_2]_s \cong \lim_{n\to\infty} H^{k+n}(K(G_1,n); G_2) \cong \lim_{n\to\infty} [K(G_1,n), K(G_2,k+n)]_*.$$

In particular, $[HG_1, HG_2]_s \cong \operatorname{Hom}(G_1, G_2)$, and the graded abelian group $[H\mathbb{Z}/2, \Sigma^k H\mathbb{Z}/2]_s$ is isomorphic to the Steenrod algebra $\mathscr{A}$. This is a graded algebra whose kth level is the abelian group of stable operations on mod 2 cohomology

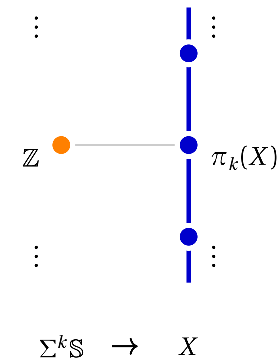
$$H^n(-;\mathbb{Z}/2) \Rightarrow H^{k+n}(-;\mathbb{Z}/2), \quad n >> 0.$$

It is generated by operations Sq^k called the **Steenrod squares**, subject to relations called the Adem relations. See [] for more details.

In a broad sense, the above results tell us that if we know the cells of X, and the homotopy groups of Y, then we have a good shot at computing $[X, Y]_s$. For instance, if X has a single cell, it is a shift of the sphere spectrum, so we get

$$[\Sigma^k \mathbb{S}, Y]_s \cong \pi_k(Y).$$

If X has just a few cells, we can use the cofiber sequence from Proposition 3.2.19 to patch together these copies of $\pi_k(Y)$ to get $[X, Y]_s$.



Example 3.2.23. The suspension spectrum $\Sigma^\infty \mathbb{RP}^2$ can be defined as the homotopy cofiber of the degree two map $\mathbb{S}^1 \to \mathbb{S}^1$. We therefore get a long exact sequence

$$\ldots \longrightarrow \pi_2(W) \xrightarrow{\ 2\ } \pi_2(W) \longrightarrow [\Sigma^\infty \mathbb{RP}^2, W]_s \longrightarrow \pi_1(W) \xrightarrow{\ 2\ } \pi_1(W) \longrightarrow \ldots$$

On the other hand, suppose Y has a single nonzero homotopy group.

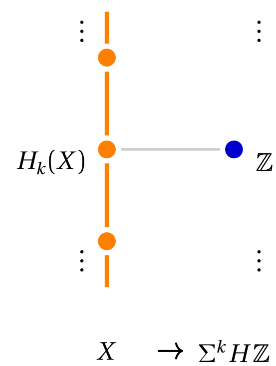
Lemma 3.2.24. *If $\pi_n(Y) = 0$ for $n \neq k$, then* $[X, Y]_s \cong H^k(X; \pi_k(Y))$.

Proof. Y is stably equivalent to the Eilenberg-Maclane spectrum $\Sigma^k H(\pi_k(Y))$ by Section 2.7, exercise 38. The conclusion follows from the definition in Example 3.2.16. □

For instance,

$$[X, \Sigma^k H\mathbb{Z}]_s \cong H^k(X).$$

Using the Postnikov tower of Y from Definition 2.6.34, we can stitch these results together to get $[X, Y]_s$ if Y has just a few nonzero homotopy groups.



Both of these approaches can be further formalized into a spectral sequence. In fact, they both give the same spectral sequence. It starts from $H^*(X; \pi_*(Y))$ and converges to $[X, Y]_s$. See **??** for the construction. An important special case is when X is a suspension spectrum, where this is called the cohomology Atiyah-Hirzebruch spectral sequence, see **??**.

You can get pretty far with the intuition that the cells or homology groups of X "hook onto" the homotopy groups of Y to create $[X, Y]_s$. So the maps from X to Y are "generated" by levels in which X has homology *and* Y has homotopy. This is admittedly rather rough intuition, but it is enough to get a start on understanding how many maps there should be from X to Y. The examples in Figure 3.2.25 illustrate this idea in a few examples that we have seen so far.

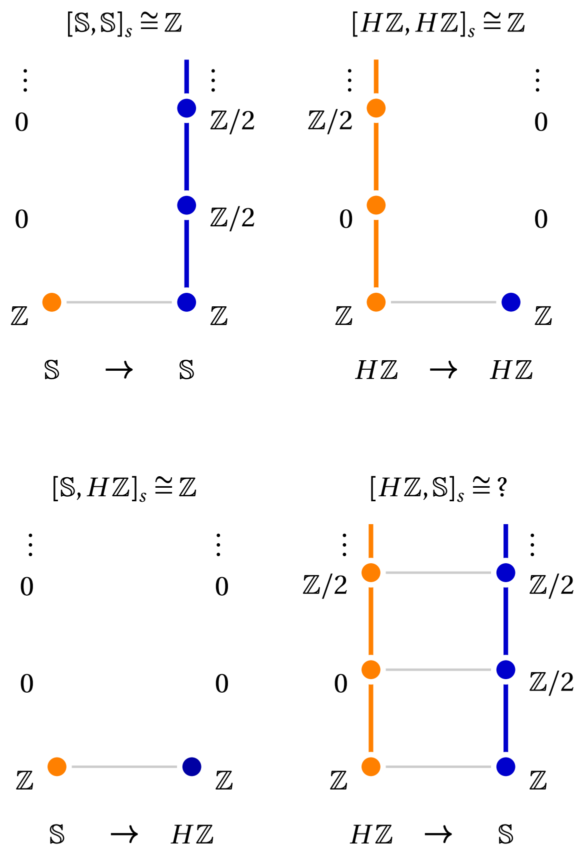


Figure 3.2.25

3.2.3 Symmetric monoidal structure

The stable homotopy category Ho **Sp** is an example of a symmetric monoidal category. This will be discussed further in Chapter 4. For now, we'll just mention that there is a product on this category called the smash product, $\wedge$: Ho **Sp** $\times$ Ho **Sp** $\to$ Ho **Sp**. You can get a rough feel for what the smash product does from the following facts:

- The smash product preserves wedge sums, cofiber sequences, homotopy pushouts, and sequential compositions in each variable separately. So if Y is built out of cells, then $X \wedge Y$ is built in the same way out of X smashed with each of those cells.
- We have $\mathbb{S}^m \wedge \mathbb{S}^n \cong \mathbb{S}^{m+n}$. More generally, each m-cell in X and n-cell in Y becomes an $(m+n)$-cell in $X \wedge Y$. Morally, this is enough to tell you the homotopy type of the smash product of every pair of spectra.
- We have

$$X \wedge \mathbb{S} \cong X, \qquad X \wedge Y \cong Y \wedge X, \qquad (X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z).$$

- The map induced on morphisms by the smash product is bilinear, so it induces maps out of the tensor product

$$[X,Y]_s \otimes [W,Z]_s \longrightarrow [X\wedge W, Y\wedge Z]_s.$$

In particular, the smash product puts a ring structure on $[\mathbb{S},\mathbb{S}]_s \cong \mathbb{Z}$, that is the usual ring structure on the integers. We'll develop these and many more properties in Chapter 4.

3.3 Functors on the homotopy category

3.3.1 Homotopy functors

Now that we have proven the fundamental properties of Ho **Sp**, it's time to start defining more operations on this category. For instance:

- homotopy groups $\pi_k\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathbf{Ab}$,
- homology groups $E_k\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathbf{Ab}$,
- cohomology groups $E^k\colon \mathrm{Ho}\,\mathbf{Sp}^{\mathrm{op}} \to \mathbf{Ab}$,
- suspension and loops $\Sigma,\Omega\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$,
- wedge sums $\bigvee\colon \mathrm{Ho}\,\mathbf{Sp}^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$,
- finite products $\prod\colon \mathrm{Ho}\,\mathbf{Sp}^{\times n} \to \mathrm{Ho}\,\mathbf{Sp}$,
- infinite products of Ω-spectra $\prod\colon (\mathrm{Ho}\,\mathbf{Sp}^{\Omega})^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$,
- shift $\mathrm{sh}^d\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$,
- evaluate at level n, $\mathrm{ev}_n\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Top}_*$,
- homotopy cofiber and fiber $C,F\colon \mathrm{Ho}\,\mathbf{Sp}^{(\bullet\to\bullet)} \to \mathrm{Ho}\,\mathbf{Sp}$,
- smashing with a cell complex $-\wedge K\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$, and
- maps from a finite complex $F(K,-)\colon \mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$.

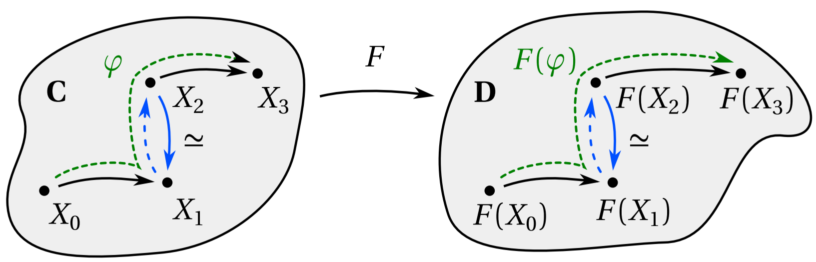
We define these using the universal property of the homotopy category from Theorem 3.1.10. Any functor $\mathbf{Sp} \to \mathbf{D}$ that sends stable equivalences to isomorphisms, must factor through Ho **Sp**. In most of the above cases, the target category is itself a homotopy category Ho **D**. When this happens, it's better to phrase the universal property this way:

Definition 3.3.1. If $\mathbf{C}$ and $\mathbf{D}$ are categories with weak equivalences, a functor $F\colon \mathbf{C} \to \mathbf{D}$ is **homotopical**, or a **homotopy functor**, if it takes each weak equivalence in $\mathbf{C}$ to a weak equivalence in $\mathbf{D}$.

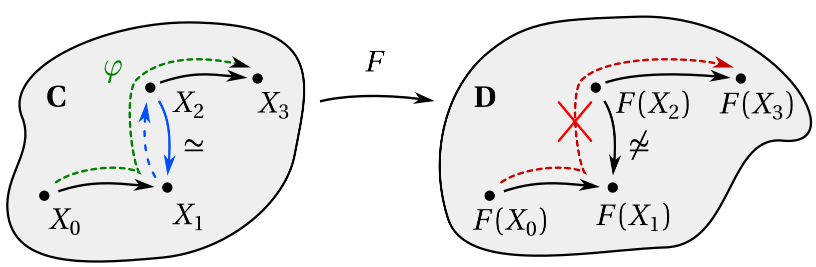
Lemma 3.3.2. *If F is a homotopy functor then it extends in a unique way to functor of homotopy categories making the following square commute.*

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d, "\delta"'] & \mathbf{D} \arrow[d, "\delta"] \\
\operatorname{Ho}\mathbf{C} \arrow[r, dashed, "\operatorname{Ho}F \text{ (or just } F)"'] & \operatorname{Ho}\mathbf{D}
\end{tikzcd}$$

Proof. This follows immediately from the universal property of $\operatorname{Ho}\mathbf{C}$ in Theorem 3.1.10. The composite $\mathbf{C} \to \mathbf{D} \to \operatorname{Ho}\mathbf{D}$ sends weak equivalences to isomorphisms, so it factors through $\operatorname{Ho}\mathbf{C}$. Concretely, this factorization takes each zig-zag in $\operatorname{Ho}\mathbf{C}$ and applies F to every object and map in the zig-zag.



We need F to be homotopical for this to work – otherwise a backwards-pointing weak equivalence may turn into a backwards map that is not a weak equivalence, so we fail to get a valid zig-zag. □



By a standard abuse of notation, we simply say F instead of $\operatorname{Ho}F$. In this case F is playing "double duty" by defining a functor both on $\mathbf{C}$ and on $\operatorname{Ho}\mathbf{C}$.

Example 3.3.3. The reduced suspension functor $\Sigma\colon \mathbf{Sp} \to \mathbf{Sp}$ preserves stable equivalences by Corollary 2.4.3. By Lemma 3.3.2, it therefore gives a functor $\operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$. We already observed this in Proposition 3.2.1.

Example 3.3.4. Smashing with a CW complex K is a homotopy functor by Corollary 2.4.23. Taking maps from a finite complex K, $F(K,-)$, is a homotopy functor by Section 2.7, exercise 26. Therefore these both define functors on the stable homotopy category, $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}$.

Example 3.3.5. Let **Ch** be the category of unbounded ($\mathbb{Z}$-graded) chain complexes of abelian groups. The weak equivalences are the quasi-isomorphisms, maps $C \to D$ inducing isomorphisms on homology $H_*(C) \xrightarrow{\cong} H_*(D)$. The homotopy category $\mathrm{Ho}\,\mathbf{Ch}$ is also called the **derived category** $\mathscr{D}(\mathbb{Z})$.

Taking tensor product with the rationals

$$(-) \otimes \mathbb{Q} \colon \mathbf{Ch} \to \mathbf{Ch}$$

preserves quasi-isomorphisms, because $\mathbb{Q}$ is flat. Therefore it induces a functor

$$(-) \otimes \mathbb{Q} \colon \mathscr{D}(\mathbb{Z}) \to \mathscr{D}(\mathbb{Z}).$$

Of course, the same is true for the category of nonnegatively graded chain complexes $\mathbf{Ch}_{\geq 0}$ and its homotopy category $\mathscr{D}_{\geq 0}(\mathbb{Z})$.

Example 3.3.6. The wedge sum of spectra preserves equivalences by Section 2.7, exercise 29. But this isn't a functor from spectra to spectra – it is a functor from *pairs* of spectra to spectra.

Let $\mathbf{Sp} \times \mathbf{Sp}$ be the product category whose objects are pairs of spectra (X, Y) and whose maps are pairs of maps $(f \colon X \to X', g \colon Y \to Y')$. The weak equivalences are those pairs where both f and g are stable equivalences. By exercise 18, inverting these gives the product of homotopy categories,

$$\mathrm{Ho}(\mathbf{Sp} \times \mathbf{Sp}) \cong (\mathrm{Ho}\,\mathbf{Sp}) \times (\mathrm{Ho}\,\mathbf{Sp}).$$

The wedge sum is a functor $\vee \colon \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$, that sends all pairs of equivalences to equivalences. By Lemma 3.3.2 it therefore passes to a functor

$$\vee \colon \mathrm{Ho}\,\mathbf{Sp} \times \mathrm{Ho}\,\mathbf{Sp} \longrightarrow \mathrm{Ho}\,\mathbf{Sp}.$$

Alternatively, we could deduce this from its universal property (Proposition 3.2.2).

Example 3.3.7. If A is any set, the A-indexed wedge sum

$$\bigvee \colon \mathrm{Ho}\,\mathbf{Sp}^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$$

is defined on the homotopy category as in the previous example. For the product, however, we either have to restrict to A finite, or restrict the spectra to $\mathbf{Sp}^{\Omega}$. This ensures that the product preserves weak equivalences, by Section 2.7, exercise 25.

Example 3.3.8. Cohomology is a contravariant functor. So it is a functor on the opposite category, where the morphisms are reversed:

$$E^k : \mathbf{Sp}^{\mathrm{op}} \to \mathbf{Ab}.$$

An equivalence in $\mathbf{Sp}^{\mathrm{op}}$ is just the reverse of an equivalence in $\mathbf{Sp}$. By exercise 19, inverting these gives the opposite of the homotopy category:

$$\mathrm{Ho}(\mathbf{Sp}^{\mathrm{op}}) = (\mathrm{Ho}\,\mathbf{Sp})^{\mathrm{op}}.$$

Therefore cohomology defines a functor $E^k : (\mathrm{Ho}\,\mathbf{Sp})^{\mathrm{op}} \to \mathbf{Ab}$.

Definition 3.3.9. For any small category $\mathbf{I}$, let $\mathbf{Sp}^{\mathbf{I}}$ be the category of $\mathbf{I}$-diagrams of spectra. A map of diagrams $X(-) \to Y(-)$ is a **pointwise equivalence** if for every $i \in \mathrm{ob}\,\mathbf{I}$ the map of spectra $X(i) \to Y(i)$ is a stable equivalence. Inverting the pointwise equivalences of diagrams gives a homotopy category $\mathrm{Ho}\,\mathbf{Sp}^{\mathbf{I}}$.

Warning 3.3.10. In general, this homotopy category of $\mathbf{I}$-diagrams of spectra is *not* the same as $\mathbf{I}$-diagrams in $\mathrm{Ho}\,\mathbf{Sp}$:

$$\mathrm{Ho}(\mathbf{Sp}^{\mathbf{I}}) \xrightarrow{\ncong} (\mathrm{Ho}\,\mathbf{Sp})^{\mathbf{I}}.$$

This only really happens if $\mathbf{I}$ is discrete: products of categories give products of homotopy categories (exercise 18). Beyond this case, the above functor is not an isomorphism of categories, because not every diagram in $\mathrm{Ho}\,\mathbf{Sp}$ lifts to a diagram in $\mathbf{Sp}$, and if it does, the lift may not be unique.

We usually prefer to use $\mathrm{Ho}(\mathbf{Sp}^{\mathbf{I}})$. In other words, consider diagrams of actual spectra. Then, invert the equivalences between such diagrams.

Example 3.3.11. Taking $I = \{\bullet \to \bullet\}$, we get the category of arrows of spectra. An object is a map $X \to Y$ and a morphism is a commuting square. The homotopy cofiber C from Example 2.3.19 defines a functor

$$C : \mathbf{Sp}^{(\bullet \to \bullet)} \to \mathbf{Sp}.$$

By the long exact sequence of Lemma 2.4.9, any weak equivalence of arrows gives a weak equivalence of homotopy cofibers:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\sim"] & Y \arrow[r] \arrow[d, "\sim"] & Cf \arrow[d, "\sim"] \\
X' \arrow[r, "g"] & Y' \arrow[r] & Cg.
\end{tikzcd}$$

Therefore C is a homotopy functor. By Lemma 3.3.2 it therefore passes to a functor

$$C : \mathrm{Ho}(\mathbf{Sp}^{(\bullet \to \bullet)}) \longrightarrow \mathrm{Ho}\,\mathbf{Sp}.$$

The same is true of the homotopy fiber F, using the long exact sequence of Lemma 2.4.8.

Remark 3.3.12. The homotopy cofiber Cf can also be defined if f is only a map in the homotopy category, see exercise 20. However, it does not define a functor $(\mathrm{Ho}\,\mathbf{Sp})^{(\bullet\to\bullet)} \to \mathrm{Ho}\,\mathbf{Sp}$, only a function on the objects of the two categories. This makes $\mathrm{Ho}\,\mathbf{Sp}$ into a **triangulated category**, see e.g. [Nee01, HPS97]. When using this language, we call each sequence of maps of the form

$$X \xrightarrow{f} Y \longrightarrow Cf \longrightarrow \Sigma X$$

a **distinguished triangle**. Since cofiber and fiber sequences coincide, these can equivalently be described as sequences of the form

$$\Omega Z \longrightarrow Fp \longrightarrow W \xrightarrow{p} Z$$

modulo a funny issue about signs, see [May01] for more details.

This point of view is appealing for some applications, but it has serious shortcomings when we want to build anything more elaborate than a cone. We can't form more general kinds of homotopy colimits, for instance. It is often better to be biased towards taking diagrams of *actual* spectra, and *then* passing to the homotopy category of such diagrams. Use $\mathrm{Ho}(\mathbf{Sp}^{\mathbf{I}})$, not $(\mathrm{Ho}\,\mathbf{Sp})^{\mathbf{I}}$.

3.3.2 Derived functors

The list in Section 3.3.1 is missing several important operations. For instance, there ought to be a suspension spectrum functor

$$\Sigma^\infty\colon \mathrm{Ho}\,\mathbf{Top}_* \to \mathrm{Ho}\,\mathbf{Sp},$$

an infinite product functor

$$\prod\colon \mathrm{Ho}\,\mathbf{Sp}^{\times A} \to \mathrm{Ho}\,\mathbf{Sp},$$

and so on.

The problem is that these functors don't preserve equivalences. For instance, the infinite product $\prod\colon \mathbf{Sp}^{\times A} \to \mathbf{Sp}$ is does not send A-indexed families of stable equivalences to stable equivalences (see Proposition 3.2.2 and Section 2.7, exercise 25).

However, the infinite product does preserve stable equivalences, if we start with a family of Ω-spectra. And every spectrum is equivalent to an Ω-spectrum. So, if we first replace everything by an equivalent Ω-spectrum, then take the product

$$\prod_{\alpha\in A} RX(\alpha),$$

we get a well-defined functor on the homotopy category! We call this the **right-derived product**

$$\mathbb{R}(\prod)\colon \operatorname{Ho}\mathbf{Sp}^{\times A} \longrightarrow \operatorname{Ho}\mathbf{Sp}.$$

To give another example, recall from Corollary 1.5.11 that the smash product $\wedge$ defines a homotopy functor on well-based spaces

$$\mathbf{Top}_*^{wb} \times \mathbf{Top}_*^{wb} \to \mathbf{Top}_*^{wb},$$

but not on all based spaces.

But this is fine! Every space X is equivalent to a space QX that is a CW complex, and therefore well-based. So, just replace our spaces by CW complexes, and *then* take their smash product. We call this the **left-derived smash product**:

$$X \wedge^{\mathbb{L}} Y := QX \wedge QY.$$

This preserves all weak homotopy equivalences, so it gives a functor on the homotopy category,

$$\wedge^{\mathbb{L}}\colon \operatorname{Ho}\mathbf{Top}_* \times \operatorname{Ho}\mathbf{Top}_* \longrightarrow \operatorname{Ho}\mathbf{Top}_*.$$

If you have seen homological algebra, this is not so different from taking a projective resolution to compute Tor groups, or an injective resolution to compute Ext groups. Here's a way to put them under a common umbrella.

Definition 3.3.13. Suppose $\mathbf{C}$ is a category with weak equivalences W. We say that W satisfies the **2 out of 3** property if for any two maps f and g whose composite $g \circ f$ is defined, if two of the maps f, g, and $g \circ f$ are in W, then so is the third.

Example 3.3.14. Isomorphisms always have the 2 out of 3 property. More generally, if $F\colon \mathbf{C} \to \mathbf{D}$ is any functor, the collection of maps in $\mathbf{C}$ that go to isomorphisms in $\mathbf{D}$, has the 2 out of 3 property. Applying this to the functor that takes all of the homotopy groups of a spectrum

$$\pi_*\colon \mathbf{Sp} \to \prod_{k\in\mathbb{Z}} \mathbf{Ab},$$

we see that the stable equivalences in $\mathbf{Sp}$ satisfy 2 out of 3.

Definition 3.3.15. Suppose $\mathbf{C}$ and $\mathbf{D}$ are categories with weak equivalences satisfying 2 out of 3, and $F\colon \mathbf{C} \to \mathbf{D}$ is any functor.

A **left deformation** of F is

- a functor $Q\colon \mathbf{C} \to \mathbf{C}$,
- a natural transformation $Q \Rightarrow \mathrm{id}$, and

- a full subcategory $\mathbf{A} \subseteq \mathbf{C}$ containing $Q(\mathbf{C})$,

such that

- $QX \xrightarrow{\sim} X$ is a weak equivalence for all X in $\mathbf{C}$, and
- F is homotopical (preserves weak equivalences) in the subcategory $\mathbf{A} \subseteq \mathbf{C}$.

We call $\mathbb{L}F = F \circ Q$ the **left-derived functor** of F.

Similarly, a **right deformation** of F is

- a functor $R\colon \mathbf{C} \to \mathbf{C}$,
- a natural transformation $\mathrm{id} \Rightarrow R$, and
- a full subcategory $\mathbf{A} \subseteq \mathbf{C}$ containing $R(\mathbf{C})$,

such that

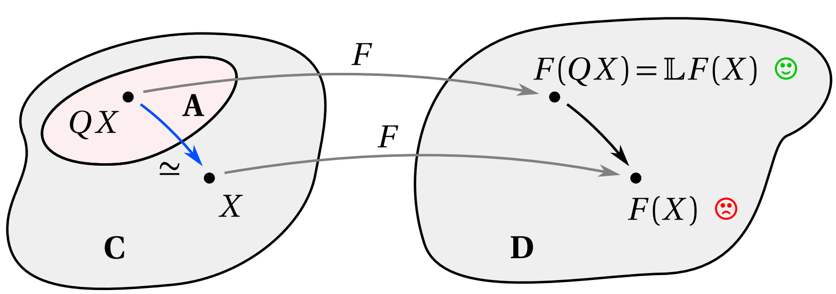
- $X \xrightarrow{\sim} RX$ is a weak equivalence for all X in $\mathbf{C}$, and
- F is homotopical in $\mathbf{A}$.

We call $\mathbb{R}F = F \circ R$ the **right-derived functor** of F.

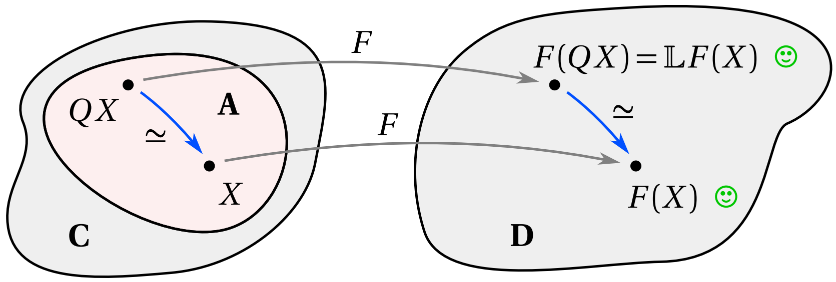
To summarize, the left-derived functor $\mathbb{L}F$ is formed by sampling the behavior of F on $\mathbf{A}$, and extending that behavior to the rest of $\mathbf{C}$. We replace each object X by something equivalent in $\mathbf{A}$, then apply F to it.

Remark 3.3.16. We call Q a "deformation" because it replaces each object X by an object QX that is "almost the same" as X. It's almost the same because $QX \to X$ is a weak equivalence. We call $\mathbb{L}F$ the "derived functor" because it is formed by taking F and changing the input *just a little bit*, so that the output is better behaved.

Think of the objects in $\mathbf{A}$ as the "nice" objects, because that's where F preserves weak equivalences. If $X \in \mathbf{A}$ then $F(X)$ is "good." If $X \notin \mathbf{A}$ then $F(X)$ is possibly bad. Though we're not sure. It's like a fruit that might be rotten, but you don't really want to bite into it to find out.



We want to keep F as it is on the good objects, but change it on the bad objects, so that it is good everywhere. This is precisely what the left-derived functor $\mathbb{L}F$ does – it is "good" all the time, and is essentially the same as F when F is behaving well, but it fixes up the times that F behaves badly. The next two lemmas make this precise.



Lemma 3.3.17. *If $X \in \mathbf{A}$, then $\mathbb{L}F(X)$ is equivalent to $F(X)$.*

Proof. The map $QX \to X$ is a weak equivalence of objects in $\mathbf{A}$, so applying F gives a weak equivalence $F(QX) \xrightarrow{\sim} F(X)$. □

Lemma 3.3.18. *If F has a left- or right-derived functor, the derived functor preserves weak equivalences. It therefore induces a map of homotopy categories* $\mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,\mathbf{D}$.

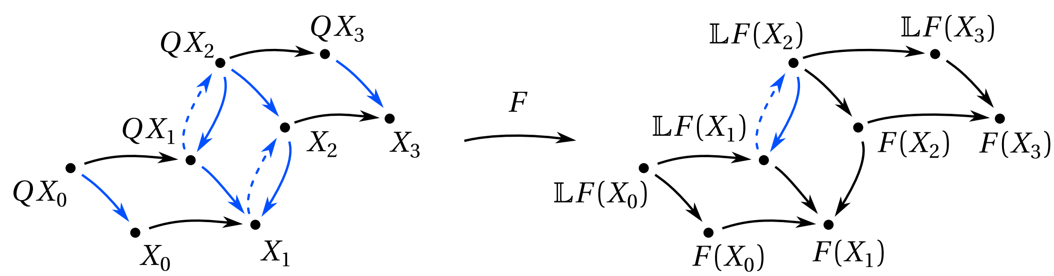
Proof. We do the left-derived case. Since Q is a functor and $QX \to X$ is a natural transformation, for any weak equivalence $X \xrightarrow{\sim} Y$, we get a commuting diagram

$$\begin{tikzcd} QX \arrow[r, "\sim"] \arrow[d] & X \arrow[d, "\sim"] \\ QY \arrow[r, "\sim"] & Y. \end{tikzcd}$$

Applying the 2 out of 3 property, twice, we see that $QX \to Y$ is a weak equivalence, and therefore $QX \to QY$ is a weak equivalence. (In fact, we've just shown that Q is homotopical – it preserves weak equivalences.)

Since QX and QY both lie in $\mathbf{A}$, $F(QX) \to F(QY)$ is a weak equivalence in $\mathbf{D}$. Therefore $\mathbb{L}F = F \circ Q$ preserves weak equivalences. □

We give one more illustration, of how $\mathbb{L}F$ gives a well-defined map on zig-zags, and therefore on the homotopy category, even though F does not.



We now turn to the examples.

Example 3.3.19. The simplest example is when F is already a homotopy functor. Then its left-derived functor is just F itself. We see this by taking

- $Q\colon \mathbf{C} \to \mathbf{C}$ to be the identity functor,
- $Q \Rightarrow \mathrm{id}$ to be the identity natural transformation, and
- $\mathbf{A} = \mathbf{C}$.

Similarly, the right-derived functor of F is just F itself. This preserves equivalences, so we get

$$F\colon \operatorname{Ho}\mathbf{C} \longrightarrow \operatorname{Ho}\mathbf{D}.$$

Example 3.3.20. For any set A, the A-indexed product of spectra $\prod\colon \mathbf{Sp}^{\times A} \to \mathbf{Sp}$ has a right deformation:

- $R\colon \mathbf{Sp}^{\times A} \to \mathbf{Sp}^{\times A}$ replaces every spectrum in the tuple with an Ω-spectrum, in other words it applies R from Proposition 2.2.9 to each slot,
- $\mathrm{id} \Rightarrow R$ is in each slot the natural map from $X(\alpha)$ to its replacement $RX(\alpha)$, and
- $\mathbf{A}$ is the category of A-tuples of Ω-spectra.

The map $(X(\alpha))_{\alpha\in A} \to (RX(\alpha))_{\alpha\in A}$ is an equivalence of A-tuples, because it is a stable equivalence in each slot. We know by Section 2.7, exercise 25 that the infinite product preserves equivalences on $\mathbf{A}$. Therefore the right-derived functor is

$$\mathbb{R}\left(\prod_{\alpha\in A}\right)X(\alpha) = \prod_{\alpha\in A} RX(\alpha).$$

We emphasize the two conclusions again: one, the right-derived product preserves equivalences, so it gives a map of homotopy categories

$$\mathbb{R}\left(\prod\right)\colon \operatorname{Ho}\mathbf{Sp}^{\times A} \longrightarrow \operatorname{Ho}\mathbf{Sp}.$$

Two, the right-derived product is equivalent to the product if the product is nice. We only changed the product in the bad cases. So the right-derived product is exactly the functor we want.

3.3.3 Examples of left-derived functors

Example 3.3.21. The smash product of two spaces $\wedge\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_*$ has a left deformation:

- $Q\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_* \times \mathbf{Top}_*$ replaces every pair of spaces (X, Y) by CW complexes (QX, QY),
- $Q \Rightarrow \mathrm{id}$ is in each slot the natural map from QX to X, and
- $\mathbf{A}$ is the category of pairs of well-based spaces.

The map $(QX, QY) \to (X, Y)$ is an equivalence of pairs. We know by Corollary 1.5.11 that the smash product preserves equivalences on $\mathbf{A}$. Therefore the left-derived functor is

$$X \wedge^{\mathbb{L}} Y := QX \wedge QY.$$

This preserves all weak homotopy equivalences, so it gives a functor on the homotopy category,

$$\wedge^{\mathbb{L}}\colon \mathrm{Ho}\,\mathbf{Top}_* \times \mathrm{Ho}\,\mathbf{Top}_* \longrightarrow \mathrm{Ho}\,\mathbf{Top}_*.$$

Example 3.3.22. To give an analogous example from homological algebra, as in Example 3.3.5, let $\mathbf{Ch}$ be the category of unbounded chain complexes and let $\mathscr{D}(\mathbb{Z})$ be its homotopy category. The tensor product

$$\otimes\colon \mathbf{Ch} \times \mathbf{Ch} \to \mathbf{Ch}$$

does not preserve quasi-isomorphisms. However, it does preserve quasi-isomorphisms if at least one of the chain complexes has free abelian groups in every degree. Letting $QC \xrightarrow{\sim} C$ denote a replacement by a levelwise free complex, we therefore get a left-derived functor

$$C \otimes^{\mathbb{L}} D := QC \otimes^{\mathbb{L}} QD,$$

giving a product on the derived category

$$\otimes^{\mathbb{L}}\colon \mathscr{D}(\mathbb{Z}) \times \mathscr{D}(\mathbb{Z}) \to \mathscr{D}(\mathbb{Z}).$$

The homology of this chain complex has the same Künneth theorem as the homology of a product of spaces (Theorem 1.3.9). Taking a derived tensor product of one-term chain complexes A and B and taking homology gives the Tor groups $\mathrm{Tor}^i(A, B)$.

Example 3.3.23. The suspension spectrum functor $\Sigma^\infty\colon \mathbf{Top}_* \to \mathbf{Sp}$ has a left deformation. As in Example 3.3.21, we take Q to be CW replacement and $\mathbf{A} \subseteq \mathbf{Top}_*$ to be well-based spaces.

We have to check that Σ^∞ preserves equivalences on **A**. If $K \to L$ is a weak equivalence of spaces, the map $K \wedge S^n \to L \wedge S^n$ is an equivalence by Corollary 1.5.11, so $\Sigma^\infty K \to \Sigma^\infty L$ is a level equivalence of spectra. Therefore Σ^∞ sends weak equivalences in **A** to stable equivalences in **Sp**.

Therefore we get a left-derived suspension spectrum functor

$$\mathbb{L}\Sigma^\infty K := \Sigma^\infty QK,$$

which defines a map of homotopy categories

$$\mathbb{L}\Sigma^\infty \colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}.$$

,

Example 3.3.24. The tensor $\wedge\colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp}$ is interesting. To derive $K \wedge X$, it is enough to replace K by a CW complex, by Corollary 2.4.23. On the other hand, we could decide to replace both K and X by CW objects. This gives two different left-derived functors that are equivalent to each other:

$$QK \wedge X \simeq QK \wedge QX.$$

We will see in Proposition 3.4.2 that this always happens – any two left-derived functors of F are equivalent.

Collecting the previous examples together, and adding a few more that follow the same argument:

Proposition 3.3.25. *The following functors can be left-derived:*

- smash product $\wedge\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_*$,
- smash product with a well-based space $K \wedge (-)\colon \mathbf{Top}_* \to \mathbf{Top}_*$,
- reduced suspension $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$,
- suspension spectrum $\Sigma^\infty\colon \mathbf{Top}_* \to \mathbf{Sp}$,
- free spectrum $F_n\colon \mathbf{Top}_* \to \mathbf{Sp}$,
- tensor $\wedge\colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp}$, and
- handicrafted smash product $\wedge\colon \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$.

In every case, the functor can be derived by making the spaces CW complexes, and the spectra into CW spectra. See exercise 22.

Example 3.3.26. Let $\mathbf{Top}^{\{\bullet \leftarrow \bullet \rightarrow \bullet\}}$ be the category of diagrams of the form

$$X \longleftarrow A \longrightarrow Y. \tag{3.3.27}$$

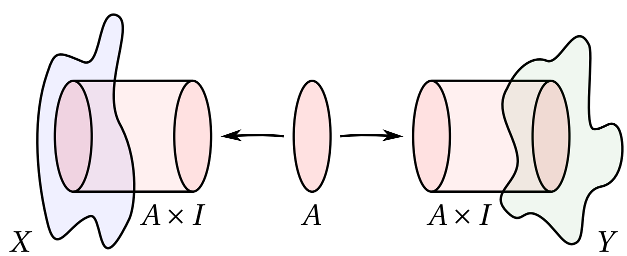
A weak equivalence of diagrams is a map that gives an equivalence on each space separately:

$$\begin{tikzcd} X \arrow[d, "\sim"] & A \arrow[l] \arrow[r] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ X' & A' \arrow[l] \arrow[r] & Y'. \end{tikzcd}$$

The colimit, or pushout of the diagram, defines a functor $\mathbf{Top}^{\{\bullet \leftarrow \bullet \rightarrow \bullet\}} \to \mathbf{Top}$. We claim that its left-derived functor is the homotopy pushout $X \cup_A (A \times I) \cup_A Y$, from Definition 1.5.1.

To see this, let $\mathbf{A}$ be the subcategory of diagrams in which the maps $A \to X$ and $A \to Y$ are cofibrations. By Lemma 1.5.2 and Theorem 1.5.10, weak equivalences between diagrams of this form go to weak equivalences of pushouts. Let Q be the operation that replaces (3.3.27) with the diagram

$$X \cup_A (A \times [0,1/2]) \longleftarrow A \times \{1/2\} \longrightarrow (A \times [1/2,1]) \cup_A Y. \tag{3.3.28}$$



By Section 1.7, exercise 12, the maps of this diagram are cofibrations, so Q lands in $\mathbf{A}$. Collapsing away the cyliders gives a weak equivalence back to the original diagram (3.3.27). Therefore, the left-derived pushout is the pushout of (3.3.28), which is the homotopy pushout $X \cup_A (A \times I) \cup_A Y$.

The same applies to pushouts of based spaces and spectra, and to the mapping telescope $\mathbf{Top}^{\{\bullet \rightarrow \bullet \rightarrow \ldots\}} \to \mathbf{Top}$. See exercises 23 and 24.

Remark 3.3.29. More generally, the colimit of a diagram of any shape can be left-derived, and the left-derived functor is equivalent to the homotopy colimit:

$$\mathbb{L}\mathrm{colim} \simeq \mathrm{hocolim}.$$

For spectra, this is the homotopy colimit at each spectrum level. We will prove this in general in **??**, but see Example 3.3.26 and exercises 23 and 24 for special cases.

Remark 3.3.30. The reduced suspension of spaces $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$ does not preserve all equivalences, see exercise 21. So it has to be left-derived. It is a bit of a miracle that the reduced suspension of spectra $\Sigma\colon \mathbf{Sp} \to \mathbf{Sp}$ preserves all equivalences (Corollary 2.4.3), and doesn't have to be left-derived. It has to do with the fact that the definition of π_* of a spectrum has Σ baked into it.

3.3.4 Examples of right-derived functors

We now turn to right-derived functors. We have already seen that the infinite product can be right-derived.

Example 3.3.31. The evaluation functor $\mathrm{ev}_n\colon \mathbf{Sp} \to \mathbf{Top}_*$ has a right deformation. We take R to be Ω-spectrum replacement and $\mathbf{A} \subseteq \mathbf{Sp}$ to be the Ω-spectra. Since a stable equivalence of Ω-spectra is a level equivalence (Lemma 2.2.5), every stable equivalence in $\mathbf{A}$ gives an equivalence after ev_n.

Therefore we get a right-derived evaluation functor

$$\mathbb{R}\mathrm{ev}_n := \mathrm{ev}_n RX,$$

which defines a map of homotopy categories

$$\mathbb{R}\mathrm{ev}_n\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}_*.$$

In the special case $n = 0$, $\mathbb{R}\mathrm{ev}_0$ is also called Ω^∞ (Definition 2.2.11).

We can also right-derive the mapping space $\mathrm{Map}(X, Y)$ from Definition 2.3.12, by replacing X by a CW spectrum QX, and replacing Y by an Ω-spectrum RY. This really is a right deformation because the maps $QX \to X$ and $Y \to RY$ give a single map $(X, Y) \to (QX, RY)$ in the category $\mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp}$.

Lemma 3.3.32. *On the subcategory of pairs (X, Y) in which X is CW and Y is an Ω-spectrum, the mapping space* $\mathrm{Map}(-,-)$ *preserves all equivalences.*

Proof. The key observation is that

$$\pi_k(\mathrm{Map}(X, Y)) \cong \pi_0(\mathrm{Map}(S^k \wedge X, Y)) \cong [\Sigma^k X, Y]_h,$$

using Section 2.7, exercise 15. By Proposition 3.1.40, this sends stable equivalences $X \to X'$ and $Y \to Y'$ to bijections, so long as X and X' are CW and Y and Y' are Ω-spectra. (It is also enough to take π_k at the basepoint of $\mathrm{Map}(X, Y)$, because $\mathrm{Map}(X, Y) \simeq \Omega\mathrm{Map}(X, \mathrm{sh}^1 Y)$ is a loop space, so its homotopy groups at all basepoints are isomorphic.) □

Proposition 3.3.33. *The following functors can be right-derived:*

- evaluation $\mathrm{ev}_n\colon \mathbf{Sp} \to \mathbf{Top}_*$,
- mapping spaces $\mathrm{Map}(-,-)\colon \mathbf{Top}^{\mathrm{op}} \times \mathbf{Top} \to \mathbf{Top}$,
- based mapping spaces $\mathrm{Map}_*(-,-)\colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Top}_* \to \mathbf{Top}_*$,
- cotensors $F(-,-)\colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Sp}$,
- cotensor with a fixed CW complex, $F(K,-)\colon \mathbf{Sp} \to \mathbf{Sp}$, and
- mapping spaces $\mathrm{Map}_*(-,-)\colon \mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Sp}$.

In every case, we use Ω-spectra, and if we are taking maps out of some object then we make it a CW object.

Example 3.3.34. This example is the dual of Example 3.3.26. Let $\mathbf{Top}^{\{\bullet\to\bullet\leftarrow\bullet\}}$ be the category of diagrams of the form

$$X \longrightarrow B \longleftarrow Y. \tag{3.3.35}$$

A weak equivalence of diagrams is a map that gives an equivalence on each space separately:

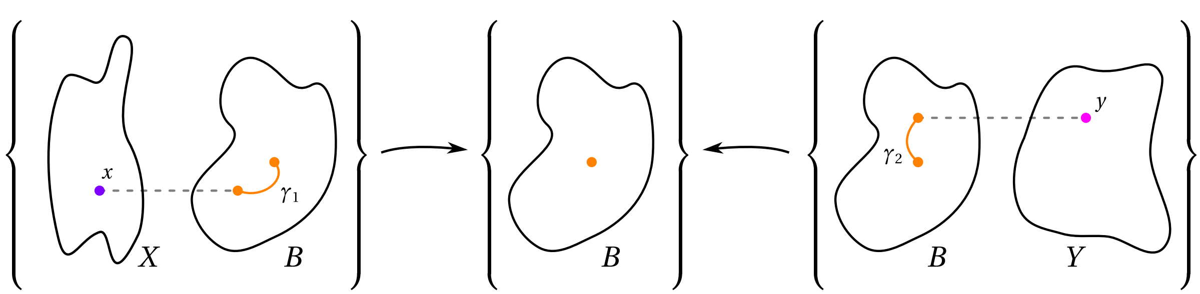
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\sim"] & B \arrow[d, "\sim"] & Y \arrow[l] \arrow[d, "\sim"] \\ X' \arrow[r] & B' & Y'. \arrow[l] \end{tikzcd}$$

The limit, or pullback of the diagram, defines a functor $\mathbf{Top}^{\{\bullet\to\bullet\leftarrow\bullet\}} \to \mathbf{Top}$. We claim that its right-derived functor is the homotopy pullback $X \times_B B^I \times_B Y$, from Definition 1.5.18.

To see this, let $\mathbf{A}$ be the subcategory of diagrams in which the maps $X \to B$ and $Y \to B$ are fibrations. By Lemma 1.5.19 and Theorem 1.5.24, weak equivalences between diagrams of this form go to weak equivalences of pushouts. Let R be the operation that replaces (3.3.27) with the diagram

$$X \times_B B^{[0,1/2]} \longrightarrow B \longleftarrow B^{[1/2,1]} \times_B Y, \tag{3.3.36}$$

where the fiber products are taken over 0 and 1, and the maps to the B in the middle evaluate at 1/2.



By Section 1.7, exercise 12, the maps of this diagram are fibrations, so R lands in $\mathbf{A}$. Collapsing away the cyliders gives a weak equivalence back to the original diagram (3.3.35). Therefore, the right-derived pullback is the pullback of (3.3.36), which is the homotopy pullback $X \times_B B^I \times_B Y$

The same applies to pullbacks of based spaces and spectra, and to sequential limits. See exercise 25.

Remark 3.3.37. More generally, the limit of a diagram of any shape can be right-derived, and the right-derived functor is equivalent to the homotopy limit:

$$\mathbb{R}\mathrm{lim} \simeq \mathrm{holim}\,.$$

For spaces, this is constructed in **??**. For spectra, the homotopy limit is defined by making the terms of the diagram into Ω-spectra, and then taking the homotopy limit of the spaces at each spectrum level. See exercise 25.

Example 3.3.38. There is a hom functor on unbounded chain complexes

$$\mathrm{Hom}(-,-)\colon \mathbf{Ch}^{\mathrm{op}} \times \mathbf{Ch} \to \mathbf{Ch}.$$

It can be right-derived by making the first chain complex projective, or the second chain complex injective. Either way, applying this to one-term chain complexes and taking homology of the result gives the Ext groups $\mathrm{Ext}^i(A,B)$.

Example 3.3.39. Let $(\mathbf{Sp}, S)$ be the category of spectra with the stable equivalences. Let $(\mathbf{Sp}, \mathbb{Q})$ be the category of spectra with the **rational equivalences** – these are the maps inducing isomorphisms on the rational stable homotopy groups $\pi_*(X) \otimes \mathbb{Q}$. Clearly every stable equivalence is a rational equivalence, so the identity functor

$$(\mathbf{Sp}, S) \xrightarrow{\mathrm{id}} (\mathbf{Sp}, \mathbb{Q}).$$

is a homotopy functor. Therefore we get a map of homotopy categories $\mathrm{Ho}(\mathbf{Sp}, S) \to \mathrm{Ho}(\mathbf{Sp}, \mathbb{Q})$. Going the other way, the identity functor

$$(\mathbf{Sp}, \mathbb{Q}) \xrightarrow{\mathrm{id}} (\mathbf{Sp}, S)$$

is not homotopical. However, it can be right-derived by the construction $X_{\mathbb{Q}}$ of Example 2.5.33, or see Section 2.7, exercise 39. The subcategory $\mathbf{A} \subseteq (\mathbf{Sp}, \mathbb{Q})$ consists of those spectra whose homotopy groups are already rational, so that $\pi_*(X) \to \pi_*(X) \otimes \mathbb{Q}$ is an isomorphism. We therefore get a map of homotopy categories $\mathrm{Ho}(\mathbf{Sp}, \mathbb{Q}) \to \mathrm{Ho}(\mathbf{Sp}, S)$ given by the right-derived identity functor, in other words the rationalization functor $X_{\mathbb{Q}}$.

Going back and forth

$$(\mathbf{Sp}, \mathbb{Q}) \xrightarrow{\mathbb{R}\mathrm{id}} (\mathbf{Sp}, S) \xrightarrow{\mathrm{id}} (\mathbf{Sp}, \mathbb{Q})$$

is equivalent to the identity, so that rational spectra sit inside spectra as a retract. The composite the other way

$$(\mathbf{Sp}, S) \xrightarrow{\mathrm{id}} (\mathbf{Sp}, \mathbb{Q}) \xrightarrow{\mathbb{R}\mathrm{id}} (\mathbf{Sp}, S)$$

is the rationalization functor $X_{\mathbb{Q}}$. It defines the retract onto this subcategory of rational spectra. See Example 5.7.8 for more examples of this kind.

3.4 Advanced properties of derived functors*

3.4.1 Uniqueness of derived functors

In this section we explain how derived functors are unique. They won't be unique on the nose – different choices of left deformation Q will lead to *weakly equivalent* functors $F \circ Q$, not isomorphic ones. So to capture this uniqueness, we'll have to consider functors up to weak equivalence.

Definition 3.4.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with weak equivalences, satisfying 2 out of 3. The **functor category** $\mathbf{F}(\mathbf{C}, \mathbf{D})$ has an object for each functor $F: \mathbf{C} \to \mathbf{D}$, and a morphism for each natural transformation $F \to G$.

We say a map of functors $F \to G$ in $\mathbf{F}(\mathbf{C}, \mathbf{D})$ is a **weak equivalence of functors** if for each $X \in \mathbf{C}$, the map $F(X) \to G(X)$ is a weak equivalence in $\mathbf{D}$.

This gives the functor category $\mathbf{F}(\mathbf{C}, \mathbf{D})$ a class of weak equivalences, so we can take its homotopy category $\mathrm{Ho}\,\mathbf{F}(\mathbf{C}, \mathbf{D})$. Note that a map in this homotopy category is a zig-zag of functors, e.g.

$$F \longrightarrow F_1 \xleftarrow{\sim} F_2 \xleftarrow{\sim} F_3 \longrightarrow F_4 \longrightarrow G.$$

A **homotopy functor** is a functor $F: \mathbf{C} \to \mathbf{D}$ that sends weak equivalences of objects of $\mathbf{C}$, $X \xrightarrow{\sim} Y$, to weak equivalences $F(X) \xrightarrow{\sim} F(Y)$.

Proposition 3.4.2. *If F has a left-derived functor $\mathbb{L}F$, then in $\mathrm{Ho}\,\mathbf{F}(\mathbf{C}, \mathbf{D})$, the functor $\mathbb{L}F$ is terminal among homotopy functors mapping to F. Dually, a right-derived functor $\mathbb{R}F$ is initial among homotopy functors receiving a map from F.*

In other words, if G is any other homotopy functor, and $G \to F$ is any zig-zag of natural transformations from G to F, then there is a unique zig-zag $G \to \mathbb{L}F$ making this triangle commute:

$$\begin{tikzcd} G \arrow[d, dashed, "\exists!"'] \arrow[dr] & \\ \mathbb{L}F \arrow[r] & F. \end{tikzcd}$$

So $\mathbb{L}F$ is the "closest homotopy functor to F" on the left.

Dually, if F has a right-derived functor $\mathbb{R}F$, then any zig-zag from F to a homotopy functor G gives a unique zig-zag $\mathbb{R}F \to G$ making this triangle commute:

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & \mathbb{R}F \arrow[d, dashed, "\exists!"] \\ & G. \end{tikzcd}$$

So $\mathbb{R}F$ is the "closest homotopy functor to F" on the right.

Proof. We do the left-derived case in detail. The right-derived case is the same except we reverse all the arrows.

Let $q\colon QX \to X$ denote the weak equivalence from Q back to the identity, and let $\varphi\colon G \to F$ denote the zig-zag from G to F. Since this is a zig-zag of natural transformations, on the map $q\colon QX \to X$ it gives a commuting square in the homotopy category of functors,

$$\begin{tikzcd} G \circ Q \arrow[r, "\varphi \circ Q"] \arrow[d, "G \circ q", "\cong"'] & F \circ Q \arrow[d, "F \circ q"] \\ G \arrow[r, "\varphi"] & F. \end{tikzcd}$$

The maps $G(QX) \to G(X)$ are weak equivalences, so the left-hand vertical map in the above square is an isomorphism in the homotopy category. The left-hand vertical and top horizontal map give the desired zig-zag from G to $\mathbb{L}F$.

To show this is unique, let $\gamma\colon G \to F \circ Q$ be any other zig-zag commuting with $q\colon F \circ Q \to F$. Applying these zig-zags to the map q produces the following commuting diagram in the homotopy category of functors.

$$\begin{tikzcd} G \circ Q \arrow[r, "\gamma \circ Q"] \arrow[rr, bend left, "\varphi \circ Q"] \arrow[d, "G \circ q", "\cong"'] & F \circ Q \circ Q \arrow[r, "F \circ q \circ Q", "\cong"'] \arrow[d, "F \circ Q \circ q", "\cong"'] & F \circ Q \arrow[d, "F \circ q"] \\ G \arrow[r, "\gamma"] \arrow[rr, bend right, "\varphi"'] & F \circ Q \arrow[r, "F \circ q"] & F. \end{tikzcd}$$

The vertical isomorphisms are because G and $F \circ Q$ are homotopy functors. For the isomorphism in the top row, $q \circ Q\colon QQX \to QX$ is a weak equivalence of objects in $\mathbf{A}$, so F sends it to a weak equivalence, hence an isomorphism in the homotopy category.

As a map in the homotopy category, γ is therefore equal to a composite of $\varphi \circ Q$ with some fixed isomorphisms. The map γ is therefore determined by φ, so it is unique.[2] □

[2] This version of the argument is inspired by the proof in [Goo03, 1.8], see also [Rie17, 6.4.11].

Corollary 3.4.3. *Any two left-derived functors of F are canonically equivalent by a zig-zag of equivalences of functors. The same is true for right-derived functors.*

There is a similar universal property that $\mathbb{L}F$ enjoys as a functor on $\operatorname{Ho}\mathbf{C}$, see exercise 29.

Corollary 3.4.4. *The two maps $QQX \rightrightarrows QX$ that cancel either the first or second copy of Q, give the same map in* $\operatorname{Ho}\mathbf{C}$.

Proof. Take $F = \mathrm{id}_{\mathbf{C}}$ to be the identity functor and $G = QQ$. The two maps $QQX \rightrightarrows QX$ are natural transformations $G \to \mathbb{L}F$ that commute with the map back to F, so by Proposition 3.4.2 they are equal in the homotopy category. □

Example 3.4.5. The smash product of spaces $\wedge\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_*$ can be left-derived by replacing the inputs (X, Y) by CW complexes. It can also be derived by "whiskering" the inputs: attach one end of an interval I to the basepoint of X, and let the other end of I be the new basepoint. This space $I \cup_* X$ is well-based, and collapsing I gives an equivalence back to X. By Corollary 3.4.3, these two derived smash products are equivalent to each other:

$$QX \wedge QY \simeq (I \cup_* X) \wedge (I \cup_* Y).$$

A similar argument applies when smashing a space K with a spectrum X, though whiskering a spectrum is a little harder to do because of the bonding maps – you have to iteratively replace each map $\Sigma X_n \to X_{n+1}$ by its mapping cylinder.

Example 3.4.6. As in Example 3.3.24, the tensor $\wedge\colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp}$ can be derived by making both K and X CW, or by making just K CW. These are equivalent by Corollary 3.4.3. This is also easy to see directly: since $QK \wedge -$ preserves all equivalences, we have an equivalence

$$QK \wedge QX \simeq QK \wedge X.$$

Example 3.4.7. In Example 3.3.26, we left-derived the pushout $X \cup_A Y$ by replacing the maps $A \to X$ and $A \to Y$ by their mapping cylinders. We could, instead, replace them by relative CW complexes $A \to X' \simeq X$ and $A \to Y' \simeq Y$ using Corollary 1.4.11. This gives a different model for the left-derived pushout, but by Corollary 3.4.3, the two models are canonically equivalent to each other.

Example 3.4.8. The hom functor on chain complexes $\operatorname{Hom}\colon \mathbf{Ch}^{\mathrm{op}} \times \mathbf{Ch} \to \mathbf{Ch}$ can be right-derived by replacing the source by a projective chain complex, or the target by an injective one. By Corollary 3.4.3, these produce equivalent results. In particular, we get the standard result in homological algebra that $\operatorname{Ext}^i(A, B)$ can be computed by taking a projective resolution of A, or by taking an injective resolution of B, before applying Hom and taking the cohomology of the result.

Example 3.4.9. In Definition 2.2.11 we define the infinite loop space $\Omega^\infty X$ of a spectrum X to be the right-derived functor of ev_0 on X. By Corollary 3.4.3, this is independent of the method we use to replace X by an Ω-spectrum.

3.4.2 "Correct" homotopy types

We can now explain what we meant by the "correct" homotopy type in Section 1.5.

Definition 3.4.10. Suppose F has a left-derived functor $\mathbb{L}F$. We say that F is **correct** on X or "has the correct homotopy type on X" if $\mathbb{L}F(X) \to F(X)$ is a weak equivalence. Correct for right-derived functors is defined similarly.

Since derived functors are unique, this doesn't depend on how we deform F.

Remark 3.4.11. F preserves all equivalences whenever it is correct. Therefore, if we take all values of X for which F is correct, we get the largest possible choice of subcategory $\mathbf{A}$ that we can use when deriving F.

Example 3.4.12. The smash product of spaces $X \wedge Y$ is correct if X and Y are CW complexes. It is also correct if X and Y are well-based. The tensor $K \wedge X$ is correct if K is a CW complex. It is also correct if K and all the spectrum levels X_n are well-based.

The pushout $X \cup_A Y$ is correct if one of the maps $A \to X$ or $A \to Y$ is a cofibration (Lemma 1.5.2). The sequential colimit is correct if the maps of the system are closed inclusions (Lemma 1.5.5). The pullback $X \times_B Y$ is correct if one of the maps $X \to B$ or $Y \to B$ is a fibration (Lemma 1.5.19). The sequential limit is correct if the maps of the system are fibrations (Lemma 1.5.22).

The coproduct of spectra $\bigvee_\alpha X(\alpha)$ is always correct. The product of spectra $\prod_\alpha X(\alpha)$ is correct if it is a finite product, or if the inputs are Ω-spectra.

3.4.3 Composites of derived functors

Suppose we have functors on categories with weak equivalences that can be composed,

$$\mathbf{C}_1 \xrightarrow{F_1} \mathbf{C}_2 \xrightarrow{F_2} \mathbf{C}_3,$$

and that

- F_1 has a left-deformation Q_1 landing in $\mathbf{A}_1 \subseteq \mathbf{C}_1$,
- F_2 has a left-deformation Q_2 landing in $\mathbf{A}_2 \subseteq \mathbf{C}_2$, and

- $F_1(\mathbf{A}_1) \subseteq \mathbf{A}_2$.

Lemma 3.4.13. *Under these hypotheses, Q_1 is also a left deformation for the composite $F_2 \circ F_1$, and we have a weak equivalence of functors*

$$\mathbb{L}(F_2 \circ F_1) \simeq (\mathbb{L}F_2) \circ (\mathbb{L}F_1).$$

A similar statement holds for composites of right-derived functors.

Proof. This is actually pretty easy. Any weak equivalence in $\mathbf{A}_1$ goes to a weak equivalence in $F(\mathbf{A}_1) \subseteq \mathbf{A}_2$, and therefore goes to a weak equivalence in $F(\mathbf{A}_2) \subseteq \mathbf{C}_3$. Therefore Q_1 is a left deformation of $F_2 \circ F_1$. Since F_2 preserves equivalences on $\mathbf{A}_2$, we get an equivalence

$$(\mathbb{L}F_2)(\mathbb{L}F_1)(X) = F_2 Q_2 F_1 Q_1 X \xrightarrow{\sim} F_2 F_1 Q_1 X = \mathbb{L}(F_2 \circ F_1)(X).$$

For right-derived functors the same argument applies. The equivalence looks like:

$$\mathbb{R}(G_2 \circ G_1)(X) = G_2 G_1 R_1 X \xrightarrow{\sim} G_2 R_2 G_1 R_1 X = (\mathbb{R}G_2)(\mathbb{R}G_1)(X).$$

□

Remark 3.4.14. The above can be summarized as "left-derived functors compose, *if* the deformations are compatible with each other." They do not compose in general if $F_1(\mathbf{A}_1) \nsubseteq \mathbf{A}_2$, see Example 3.4.19.

Example 3.4.15. The suspension spectrum and disjoint basepoint functors

$$\mathbf{Top} \xrightarrow{(-)_+} \mathbf{Top}_* \xrightarrow{\Sigma^\infty} \mathbf{Sp}$$

satisfy the hypotheses of Lemma 3.4.13. Therefore the left-derived functors compose:

$$\begin{tikzcd} \mathrm{Ho}\,\mathbf{Top} \arrow[r, "(-)_+"] \arrow[rr, bend right, "\mathbb{L}(\Sigma^\infty_+)"'] & \mathrm{Ho}\,\mathbf{Top}_* \arrow[r, "\mathbb{L}\Sigma^\infty"] & \mathbf{Sp} \end{tikzcd}$$

The same applies to the right adjoints (exercise 30).

Example 3.4.16. The suspension functor commutes with suspension spectrum, up to isomorphism:

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, "\Sigma^\infty"] \arrow[d, "\Sigma"'] & \mathbf{Sp} \arrow[d, "\Sigma"] \\ \mathbf{Top}_* \arrow[r, "\Sigma^\infty"] & \mathbf{Sp} \end{tikzcd}$$

Thinking of this as two different factorizations of $\Sigma^\infty \circ \Sigma \cong \Sigma \circ \Sigma^\infty$, the hypotheses of Lemma 3.4.13 are satisfied, so we get

$$(\mathbb{L}\Sigma^\infty)(\mathbb{L}\Sigma) \simeq \mathbb{L}(\Sigma^\infty \circ \Sigma) \cong \mathbb{L}(\Sigma \circ \Sigma^\infty) \simeq (\mathbb{L}\Sigma)(\mathbb{L}\Sigma^\infty).$$

In other words, the corresponding square of left-derived functors on the homotopy category also commutes, up to isomorphism:

$$\begin{tikzcd} \operatorname{Ho}\mathbf{Top}_* \arrow[r, "\mathbb{L}\Sigma^\infty"] \arrow[d, "\mathbb{L}\Sigma"'] & \operatorname{Ho}\mathbf{Sp} \arrow[d, "\mathbb{L}\Sigma=\Sigma"] \\ \operatorname{Ho}\mathbf{Top}_* \arrow[r, "\mathbb{L}\Sigma^\infty"] & \operatorname{Ho}\mathbf{Sp} \end{tikzcd}$$

Example 3.4.17. As in the previous exercise, the loopspace functor commutes with evaluation

$$\begin{tikzcd} \mathbf{Top}_* \arrow[d, "\Omega"'] & \mathbf{Sp} \arrow[l, "\mathrm{ev}_0"'] \arrow[d, "\Omega"] \\ \mathbf{Top}_* & \mathbf{Sp}, \arrow[l, "\mathrm{ev}_0"'] \end{tikzcd}$$

giving a commuting square of right-derived functors

$$\begin{tikzcd} \operatorname{Ho}\mathbf{Top}_* \arrow[r, "\Omega^\infty"] \arrow[d, "\Omega"'] & \operatorname{Ho}\mathbf{Sp} \arrow[d, "\Omega"] \\ \operatorname{Ho}\mathbf{Top}_* \arrow[r, "\Omega^\infty"] & \operatorname{Ho}\mathbf{Sp} \end{tikzcd}$$

Example 3.4.18. The smash product of based spaces is associative, $(X \wedge Y) \wedge Z) \cong X \wedge (Y \wedge Z)$. We can draw this as a commuting square (up to isomorphism) of functors

$$\begin{tikzcd} \mathbf{Top}_* \times \mathbf{Top}_* \times \mathbf{Top}_* \arrow[r, "\mathrm{id}\times\wedge"] \arrow[d, "\wedge\times\mathrm{id}"'] & \mathbf{Top}_* \times \mathbf{Top}_* \arrow[d, "\wedge"] \\ \mathbf{Top}_* \times \mathbf{Top}_* \arrow[r, "\wedge"] & \mathbf{Top}_*. \end{tikzcd}$$

The hypotheses of Lemma 3.4.13 are satisfied, so we get a commuting square of left-derived smash products

$$\begin{tikzcd} \operatorname{Ho}\mathbf{Top}_* \times \operatorname{Ho}\mathbf{Top}_* \times \operatorname{Ho}\mathbf{Top}_* \arrow[r, "\mathrm{id}\times\wedge^{\mathbb{L}}"] \arrow[d, "\wedge^{\mathbb{L}}\times\mathrm{id}"'] & \operatorname{Ho}\mathbf{Top}_* \times \operatorname{Ho}\mathbf{Top}_* \arrow[d, "\wedge^{\mathbb{L}}"] \\ \operatorname{Ho}\mathbf{Top}_* \times \operatorname{Ho}\mathbf{Top}_* \arrow[r, "\wedge^{\mathbb{L}}"] & \operatorname{Ho}\mathbf{Top}_* \end{tikzcd}$$

giving an isomorphism on the homotopy category

$$(X \wedge^{\mathbb{L}} Y) \wedge^{\mathbb{L}} Z) \cong X \wedge^{\mathbb{L}} (Y \wedge^{\mathbb{L}} Z).$$

If we proceed this way with the unit isomorphisms $S^0 \wedge X \cong X$ and symmetry isomorphisms $X \wedge Y \cong Y \wedge X$, we can prove that $\operatorname{Ho}\mathbf{Top}_*$ is a symmetric monoidal category. See Lemma 4.1.7. The argument will apply to $\operatorname{Ho}\mathbf{Sp}$ as well, as soon as we actually have a smash product of spectra that has isomorphisms $(X \wedge Y) \wedge Z) \cong X \wedge (Y \wedge Z)$, $\mathbb{S} \wedge X \cong X$, and $X \wedge Y \cong Y \wedge X$. See Section 6.2.

Example 3.4.19. The colimit of a diagram of spaces $X\colon \mathbf{I} \to \mathbf{Top}$ can be defined in two stages, by first forming the coequalizer diagram

$$\coprod_{i\to j} X(i) \rightrightarrows \coprod_i X(i)$$

and then taking its colimit. In other words, a colimit can always be built out of coproducts and coequalizers:

$$\begin{tikzcd} \mathbf{Top}^{\mathbf{I}} \arrow[r, "\text{coproducts}"] \arrow[rr, bend right, "\text{colimit}"'] & \mathbf{Top}^{\{\bullet\rightrightarrows\bullet\}} \arrow[r, "\text{coequalizer}"] & \mathbf{Top} \end{tikzcd}$$

However, the *homotopy* colimit is not the *homotopy* coequalizer of the same diagram. In other words, we do *not* have an agreement up to isomorphism between the following functors:

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}) \arrow[r, "\mathbb{L}(\text{coproducts})"] \arrow[rr, bend right, "\mathbb{L}(\text{colimit})"'] & \mathrm{Ho}(\mathbf{Top}^{\{\bullet\rightrightarrows\bullet\}}) \arrow[r, "\mathbb{L}(\text{coequalizer})"] & \mathrm{Ho}\,\mathbf{Top}. \end{tikzcd}$$

In particular, this means the conditions of Lemma 3.4.13 are not satisfied.

3.4.4 Adjunctions between derived functors

Another common situation is that left and right deformable functors come in adjoint pairs. When this happens, their derived functors also form an adjoint pair:

Proposition 3.4.20. *Suppose that $(F \dashv G)$ is a pair of adjoint functors*

$$F\colon \mathbf{C} \to \mathbf{D}, \qquad G\colon \mathbf{D} \to \mathbf{C},$$

that F has a left deformation, and that G has right deformation. Then the derived functors

$$\mathbb{L}F\colon \mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,\mathbf{D}, \qquad \mathbb{R}G\colon \mathrm{Ho}\,\mathbf{D} \to \mathrm{Ho}\,\mathbf{C}$$

are adjoint as well.

Proof. Recall that an adjunction can be described by a pair of natural transformations

$$\eta\colon \mathrm{id}_{\mathbf{C}} \to GF, \qquad \epsilon\colon FG \to \mathrm{id}_{\mathbf{D}},$$

satisfying the "triangle identities"

$$\begin{tikzcd} F \arrow[r, "1\eta"] \arrow[dr, equal] & FGF \arrow[d, "\epsilon 1"] \\ & F, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta 1"] \arrow[dr, equal] & GFG \arrow[d, "1\epsilon"] \\ & G. \end{tikzcd}$$

We define maps in the homotopy category of functors

$$\widetilde{\eta}\colon \mathrm{id}_{\mathbf{C}} \to \mathbb{R}G\mathbb{L}F = G\circ R\circ F\circ Q, \qquad \widetilde{\epsilon}\colon F\circ Q\circ G\circ R = \mathbb{L}F\mathbb{R}G \to \mathrm{id}_{\mathbf{D}},$$

satisfying the same triangle identities in the homotopy category. We define them as the zig-zags

$$\mathrm{id}_{\mathbf{C}} \xleftarrow{q} Q \xrightarrow{\eta Q} GFQ \xrightarrow{GrFQ} GRFQ, \qquad FQGR \xrightarrow{q} FGR \xrightarrow{\epsilon R} R \xleftarrow{r} \mathrm{id}_{\mathbf{D}}.$$

We check that the two diagrams below commute in the homotopy category of functors, which verifies the triangle identities in the homotopy category. The two possible descriptions of the map $FQQ \to FQ$ are equal by Corollary 3.4.4, and similarly for $GR \to GRR$ in the second diagram.

$$\begin{tikzcd}
FQQ \arrow[r, "\eta"] \arrow[d, "1q=q1", "\sim"'] & FQGFQ \arrow[r, "r"] \arrow[d, "q"] & FQGRFQ \arrow[d, "q"] \\
FQ \arrow[r, "\eta"] \arrow[dr, equal] & FGFQ \arrow[r, "r"] \arrow[d, "\epsilon"] & FGRFQ \arrow[d, "\epsilon"] \\
 & FQ \arrow[r, "r", "\sim"'] & RFQ
\end{tikzcd}
\qquad
\begin{tikzcd}
QGR \arrow[r, "\eta"] \arrow[d, "q", "\sim"'] & GFQGR \arrow[r, "r"] \arrow[d, "q"] & GRFQGR \arrow[d, "q"] \\
GR \arrow[r, "\eta"] \arrow[dr, equal] & GFGR \arrow[r, "r"] \arrow[d, "\epsilon"] & GRFGR \arrow[d, "\epsilon"] \\
 & GR \arrow[r, "1r=r1", "\sim"'] & GRR
\end{tikzcd}$$

□

Example 3.4.21. This gives the following adjunctions of homotopy categories. (K must be a CW complex.)

Left adjoint	Right adjoint
Left-derived suspension $\mathbb{L}\Sigma\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Top}_*$	Loopspace $\Omega\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Top}_*$
Disjoint basepoint $(-)_+\colon \operatorname{Ho}\mathbf{Top} \to \mathbf{Top}_*$	Forget basepoint $U\colon \operatorname{Ho}\mathbf{Top}_* \to \mathbf{Top}$

Left adjoint	**Right adjoint**
Suspension $\Sigma\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$	Loopspace $\Omega\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$
Left-derived suspension spectrum $\mathbb{L}\Sigma^\infty\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$	Right-derived 0th space $\Omega^\infty = \mathbb{R}\mathrm{ev}_0\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}_*$
Left-derived suspension spectrum $\mathbb{L}\Sigma^\infty_+\colon \operatorname{Ho}\mathbf{Top} \to \operatorname{Ho}\mathbf{Sp}$	Right-derived 0th space $U\Omega^\infty = \mathbb{R}(U\mathrm{ev}_0)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}$
Left-derived free spectrum $\mathbb{L}F_n\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$	Right-derived nth space $\mathbb{R}\mathrm{ev}_n\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}_*$
Left-derived free spectrum $\mathbb{L}F_n(-)_+\colon \operatorname{Ho}\mathbf{Top} \to \operatorname{Ho}\mathbf{Sp}$	Right-derived nth space $\mathbb{R}(U\mathrm{ev}_n)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}$
Tensor $K \wedge (-)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$ (K is a CW complex)	Right-derived cotensor $\mathbb{R}F(K,-)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$
Left-derived tensor $(-)\wedge^{\mathbb{L}} X\colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$ (X is a CW spectrum)	Right-derived mapping space $\mathbb{R}\mathrm{Map}_*(X,-)\colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$

The last two can be summarized like this: for any space K and spectra X and Y, there are bijections

$$[X, \mathbb{R}F(K,Y)]_s \cong [K \wedge^{\mathbb{L}} X, Y]_s \cong [K, \mathbb{R}\mathrm{Map}_*(X,Y)]_*. \tag{3.4.22}$$

In other words, the tensor has two right adjoints, depending on which variable we focus on, and the left-derived tensor has the same two (right-derived) right adjoints. In fact, this always happens when we have a functor with two right adjoints, and compatible deformations. See Remark 3.4.25 for more details.

Example 3.4.23. The universal property of a coproduct can be stated as follows: a map $\coprod_\alpha X_\alpha \to Y$ is the same thing as a map of tuples $(X_\alpha) \to (Y)$, to the constant tuple that is Y in every slot. Therefore, the coproduct is the left adjoint of the functor that creates constant tuples.

These adjunctions pass to the homotopy category of spaces and spectra:

Left adjoint	**Right adjoint**
Disjoint union $\coprod\colon \operatorname{Ho}\mathbf{Top}^{\times A} \to \operatorname{Ho}\mathbf{Top}$	Constant tuple $\operatorname{Ho}\mathbf{Top} \to \operatorname{Ho}\mathbf{Top}^{\times A}$
Wedge sum $\bigvee\colon \operatorname{Ho}\mathbf{Top}_*^{\times A} \to \operatorname{Ho}\mathbf{Top}_*$	Constant tuple $\operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Top}_*^{\times A}$
Wedge sum $\bigvee\colon \operatorname{Ho}\mathbf{Sp}^{\times A} \to \operatorname{Ho}\mathbf{Sp}$	Constant tuple $\operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}^{\times A}$

Dually, the product is the right adjoint of the functor that creates constant tuples: a map $Y \to \prod_\alpha X_\alpha$ is the same thing as a map of tuples $(Y) \to (X_\alpha)$. Passing to the homotopy category:

Left adjoint	**Right adjoint**
Constant tuple $\mathrm{Ho}\,\mathbf{Top} \to \mathrm{Ho}\,\mathbf{Top}^{\times A}$	Product $\prod$: $\mathrm{Ho}\,\mathbf{Top}^{\times A} \to \mathrm{Ho}\,\mathbf{Top}$
Constant tuple $\mathrm{Ho}\,\mathbf{Top}_* \to \mathrm{Ho}\,\mathbf{Top}_*^{\times A}$	Product $\prod$: $\mathrm{Ho}\,\mathbf{Top}_*^{\times A} \to \mathrm{Ho}\,\mathbf{Top}_*$
Constant tuple $\mathrm{Ho}\,\mathbf{Sp} \to \mathrm{Ho}\,\mathbf{Sp}^{\times A}$	Right-derived product $\mathbb{R}\prod$: $\mathrm{Ho}\,\mathbf{Sp}^{\times A} \to \mathrm{Ho}\,\mathbf{Sp}$

In particular, we have recovered Proposition 3.2.2 by a different method!

Example 3.4.24. The colimit functor $\mathrm{colim}\colon \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ and constant diagram $\mathbf{Top} \to \mathbf{Top}^{\mathbf{I}}$ are adjoints. We will see in **??** that the left-derived functor of colim is hocolim. Clearly constant diagram preserves equivalences, so it is its own right-derived functor. Therefore we get an adunction: the homotopy colimit $\mathrm{hocolim}\colon \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}}) \to \mathrm{Ho}\,\mathbf{Top}$ is the left adjoint to the constant diagram $\mathrm{Ho}\,\mathbf{Top} \to \mathrm{Ho}(\mathbf{Top}^{\mathbf{I}})$. The same holds for diagrams of based spaces, and diagrams of spectra.

Remark 3.4.25. The tensor has a right adjoint in each variable, the cotensor and the mapping space

$$\begin{aligned} \wedge\colon \mathbf{Top}_* \times \mathbf{Sp} &\to \mathbf{Sp} \\ F\colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Sp} &\to \mathbf{Sp} \\ \mathrm{Map}_*\colon \mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp} &\to \mathbf{Top}_*. \end{aligned}$$

This is an example of a **two-variable adjunction**: a collection of three functors

$$\begin{aligned} F(-,-)\colon \mathbf{C} \times \mathbf{D} &\to \mathbf{E}, \\ G(-,-)\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{E} &\to \mathbf{D}, \\ H(-,-)\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{E} &\to \mathbf{C} \end{aligned}$$

and natural isomorphisms of functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{Set}$,

$$\mathbf{C}(c, H(d,e)) \cong \mathbf{E}(F(c,d), e) \cong \mathbf{D}(d, G(c,e)).$$

In other words, G is the adjoint of F in one slot and H is the adjoint of F in the other slot.

It turns out that these can be deformed as well. Any time we pick

- a left deformation $Pc \xrightarrow{\sim} c$ landing in $\mathbf{A}_1 \subseteq \mathbf{C}$,

- a left deformation $Qd \xrightarrow{\sim} d$ landing in $\mathbf{A}_2 \subseteq \mathbf{D}$, and
- a right deformation $e \xrightarrow{\sim} Re$ landing in $\mathbf{B} \subseteq \mathbf{E}$,

such that

- $(Pc, Qd) \xrightarrow{\sim} (c, d)$ is a left deformation of F,
- $(c, e) \xrightarrow{\sim} (Pc, Re)$ is a right deformation of G, and
- $(d, e) \xrightarrow{\sim} (Qd, Re)$ is a right deformation of H,

then the derived functors

$$\begin{aligned}\mathbb{L}F(-,-)&\colon \operatorname{Ho}\mathbf{C} \times \operatorname{Ho}\mathbf{D} \to \operatorname{Ho}\mathbf{E},\\ \mathbb{R}G(-,-)&\colon \operatorname{Ho}\mathbf{C}^{\mathrm{op}} \times \operatorname{Ho}\mathbf{E} \to \operatorname{Ho}\mathbf{D},\\ \mathbb{R}H(-,-)&\colon \operatorname{Ho}\mathbf{D}^{\mathrm{op}} \times \operatorname{Ho}\mathbf{E} \to \operatorname{Ho}\mathbf{C}\end{aligned}$$

also form a two-variable adjunction on the homotopy categories. So we get natural bijections for $(c, d, e) \in \operatorname{Ho}\mathbf{C}^{\mathrm{op}} \times \operatorname{Ho}\mathbf{D} \times \operatorname{Ho}\mathbf{E}$,

$$[c, \mathbb{R}H(d, e))]_{\operatorname{Ho}\mathbf{C}} \cong [\mathbb{L}F(c, d), e]_{\operatorname{Ho}\mathbf{E}} \cong [d, \mathbb{R}G(c, e))]_{\operatorname{Ho}\mathbf{D}}.$$

It is not so hard to see that such isomorphisms exist by applying Proposition 3.4.20 for each fixed value of c and d. However, to prove that they are natural in c and d involves no small amount of diagram-chasing. We omit the full proof.

3.5 Exercises

1. (a) Suppose $F\colon \mathbf{C} \to \mathbf{D}$ is a functor. Prove that F gives a well-defined function

   $$\{\text{isom. classes of objects in } \mathbf{C}\} \to \{\text{isom. classes of objects in } \mathbf{D}\}.$$

   (b) Suppose that F is fully faithful. Prove that this map of isomorphism classes is injective.

2. Modify the universal property for the homotopy category by allowing the triangle (3.1.7) to commute up to natural isomorphism. Show that the homotopy category constructed in Definition 3.1.9 (or Definition 3.1.1) also satisfies this property, and that the homotopy category in this sense is unique up to equivalence of categories, rather than isomorphism of categories.

3. Prove Lemma 3.1.22.

4. Suppose $f: X \to Y$ is a stable equivalence of spectra. Explain why composition with f gives isomorphisms

$$[W, X]_s \xrightarrow{\cong} [W, Y]_s, \qquad [Y, W]_s \xrightarrow{\cong} [X, W]_s.$$

5. (a) Define $h\mathbf{Top}$ by taking all topological spaces and homotopy classes of maps. Explain why there is a functor $h\mathbf{Top} \to \operatorname{Ho}\mathbf{Top}$, and why it is not an equivalence of categories. How does the proof in Proposition 3.1.26 that $h\mathbf{CW} \simeq \operatorname{Ho}\mathbf{CW}$ fail?

   (b) Similarly, let $h\mathbf{Sp}$ denote the category of spectra and homotopy classes of maps between them, $[X, Y]_h$. Explain why there is a functor $h\mathbf{Sp} \to \operatorname{Ho}\mathbf{Sp}$, and why this functor is *not* an equivalence of categories.

6. Finish the proof of Proposition 3.1.26 by showing that the given maps $\operatorname{Ho}\mathbf{Top}(X, Y) \to \operatorname{Ho}\mathbf{CW}(X, Y)$ and $\operatorname{Ho}\mathbf{CW}(X, Y) \to \operatorname{Ho}\mathbf{Top}(X, Y)$ are inverses of each other. You may find it helpful to think of the commuting diagram at the end of the proof as a commuting diagram in $\operatorname{Ho}\mathbf{Top}$ or $\operatorname{Ho}\mathbf{CW}$, instead of using the equivalence relation on zig-zags directly.

7. Let $f: X \to Y$ be any map of spectra. Explain why the image in the homotopy category $\delta(f) = \left(\xrightarrow{f}\right)$ is an isomorphism iff f is a stable equivalence. In other words, the stable equivalences in spectra are **saturated**. (You might have to use Proposition 3.1.40).

8. Give a second proof of the Whitehead theorem for spectra (Proposition 2.6.16) by defining the deformation retract of M to X one stable cell at a time, as in the proof of Proposition 3.1.38.

9. Finish the proof of Proposition 3.1.40 by showing that the map (2) is an isomorphism.

10. Finish the proof of Theorem 3.2.5 by proving the composition is bilinear, that the zero map is an identity element for $+$, and that flipping the suspension coordinate gives inverse elements.

11. Use Proposition 3.2.7 to show that the different models for the degree n map $X \xrightarrow{n} X$ in Example 3.1.4 give the same map in $[X, X]_s$.

12. Compute the set of endomorphisms of $\Sigma^\infty_+ S^1 \simeq \mathbb{S}^0 \vee \mathbb{S}^1$ in the stable homotopy category, using that $\pi_0(\mathbb{S}) = \mathbb{Z}$, $\pi_1(\mathbb{S}) = \mathbb{Z}/2$, and $\pi_{-1}(\mathbb{S}) = 0$. Describe this set as a ring, with multiplication coming from composition.

13. If Y is an Ω-spectrum, prove that the map $F_n S^n \to F_0 S^0$ adjoint to the identity of S^n induces a bijection $[F_0 S^0, Y]_s \to [F_n S^n, Y]_s$.

14. Prove that there is a stable equivalence $F_k S^0 \simeq \Omega^k \mathbb{S}$. (Hint: use Section 2.7, exercise 19 and the fact that Σ is an equivalence of categories to argue that Ω is equivalent to sh^{-1}.)

15. (a) Recall the $\lim^1$ exact sequence for spaces from Section 1.5, equation (1.5.23):

$$0 \longrightarrow \lim_{n\to\infty}{}^1 \pi_{k+1}(X_n) \longrightarrow \pi_k\left(\operatorname*{holim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty} \pi_k(X_n) \longrightarrow 0.$$

Use this to prove the $\lim^1$ exact sequence for spectra (Proposition 3.2.20).

(b) Prove that if Y is the homotopy limit of a sequence of Ω-spectra

$$Y = \lim_{n\to\infty} Y^{(n)} \longrightarrow \ldots \longrightarrow Y^{(1)} \longrightarrow Y^{(0)} \longrightarrow *,$$

and X is another spectrum, then there is a short exact sequence

$$0 \longrightarrow \lim{}^1 [\Sigma X, Y^{(n)}]_s \longrightarrow [X, Y]_s \longrightarrow \lim [X, Y^{(n)}]_s \longrightarrow 0.$$

16. Use the $\lim^1$ sequence from Proposition 3.2.20 and the isomorphism $[X, \Sigma^k E]_s \cong E^k(X)$ from Example 3.2.16 to show that for any extraordinary cohomology theory E and any sequence of spaces X_n, we have a $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{n\to\infty}{}^1 E^{k-1}(X_n) \longrightarrow E^k\left(\operatorname*{hocolim}_{n\to\infty} X_n\right) \longrightarrow \lim_{n\to\infty} E^k(X_n) \longrightarrow 0.$$

Note the similarity to (1.5.6) for ordinary homology, and the difference with (3.2.21), since the X_n simply form a diagram of spaces, not a spectrum.

17. Prove that the Postnikov tower of a spectrum X is unique up to canonical isomorphism as a diagram in the stable homotopy category. (Hint: Use Lemma 3.2.12 repeatedly.) This also implies that connective covers of X are unique up to equivalence.

18. (a) If $\mathbf{C}$ and $\mathbf{D}$ are categories with weak equivalences, prove that

$$\mathrm{Ho}(\mathbf{C} \times \mathbf{D}) \cong (\mathrm{Ho}\,\mathbf{C}) \times (\mathrm{Ho}\,\mathbf{D}).$$

For instance, you might prove this by showing that $(\mathrm{Ho}\,\mathbf{C}) \times (\mathrm{Ho}\,\mathbf{D})$ has the correct universal property.

(b) Generalize to products of an arbitrary number of categories.

19. Recall that the opposite category $\mathbf{C}^{\mathrm{op}}$ has the same objects as $\mathbf{C}$, but the morphisms are reversed, $\mathbf{C}^{\mathrm{op}}(X, Y) = \mathbf{C}(Y, X)$. Prove that the homotopy category commutes with the opposite category,

$$\mathrm{Ho}(\mathbf{C}^{\mathrm{op}}) \cong (\mathrm{Ho}\,\mathbf{C})^{\mathrm{op}}.$$

20. Prove that every morphism in the stable homotopy category has a well-defined homotopy cofiber. In other words, any two maps of spectra $f: X \to Y$ and $g: W \to Z$, whose images in $\mathrm{Ho}\,\mathbf{Sp}$ are isomorphic to each other, have stably equivalent cofibers, $Cf \simeq Cg$.

21. Prove that $\Sigma(-)$ does not preserve all weak equivalences of based spaces. (Hint: Consider the subspace $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. Its suspension is an infinite shrinking wedge of circles.)

22. Complete the list of left-derived functors from Proposition 3.3.25:

    (a) Prove that the free spectrum functor $F_n : \mathbf{Top}_* \to \mathbf{Sp}$ can be left-derived.

    (b) Prove that the handicrafted smash product $\wedge : \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$ can be left-derived.

23. (a) Extend Example 3.3.26 to based spaces. Show that the pushout operation

$$\mathbf{Top}_*^{\{\bullet \leftarrow \bullet \to \bullet\}} \to \mathbf{Top}_*$$

    can be left-deformed. Its left-derived functor is obtained by *first* making the spaces A, X, and Y well-based, *then* taking the homotopy pushout.

    (b) Extend Example 3.3.26 to spectra. Show that the pushout operation

$$\mathbf{Sp}^{\{\bullet \leftarrow \bullet \to \bullet\}} \to \mathbf{Sp}$$

    can be left-deformed. Its left-derived functor is the homotopy pushout. (No assumptions are needed on the basepoints.)

24. (a) As in Example 3.3.26, show that the sequential colimit functor

$$\operatorname*{colim}_{n \to \infty} : \mathbf{Top}^{\{\bullet \to \bullet \to \ldots\}} \to \mathbf{Top}$$

    is left-deformable, and its left-derived functor is the mapping telescope of Definition 1.5.4.

    (b) As in exercise 23, extend this to based spaces and spectra. No assumptions are needed on the basepoints.

25. (a) Extend Example 3.3.34 to based spaces and spectra. Show that the pullback is right-deformable, and its right-derived functor is the homotopy pullback. No assumptions are needed on the basepoints.

    (b) Show that the sequential limit functor

$$\lim_{n \to \infty} : \mathbf{Top}^{\{\ldots \to \bullet \to \bullet\}} \to \mathbf{Top}$$

    is right-deformable, and its right-derived functor is the mapping telescope of Definition 1.5.21.

(c) The previous part works equally well for based spaces. However, for spectra, show that the right-derived functor of

$$\lim_{n\to\infty} : \mathbf{Sp}^{\{\dots\to\bullet\to\bullet\}} \to \mathbf{Sp}$$

is obtained by *first* making the spectra $X(i)$ into Ω-spectra, *then* taking the homotopy limit at each spectrum level. We call this the "homotopy limit in spectra" – if we don't take Ω-spectra first, it is not the true homotopy limit.

26. We lift Section 2.7, exercise 34 to the homotopy category.

(a) Prove that the left-derived suspension spectrum functor $\mathbb{L}\Sigma^\infty \colon \operatorname{Ho}\mathbf{Top}_* \to \operatorname{Ho}\mathbf{Sp}$ sends cofiber sequences of spaces to cofiber/fiber sequences of spectra.

(b) Prove that the right-derived evaluation functor $\Omega^\infty = \mathbb{R}\mathrm{ev}_0 \colon \operatorname{Ho}\mathbf{Sp} \to \operatorname{Ho}\mathbf{Top}_*$ sends cofiber/fiber sequences of spectra to fiber sequences of spaces.

27. Give an example of a functor F on a category with weak equivalences satisfying 2 out of 3, such that:

(a) F has a left-derived functor but no right-derived functor. (With proof.)

(b) F has neither a left- nor a right-derived functor.

(c) F has a left- and a right-derived functor, and the composite $\mathbb{L}F \to F \to \mathbb{R}F$ is not an equivalence.

(d) F has a left- and a right-derived functor, and the composite $\mathbb{L}F \to F \to \mathbb{R}F$ is an equivalence, but F is not homotopical.

(Hint: Make the target category the category of spaces, and make the source category as small as possible.)

28. Let $\mathbf{C}$ be a category and W a class of morphisms in $\mathbf{C}$. We say W satisfies the **2 out of 6** property if for any three composable maps f, g, h, if $g \circ f$ and $h \circ g$ are in W as shown below, then f, g, h, and $h \circ g \circ f$ are also in W.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "\sim"'] & B \arrow[d, "g"] \arrow[dr, "\sim"] & \\
& C \arrow[r, "h"] & D
\end{tikzcd}$$

(a) Prove that 2-out-of-6 implies 2-out-of-3. So it is a stronger condition.

(b) As in Example 3.3.14, prove that if $W = F^{-1}(\text{isomorphisms})$ for some functor F, then W has 2 out of 6. Conclude that the class of stable equivalences in $\mathbf{Sp}$ has 2 out of 6.

(c) Show that any class of weak equivalences W can be enlarged to have 2 out of 6, without changing the resulting homotopy category $\mathbf{C}[W^{-1}]$.

29. Suppose F has a left-derived functor. Adapt the proof of Proposition 3.4.2 to show that $\mathbb{L}F$ is terminal among functors $G\colon \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}$ with a natural transformation $G \Rightarrow F$ as functors $\mathbf{C} \to \operatorname{Ho}\mathbf{D}$. In other words, given any other such G, there is a unique natural transformation $G \Rightarrow \mathbb{L}F$ on $\operatorname{Ho}\mathbf{C}$, that on $\mathbf{C}$ commutes with the given natural transformations to F. We also say that $\mathbb{L}F$ is the *right Kan extension* of F along $\delta\colon \mathbf{C} \to \operatorname{Ho}\mathbf{C}$.

30. Prove that the evaluation functor and underlying unbased space

$$\mathbf{Sp} \xrightarrow{\mathrm{ev}_n} \mathbf{Top}_* \xrightarrow{U} \mathbf{Top}$$

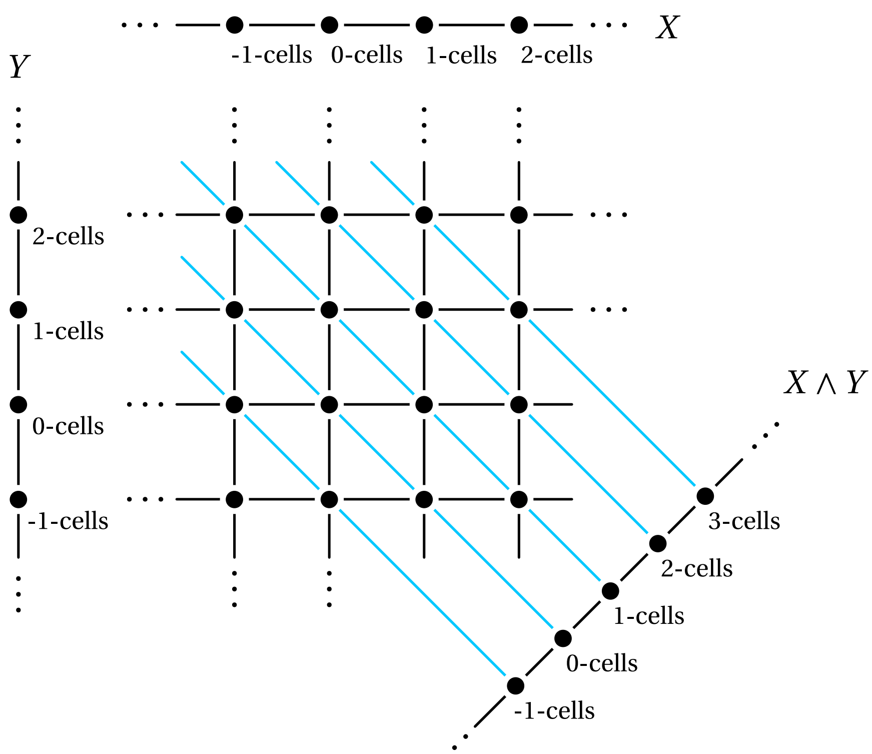
satisfy the hypotheses of Lemma 3.4.13, and therefore their right-derived functors compose, as in Example 3.4.15.

31. Define rationalization of X as the hocolim of $X \xrightarrow{2} X \xrightarrow{3} X \xrightarrow{4} X \ldots$, where n denotes the identity map of X added to itself n times. Prove it rationalizes the homotopy groups. Same for localization at p, just eliminate all multiples of p from the hocolim. Prove it coincides with smashing with the localized sphere. Show $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$ by uniqueness of Eilenberg-Maclane spectra (Section 2.7, 38). The same is not true for p-localization.

Chapter 4

Properties of the smash product

The smash product of spectra $\wedge$ plays the role of the tensor product $\otimes$ in higher algebra. The fundamental intuition about $X \wedge Y$ is that it has a stable $(m+n)$-cell for each choice of a stable m-cell of X and a stable n-cell of Y:



This is enough to get a feel for what $X \wedge Y$ looks like, at least up to stable equivalence.

We defined the smash product already in Definition 2.3.23, by picking a sequence of values of p and q and defining $(X \wedge Y)_{p+q} = X_p \wedge Y_q$. Morally, this ought to make spectra into a **symmetric monoidal category**, meaning we have isomorphisms

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z), \qquad X \wedge Y \cong Y \wedge X, \qquad X \wedge \mathbb{S} \cong X.$$

The problem is, the construction we gave depends on choices, and different choices give spectra that are not even isomorphic, only equivalent. As a result, it is impossible to prove things like $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ in the category of spectra **Sp**, only on the homotopy category Ho **Sp**.

In Chapter 6 we will give an explicit solution to this problem, modifying the category of spectra **Sp** to the category of **symmetric spectra** $\mathbf{Sp}^{\Sigma}$, and making $\mathbf{Sp}^{\Sigma}$ into a symmetric monoidal category. Unfortunately, doing this in detail requires a long detour through the theory of model categories, so that we can show that symmetric spectra are equivalent to sequential spectra, and prove that the smash product on $\mathbf{Sp}^{\Sigma}$ preserves stable equivalences.

It's important not to get lost in the theoretical machinery, and to remember what's important, why we care about the smash product. So in this chapter we give the propaganda for $\wedge$. We treat the smash product of spectra as a black box, assuming it exists, and showing what you can do with it: define rings and modules, and take duals and traces.

We also open up the box a little bit and define the smash product explicitly in the **Spanier-Whitehead category**, the subcategory of the homotopy category of spectra Ho **Sp** consisting of only the finite spectra. Even here the definition is a little complicated, signaling the difficulties that lie ahead in defining the smash product on the entire category of spectra **Sp**.

We also show how to lift the Poincaré duality theorem to a theorem about spectra called Atiyah duality, and generalize it to a result called Spanier-Whitehead duality. We get as a corollary a Poincaré duality theorem that holds with coefficients in any extraordinary homology theory E, and a topological proof of the Lefschetz fixed point theorem.

4.1 Symmetric monoidal categories

The most important thing to know about the smash product is that it makes the stable homotopy category Ho **Sp** into a symmetric monoidal category. We begin by explaining what that means, and giving other examples, to help develop intuition for the case of spectra.

4.1.1 Definition of a symmetric monoidal category

Definition 4.1.1. A **symmetric monoidal category** is a category $\mathbf{C}$, a functor

$$\otimes \colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C},$$

a unit object $I \in \operatorname{ob} \mathbf{C}$, and natural isomorphisms

$$\alpha \colon (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \qquad \gamma \colon X \otimes Y \cong Y \otimes X, \qquad \rho \colon X \otimes I \cong X,$$

that are coherent. We usually drop the notation for these isomorphisms and just say "the canonical isomorphism" – they tend to be obvious in most examples.

Example 4.1.2. We list several symmetric monoidal categories, with their tensor and unit object.

- $\mathbf{Set}, \amalg, \emptyset$
- $\mathbf{Set}, \times, *$
- $\mathbf{Top}, \amalg, \emptyset$
- $\mathbf{Top}, \times, *$
- $\mathbf{Top}_*, \vee, *$
- $\mathbf{Top}_*, \times, *$
- $\mathbf{Top}_*, \wedge, S^0$
- $\mathbf{Sp}, \vee, *$
- $\mathbf{Ab}, \oplus, 0$
- $\mathbf{Ab}, \otimes, \mathbb{Z}$
- $\mathbf{Mod}_k, \oplus, 0$
- $\mathbf{Mod}_k, \otimes_k, k$
- $\mathbf{GrMod}_k, \otimes_k, k[0]$
- $\mathbf{Ch}_{\geq 0}(k), \otimes_k, k[0]$
- $\mathbf{Ch}(k), \otimes_k, k[0]$
- $\mathbf{Ch}(k), \oplus, 0$
- $\operatorname{Ho} \mathbf{Top}, \amalg, \emptyset$
- $\operatorname{Ho} \mathbf{Top}, \times, *$
- $\operatorname{Ho} \mathbf{Top}_*, \vee^{\mathbb{L}}, *$
- $\operatorname{Ho} \mathbf{Top}_*, \times, *$
- $\operatorname{Ho} \mathbf{Top}_*, \wedge^{\mathbb{L}}, S^0$
- $\mathscr{D}(k)_{\geq 0}, \otimes_k^{\mathbb{L}}, k[0]$
- $\mathscr{D}(k), \otimes_k^{\mathbb{L}}, k[0]$
- $\mathscr{D}(k), \oplus, 0$

Here k is any field or commutative ring, and $k[0]$ refers to the chain complex or graded module that has just a k in degree 0. The category $\mathbf{GrMod}_k$ means graded modules, and $\mathscr{D}(k)$ is the derived category, in other words the homotopy category of chain complexes of k-modules $\operatorname{Ho} \mathbf{Ch}(k)$.

So for instance, in $\mathbf{Ab}$ we have the isomorphism $X \otimes \mathbb{Z} \cong X$, while in $\mathbf{Top}_*$ we have the isomorphism $X \wedge S^0 \cong X$. In this sense the 0-sphere S^0 is like the integers $\mathbb{Z}$, because each one is an identity element for its respective product.

Notice that the same category can have several different symmetric monoidal structures. A symmetric monoidal category is a category with an extra *structure* placed on top of it, not just a *property* of the category.

Remark 4.1.3. The choice of isomorphisms in Definition 4.1.1 are also part of the structure – we could keep the same product, but change the *isomorphisms*, and get a different structure! For instance, in graded k-modules $\mathbf{GrMod}_k$, we could define the symmetry isomorphism $X \otimes Y \cong Y \otimes X$ by sending each tensor of homogeneous terms $a \otimes b$ to either $b \otimes a$, or to $(-1)^{|a||b|}(b \otimes a)$. The first choice is the most obvious isomorphism, but the second choice is the one that obeys the Koszul sign rule.

When we pass to chain complexes $\mathbf{Ch}$, we have to make the second choice here and send $a \otimes b$ to $(-1)^{|a||b|}(b \otimes a)$. If we didn't, the isomorphism $X \otimes Y \cong Y \otimes X$ would fail to be a map of chain complexes! See exercise 1.

In Definition 4.1.1, we should explain what we mean by "coherent." If we take any string of distinct objects $X_1, \ldots, X_n$ and apply strings of the above isomorphisms to rearrange the objects, add and remove units, and regroup parentheses, if we ever come back to the same expression, the composite map must be the identity. For example, the following composites have to be the identity:

$$\begin{tikzcd}
 & & (X \otimes (Y \otimes Z)) \otimes W \arrow[dll, leftrightarrow, "\cong", "\alpha"'] \arrow[drr, leftrightarrow, "\cong", "\alpha"'] & & \\
((X \otimes Y) \otimes Z) \otimes W \arrow[dr, leftrightarrow, "\cong", "\alpha"'] & & & & X \otimes ((Y \otimes Z) \otimes W) \arrow[dl, leftrightarrow, "\cong"', "\alpha"] \\
 & (X \otimes Y) \otimes (Z \otimes W) \arrow[rr, leftrightarrow, "\cong", "\alpha"'] & & X \otimes (Y \otimes (Z \otimes W)) &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[r, leftrightarrow, "\cong", "\alpha"'] \arrow[d, leftrightarrow, "\rho"', "\cong"] & X \otimes (I \otimes Y) \arrow[d, leftrightarrow, "\gamma"', "\cong"] \\
X \otimes Y \arrow[r, leftrightarrow, "\cong", "\rho"'] & X \otimes (Y \otimes I)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, leftrightarrow, "\cong", "\gamma"'] \arrow[d, leftrightarrow, "\alpha"', "\cong"] & (Y \otimes X) \otimes Z \arrow[r, leftrightarrow, "\cong", "\alpha"'] & Y \otimes (X \otimes Z) \arrow[d, leftrightarrow, "\gamma"', "\cong"] \\
X \otimes (Y \otimes Z) \arrow[r, leftrightarrow, "\cong", "\gamma"'] & (Y \otimes Z) \otimes X \arrow[r, leftrightarrow, "\cong", "\alpha"'] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\cong", "\gamma"'] \arrow[rr, bend right, "\mathrm{id}"'] & Y \otimes X \arrow[r, "\cong", "\gamma"'] & X \otimes Y
\end{tikzcd}$$

In fact, these diagrams suffice. As soon as they all commute, any other diagram of the same form must commute.

Theorem 4.1.4 (MacLane)**.** *If the above diagrams commute, then all similar diagrams commute.*

In practical terms, coherence means that we are allowed to drop parentheses and just write $X \otimes Y \otimes Z$, and similarly we don't have a choose an ordering when we multiply distinct terms together:

$$X \otimes Y \otimes Z \cong Z \otimes X \otimes Y.$$

We *do* have to keep track of order when we have repeated terms. The swap map $\gamma\colon X\otimes X\cong X\otimes X$ is often not the identity. In **Top**, $\times$ for instance, it's the self-map of the space $X\times X$ that switches the two coordinates. In $\mathbf{Ch}_{\geq 0}(\mathbb{Z}),\otimes$, it switches the tensors around and applies minus signs whenever we switch terms of odd degree.

In general, we therefore have to specify what permutation we apply to go between two different orderings of a given expression. (But, we don't have to explain how to get that permutation from instances of the map γ.)

Example 4.1.5. If **C** is a category with all finite products $\times$ and terminal object $*$, then $(\mathbf{C},\times,*)$ is a symmetric monoidal category in a canonical way (exercise 2). We say **C** is **cartesian monoidal** in this case. For instance, **Set**, **Top**, $\mathbf{Top}_*$, and **Ab** are all cartesian monoidal.

Dually, if **C** has all finite coproducts and an initial object $\emptyset$, then $(\mathbf{C},\amalg,\emptyset)$ is a symmetric monoidal category – in this case **C** is **cocartesian monoidal**.

Remark 4.1.6. Often a category **C** has two symmetric monoidal structures $\oplus$ and $\otimes$, and the tensor distributes over the direct sum:

$$A\otimes(B_1\oplus B_2)\cong(A\otimes B_1)\oplus(A\otimes B_2).$$

For instance, if **C** has coproducts and $\otimes$ preserves coproducts in each variable, then we get isomorphisms as above. We'll frequently encounter the situation that **C** is an **additive category** with direct sum $\oplus$ (see Theorem 3.2.5), and also a symmetric monoidal category under some tensor product $\otimes$. In this situation, we ask that the tensor preserves finite sums in each variable, so that we get isomorphisms as above. See exercise 5 for more details.

Several of the examples in Example 4.1.2 are homotopy categories. To explain how this works, suppose $(\mathbf{C},\otimes,I)$ is a symmetric monoidal category, and **C** has a class of weak equivalences W satisfying 2 out of 3 (Definition 3.3.13), so that we can form the homotopy category $\operatorname{Ho}\mathbf{C}=\mathbf{C}[W^{-1}]$.

Lemma 4.1.7. *Suppose there is a full subcategory* $\mathbf{A}\subseteq\mathbf{C}$ *such that*

- $\otimes$ *preserves weak equivalences between objects in* **A**,
- *if* $X,Y\in\mathbf{A}$ *then* $X\otimes Y\in\mathbf{A}$,
- *the unit* I *is also in* **A**,[1]
- *there is a functor* $Q\colon\mathbf{C}\to\mathbf{C}$ *landing in* **A**, *and*

[1]It's actually enough if the map $QI\xrightarrow{\sim}I$ induces an equivalence $QI\otimes QX\xrightarrow{\sim}I\otimes QX\cong QX$.

- *there is a natural weak equivalence* $q\colon QX \xrightarrow{\sim} X$.

Then the left-derived tensor $X \otimes^{\mathbb{L}} Y = QX \otimes QY$ *and the unit* $I \in \mathbf{C}$ *make the homotopy category* $\operatorname{Ho}\mathbf{C}$ *into a symmetric monoidal category.*

For instance, these conditions are satisfied in $(\mathbf{Top}_*, \wedge, S^0)$, where $\mathbf{A}$ is the subcategory of based CW complexes. So $(\operatorname{Ho}\mathbf{Top}_*, \wedge^{\mathbb{L}}, S^0)$ is a symmetric monoidal category.

These conditions are also satisfied in $(\mathbf{Ch}_{\geq 0}(\mathbb{Z}), \otimes, \mathbb{Z}[0])$, where $\mathbf{A}$ is the subcategory of levelwise projective chain complexes. So $(\mathscr{D}_{\geq 0}(\mathbb{Z}), \otimes^{\mathbb{L}}, \mathbb{Z}[0])$ is also a symmetric monoidal category.

Proof. As in the proof of Lemma 3.3.18, the left-derived tensor $X \otimes^{\mathbb{L}} Y = QX \otimes QY$ preserves equivalences, because Q preserves equivalences and $\otimes$ preserves equivalences between objects in $\mathbf{A}$. Therefore it defines a functor

$$\otimes^{\mathbb{L}}\colon \operatorname{Ho}\mathbf{C} \times \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{C}.$$

We define the associativity isomorphism by removing extra Qs and then using the associativity isomorphism for $\otimes$:

$$\begin{tikzcd}
(X \otimes^{\mathbb{L}} Y) \otimes^{\mathbb{L}} Z \arrow[r, equal] & Q(QX \otimes QY) \otimes QZ \arrow[d, "q\otimes 1", "\sim"'] & QX \otimes Q(QY \otimes QZ) \arrow[r, equal] \arrow[d, "1\otimes q", "\sim"'] & X \otimes^{\mathbb{L}} (Y \otimes^{\mathbb{L}} Z) \\
& (QX \otimes QY) \otimes QZ \arrow[r, leftrightarrow, "\cong"] & QX \otimes (QY \otimes QZ) &
\end{tikzcd}$$

The vertical maps are equivalences because q is an equivalence between objects in $\mathbf{A}$, so $\otimes$ sends this to an equivalence. Therefore it is an isomorphism in the homotopy category $\operatorname{Ho}\mathbf{C}$. The composite of these maps is therefore an isomorphism in the homotopy category. (See also Example 3.4.18, which constructs the same isomorphism in a different way.)

We construct the other two isomorphisms in a similar way. It is an exercise to see that their coherence follows from the coherence of the same isomorphisms in $\mathbf{C}$. $\square$

Remark 4.1.8. On the equivalent subcategory $\operatorname{Ho}\mathbf{A} \subseteq \operatorname{Ho}\mathbf{C}$, this symmetric monoidal structure is isomorphic to a simpler one: just use the product $X \otimes Y$. It passes to the homotopy category $\operatorname{Ho}\mathbf{A}$ because it preserves all equivalences. The natural isomorphism $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ in $\mathbf{A}$ also gives a natural isomorphism in $\operatorname{Ho}\mathbf{A}$, and similarly for the other isomorphisms.

All of the examples in the right-hand column of Example 4.1.2 arise this way. For instance, the homotopy category of based spaces $\operatorname{Ho}\mathbf{Top}_*$ is symmetric monoidal under the left-derived smash product $X \wedge^{\mathbb{L}} Y = QX \wedge QY$, where QX is a cell complex equivalent to X. If we restrict attention to cell complexes, or well-based spaces, we can just use the smash product $X \wedge Y$ instead of the left-derived smash product.

Example 4.1.9 (Smash product as a black box, part 1)**.** As we've already discussed, in this chapter, we develop the smash product of spectra as a black box. Here's what's written on the label of that box:

- The category of spectra **Sp** is a symmetric monoidal category, under some operation called the smash product $\wedge$, that we will not define explicitly.
- The unit is the sphere spectrum $\mathbb{S}$. More generally, smashing with any suspension spectrum $\Sigma^\infty K$ gives the tensor from Definition 2.3.6:
$$(\Sigma^\infty K) \wedge X \cong K \wedge X.$$
In particular, the smash product of two suspension spectra is a suspension spectrum,
$$(\Sigma^\infty A) \wedge (\Sigma^\infty B) \cong \Sigma^\infty (A \wedge B).$$
These claims are stronger than those in Example 2.3.25 because we are claiming *isomorphisms*, not just stable equivalences.
- The smash product preserves colimits in each variable. It follows that the smash product preserves wedge sums, homotopy pushouts, and mapping telescopes in each variable as well (exercise 19).
- We have isomorphisms
$$F_d A \wedge F_e B \cong F_{d+e}(A \wedge B).$$
It follows that the smash product of cellular spectra is a cellular spectrum (exercise 20).
- Not only does the smash product preserve cellular spectra, it also preserves all stable equivalences between them. So the hypotheses of Lemma 4.1.7 are satisfied, making Ho **Sp** into a symmetric monoidal category as well. The unit is again $\mathbb{S}$, and the product is the left-derived smash product
$$X \wedge^{\mathbb{L}} Y = QX \wedge QY,$$
where Q is any replacement by a cellular spectrum, for instance the one in Theorem 2.6.12. We also get that the smash product preserves cofiber/fiber sequences in each variable (exercise 21).

In summary, we get two symmetric monoidal categories $(\mathbf{Sp}, \wedge, \mathbb{S})$ and $(\mathrm{Ho}\,\mathbf{Sp}, \wedge^{\mathbb{L}}, \mathbb{S})$.

The fact that the smash product preserves cofiber sequences in each slot is often captured by saying that Ho **Sp** is a **tensor triangulated category**. See [HPS97] for more details on this point of view.

Remark 4.1.10. This black box is a little bit of a lie, because there is no such smash product functor $\wedge$, if we interpret **Sp** to mean the category of (sequential) spectra. It becomes correct when we change definitions and use **Sp** to refer instead to the category of symmetric spectra or orthogonal spectra.

4.1.2 Rings and modules

Symmetric monoidal categories are a setting in which we can talk abstractly about rings and modules over them.

Definition 4.1.11. Let $\mathbf{C}, \otimes, I$ be a symmetric monoidal category. A **monoid** (or **ring**) in $\mathbf{C}$ is an object R together with a multiplication map and a unit map

$$\mu\colon R\otimes R \longrightarrow R, \qquad \eta\colon I \longrightarrow R,$$

such that the following two diagrams commute.

$$\begin{tikzcd}
R\otimes R\otimes R \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & R\otimes R \arrow[d, "\mu"] \\
R\otimes R \arrow[r, "\mu"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes R \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & R\otimes R \arrow[d, "\mu"] & R\otimes I \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\
& R. &
\end{tikzcd}$$

You should think of these as capturing the idea that $(ab)c = a(bc)$, and $1a = a = a1$. A **left module** in over R is another object M together with an action map

$$\alpha\colon R\otimes M \longrightarrow M,$$

such that the following two diagrams commute.

$$\begin{tikzcd}
R\otimes R\otimes M \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\alpha"'] & R\otimes M \arrow[d, "\mu"] \\
R\otimes M \arrow[r, "\mu"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes M \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & R\otimes M \arrow[d, "\alpha"] \\
& M.
\end{tikzcd}$$

Again, this captures the idea that $(ab)m = a(bm)$, and $1m = m$. A **right module** is defined the same way, except that the R is on the right, so we get $m(ab) = (ma)b$ instead of $(mb)a$.

We typically only consider these in the "multiplicative" examples of symmetric monoidal categories, because in the "additive" examples these concepts are rather degenerate; see exercise 8.

Example 4.1.12. We consider monoids and modules in several examples.

- In **Set**, $\times, *$, a monoid object is the same thing as a monoid G. A left module over G is the same thing as a set with left G-action (Definition 5.3.8).
- In **Ab**, $\otimes, \mathbb{Z}$, a monoid object is the same thing as a ring R. The fact that the multiplication is a map $R \otimes R \to R$ makes it bilinear in each slot, so that the axioms of a ring are satisfied. (This is a very concise way of remembering the axioms for a ring!) A left module over R is the same thing as a left module in the usual sense.
- In **Top**, $\times, *$, a monoid object is the same thing as a topological monoid G. A left module over G is the same thing as a space with a continuous left G-action (Definition 5.3.8).
- In $\mathbf{Top}_*, \wedge, S^0$, a monoid object is the same thing as a based topological monoid G. These are spaces with associative multiplication, and two special points, the basepoint 0 and the unit point 1. We have $0 * a = 0$ and $1 * a = a$ for every $a \in G$.
- In $\mathbf{Mod}_k, \otimes_k, k$, a monoid is the same thing as a k-algebra.
- In $\mathbf{GrMod}_{\mathbb{Z}}, \otimes, \mathbb{Z}[0]$, a monoid is the same thing as a graded ring, and a module is a graded module.
- In $\mathbf{Ch}_{\geq 0}(\mathbb{Z}), \otimes, \mathbb{Z}[0]$, a monoid is the same thing as a differential graded algebra (DGA).
- In Ho **Top**, $\times, *$, a monoid is the same thing as an associative H-space – a space that has a multiplication $X \times X \to X$ that is associative and unital up to homotopy.

Definition 4.1.13. A monoid R in **C** is **commutative** if the following diagram commutes.

$$\begin{tikzcd}
R \otimes R \arrow[rr, leftrightarrow, "\cong", "\gamma"'] \arrow[dr, "\mu"'] & & R \otimes R \arrow[dl, "\mu"] \\
& R &
\end{tikzcd}$$

Lemma 4.1.14. *If R is commutative then every left module over R is also a right module and vice-versa.*

Example 4.1.15.

- In **Set**, $\times, *$, this gives commutative monoids.
- In **Ab**, $\otimes, \mathbb{Z}$, this gives commutative rings.
- In $\mathbf{GrMod}_{\mathbb{Z}}, \otimes, \mathbb{Z}[0]$, this gives commutative graded rings. Whether the commutativity is $ab = ba$ or $ab = (-1)^{|a||b|}(ba)$ depends on which symmetric monoidal structure we choose.
- In $\mathbf{Ch}_{\geq 0}(k), \otimes, k[0]$, this gives commutative differential graded algebras (CDGAs). Again, we have to use the structure in which $ab = (-1)^{|a||b|}(ba)$, otherwise the ring multiplication doesn't give a map of chain complexes.

Definition 4.1.16. A **ring spectrum** is a monoid object in $(\mathbf{Sp}, \wedge, \mathbb{S})$. So it is a spectrum R with multiplication and unit maps

$$\mu\colon R \wedge R \longrightarrow R, \qquad \eta\colon \mathbb{S} \longrightarrow R,$$

such that the diagrams in Definition 4.1.11 commute. A **commutative ring spectrum** is a commutative monoid object in $(\mathbf{Sp}, \wedge, \mathbb{S})$, and a **module spectrum** is a module object over R.

Remark 4.1.17. It used to be that the term "ring spectrum" referred to a monoid object in the homotopy category $(\operatorname{Ho}\mathbf{Sp}, \wedge^{\mathbb{L}}, \mathbb{S})$, and a "highly structured ring spectrum" was a monoid in $(\mathbf{Sp}, \wedge, \mathbb{S})$. However, this terminology has fallen out of favor. Monoids in spectra turn out to be way more useful than monoids in the homotopy category, so we don't spend much time talking about monoids in the homotopy category anymore.

4.1.3 Symmetric monoidal functors

A symmetric monoidal functor is a map of symmetric monoidal categories that respects the tensor product. There are different levels to this:

Definition 4.1.18. Suppose $(\mathbf{C}, \otimes_{\mathbf{C}}, I_{\mathbf{C}})$ and $(\mathbf{D}, \otimes_{\mathbf{D}}, I_{\mathbf{D}})$ are two symmetric monoidal categories. A **lax symmetric monoidal functor** is

- A functor $F\colon \mathbf{C} \to \mathbf{D}$,
- A natural map $m\colon F(X) \otimes_{\mathbf{D}} F(Y) \to F(X \otimes_{\mathbf{C}} Y)$, and
- a map $i\colon I_{\mathbf{D}} \to F(I_{\mathbf{C}})$,

that are coherent. A **strong symmetric monoidal functor** is the same thing except that the maps m and i are isomorphisms, i.e.

$$F(X) \otimes_{\mathbf{D}} F(Y) \cong F(X \otimes_{\mathbf{C}} Y), \qquad I_{\mathbf{D}} \cong F(I_{\mathbf{C}}).$$

Example 4.1.19. The forgetful functor $(\mathbf{Top}, \times, *) \to (\mathbf{Set}, \times, *)$ is strong symmetric monoidal. The product of spaces is the product on the underyling set.

Its left adjoint takes each set and gives it the discrete topology – this is also strong symmetric monoidal, because a product of discrete spaces is discrete.

Example 4.1.20. The forgetful functor $(\mathbf{Top}_*, \wedge, S^0) \to (\mathbf{Top}, \times, *)$ is lax symmetric monoidal. In other words, for based spaces X and Y, there is a canonical map $X \times Y \to X \wedge Y$, but it isn't an isomorphism.

Its left adjoint adds a disjoint basepoint $(-)_+$. This is strong symmetric monoidal, since

$$X_+ \wedge Y_+ \cong (X \times Y)_+, \qquad S^0 \cong (*)_+.$$

Example 4.1.21. The forgetful functor $(\mathbf{Ab}, \otimes, *) \to (\mathbf{Set}, \times, *)$ is lax symmetric monoidal. In other words, for abelian groups A and B, there is a canonical map of sets $A \times B \to A \otimes B$, but it isn't an isomorphism.

The left adjoint is the free abelian group functor $(\mathbf{Set}, \times, *) \to (\mathbf{Ab}, \otimes, *)$. It takes the set S to the free abelian group $\mathbb{Z}^{\oplus S} = \bigoplus_{s \in S} \mathbb{Z}$. This is strong symmetric monoidal, using the fact that tensor distributes over direct sum:

$$\mathbb{Z}^{\oplus S} \otimes \mathbb{Z}^{\oplus T} \cong \mathbb{Z}^{\oplus (S \times T)}, \qquad \mathbb{Z} \cong \mathbb{Z}^{\oplus \{*\}}.$$

Remark 4.1.22. There is a pattern to these examples: we have an adjoint pair $(F \dashv G)$ in which F is strong symmetric monoidal and G is at least lax symmetric monoidal. In fact, this is always true – if F is strong symmetric monoidal and has a right adjoint G, then G must be lax symmetric monoidal. See exercise 12.

Again, coherence means that if we take any expression $F(X_1) \otimes \cdots \otimes F(X_n)$ and apply composites of the above maps, any two such composites that end at the same expression, must be the same map. As for symmetric monoidal categories, it suffices to check that the following diagrams commute.

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "m"] \arrow[d, leftrightarrow, "\cong"', "\alpha"] & F(X \otimes Y) \otimes F(Z) \arrow[r, "m"] & F((X \otimes Y) \otimes Z) \arrow[d, leftrightarrow, "\cong"', "\alpha"] \\
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "m"] & F(X) \otimes F(Y \otimes Z) \arrow[r, "m"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \otimes I_{\mathbf{D}} \arrow[r, "i"] \arrow[d, leftrightarrow, "\cong"', "\rho"] & F(X) \otimes F(I_{\mathbf{C}}) \arrow[d, "m"] \\
F(X) \arrow[r, leftrightarrow, "\rho", "\cong"'] & F(X \otimes I_{\mathbf{C}})
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, leftrightarrow, "\gamma", "\cong"'] \arrow[d, "m"] & F(Y) \otimes F(X) \arrow[d, "m"] \\
F(X \otimes Y) \arrow[r, leftrightarrow, "\gamma", "\cong"'] & F(Y \otimes X)
\end{tikzcd}$$

For strong symmetric monoidal functors, you can remember this by saying that, up to canonical isomorphism, there's only one way to tensor together a bunch of objects in $\mathbf{C}$ and apply F to them. Whether you apply F to them first, then tensor, or whether you tensor them first, then apply F, even if you add units in the middle for some reason, everything is identified up to canonical isomorphism.

Proposition 4.1.23. *If $F : \mathbf{C} \to \mathbf{D}$ is lax symmetric monoidal, and R is a monoid in $\mathbf{C}$, then $F(R)$ is a monoid in $\mathbf{D}$.*

Similarly, F takes modules over R to modules over $F(R)$, and takes commutative monoids to commutative monoids.

Proof. This is fairly easy – the multiplication on $F(R)$ is defined to be

$$F(R) \otimes_{\mathbf{D}} F(R) \xrightarrow{m} F(R \otimes_{\mathbf{C}} R) \xrightarrow{F(\mu)} F(R),$$

and the unit map is

$$I_{\mathbf{D}} \xrightarrow{i} F(I_{\mathbf{C}}) \xrightarrow{F(\eta)} F(R).$$

The required diagrams commute by coherence and the fact that they commuted in **C**. See exercise 15. The case of modules and commutative monoids is similar. □

Example 4.1.24.
- The forgetful functor $(\mathbf{Top}, \times, *) \to (\mathbf{Set}, \times, *)$ takes topological monoids to ordinary monoids.
- By the adjunction between $(\mathbf{Top}_*, \wedge, S^0)$ and $(\mathbf{Top}, \times, *)$, every monoid in the based sense is also a monoid in the unbased sense, and if G is a monoid in the unbased sense then G_+ is a monoid in the based sense.
- The free abelian group functor $(\mathbf{Set}, \times, *) \to (\mathbf{Ab}, \otimes, *)$ takes each monoid or group G to the ring $\mathbb{Z}[G]$, often called the **monoid ring** or **group ring**.

Example 4.1.25. Homology is lax symmetric monoidal, either as a functor

$$(\mathbf{Top}, \times, *) \to (\mathbf{GrAb}, \otimes, \mathbb{Z}[0])$$

or as a functor

$$(\mathbf{Ch}(\mathbb{Z}), \otimes, \mathbb{Z}[0]) \to (\mathbf{GrAb}, \otimes, \mathbb{Z}[0]).$$

In particular, we get maps $H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$, that aren't isomorphisms in general (Theorem 1.3.9). Therefore, if X is a topological monoid, this makes $H_*(X)$ into a graded ring. This product is called the **Pontryagin product**.

Example 4.1.26. For any symmetric monoidal category **C** satisfying the assumptions of Lemma 4.1.7, the map to the homotopy category $\mathbf{C} \to \operatorname{Ho}\mathbf{C}$ is lax symmetric monoidal. Specifically, the maps $q\colon QX \to X$ give the natural map

$$X \otimes^{\mathbb{L}} Y = QX \otimes QY \longrightarrow X \otimes Y.$$

Therefore, every monoid in **C** becomes a monoid in the homotopy category $\operatorname{Ho}\mathbf{C}$.

In particular, every topological monoid in **Top** is also an associative H-space (a monoid in $\operatorname{Ho}\mathbf{Top}$). Every ring spectrum in **Sp** is also a monoid in the stable homotopy category $\operatorname{Ho}\mathbf{Sp}$.

Example 4.1.27 (Smash product as a black box, part 2)**.** We extend the black box from Example 4.1.9 by claiming that the following three functors are symmetric monoidal.

- First we consider the suspension spectrum functor, either from based or unbased spaces:

$$\begin{tikzcd} (\mathbf{Top}, \times, *) \arrow[r, "(-)_+"] \arrow[rr, bend right, "\Sigma^\infty_+"'] & (\mathbf{Top}, \times, *) \arrow[r, "\Sigma^\infty"] & (\mathbf{Sp}, \wedge, \mathbb{S}). \end{tikzcd}$$

We claim that in both cases this functor is strong symmetric monoidal. In particular, we have isomorphisms

$$(\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y) \cong \Sigma^\infty_+ (X \times Y),^{2} \qquad \mathbb{S} \cong \Sigma^\infty_+(*).$$

If G is a group or monoid, the suspension spectrum $\Sigma^\infty_+ G$ is therefore a ring spectrum. This is sometimes called the **spherical group ring** and denoted $\mathbb{S}[G]$ to emphasize the analogy with $\mathbb{Z}[G]$. Concretely, the multiplication is by

$$(\Sigma^\infty_+ G) \wedge (\Sigma^\infty_+ G) \cong \Sigma^\infty_+ (G \times G) \xrightarrow{\Sigma^\infty_+(\mu)} \Sigma^\infty_+ G$$

and the unit is by

$$\mathbb{S} \cong \Sigma^\infty_+(*) \xrightarrow{\Sigma^\infty_+(\eta)} \Sigma^\infty_+ G.$$

If G is commutative, then $\Sigma^\infty_+ G$ is a commutative ring spectrum.

- Next, we claim that the Eilenberg-Maclane spectrum

$$(\mathbf{Ab}, \otimes, \mathbb{Z}) \xrightarrow{H(-)} (\mathbf{Sp}, \wedge, \mathbb{S})$$

is lax symmetric monoidal. So, there are natural maps

$$HA \wedge HB \longrightarrow H(A \otimes B), \qquad \mathbb{S} \longrightarrow H\mathbb{Z}.$$

Once we believe this claim, Proposition 4.1.23 tells us that for each ring R, its Eilenberg-Maclane spectrum HR is a ring spectrum. Spelling it out, the multiplication is

$$HR \wedge HR \longrightarrow H(R \otimes R) \xrightarrow{H(\mu)} HR$$

and the unit is

$$\mathbb{S} \longrightarrow H\mathbb{Z} \xrightarrow{H(\eta)} HR.$$

Furthermore, if R is commutative then HR is a commutative ring spectrum.

- Finally, we claim that the stable homotopy groups π_* are lax symmetric monoidal as a functor on the homotopy category of spectra:

$$(\mathrm{Ho}\,\mathbf{Sp}, \wedge^{\mathbb{L}}, \mathbb{S}) \xrightarrow{\pi_*} (\mathbf{GrAb}, \otimes, \mathbb{Z}[0]).$$

[2] We already asserted the existence of this isomorphism in Example 4.1.9, but the claim here is stronger. Not only does this isomorphism exist, but along this isomorphism, the associativity and commutativity isomorphisms for the smash product of spectra agree with those same isomorphisms for topological spaces. We need this stronger claim to prove that $\Sigma^\infty_+ G$ is a ring – for instance, when checking that its associativity in **Sp** follows from the associativity of G in **Top**.

For this we use the symmetric monoidal structure on **GrAb** in which the flip of $a \otimes b$ is $(-1)^{|a||b|} b \otimes a$.

It follows that π_* is also lax symmetric monoidal as a functor on the category of spectra. Either way, there are natural maps

$$\pi_*(X) \otimes \pi_*(Y) \longrightarrow \pi_*(X \wedge Y), \qquad \mathbb{Z} \longrightarrow \pi_0(\mathbb{S}).$$

It also follows that if R is a ring spectrum, then $\pi_*(R)$ is a graded ring. If R is commutative then $\pi_*(R)$ is graded-commutative (commutative but using the Koszul sign rule).

In particular, $\pi_*(HR) = R[0]$ gives our original ring back, and the stable homotopy groups of spheres $\pi_*(\mathbb{S})$ are a graded-commutative ring.

Remark 4.1.28. When $F\colon \mathbf{C} \to \mathbf{D}$ is a strong symmetric monoidal functor, it is helpful to think of $\mathbf{C}$ and $\mathbf{D}$ as "commutative rings" in categories, and F as a ring homomorphism, making $\mathbf{D}$ into an "algebra" over $\mathbf{C}$.

In particular, $\mathbf{D}$ is a "module" over $\mathbf{C}$, where objects in $\mathbf{C}$ act on the category $\mathbf{D}$ by the formula

$$X \cdot Y := F(X) \otimes_{\mathbf{D}} Y, \qquad X \in \mathbf{C},\ Y \in \mathbf{D}.$$

In particular, the suspension spectrum functor Σ^∞_+ is strong symmetric monoidal, so we can think of spectra as an "algebra" or "module" over unbased spaces. An unbased space A acts on spectra by sending each spectrum X to

$$A_+ \wedge X \cong (\Sigma^\infty_+ A) \wedge X.$$

This has all of the properties that you would expect for a commutative ring acting on a module, including associativity (up to isomorphism).

4.1.4 Closed symmetric monoidal categories

Definition 4.1.29. A **closed symmetric monoidal category C** is a symmetric monoidal category in which $X \otimes -$ always has a right adjoint $\mathrm{Hom}(X, -)$. In other words, a map $X \otimes Y \to Z$ is the same thing as a map $Y \to \mathrm{Hom}(X, Z)$.

We call $\mathrm{Hom}(-,-)$ the **internal hom** of $\mathbf{C}$. By formal category theory, it defines a functor

$$\mathrm{Hom}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{C}$$

and we have natural bijections

$$\mathbf{C}(X, \mathrm{Hom}(Y, Z)) \cong \mathbf{C}(X \otimes Y, Z) \cong \mathbf{C}(Y, \mathrm{Hom}(X, Z)).$$

Example 4.1.30.
- The category of k-vector spaces, or more generally k-modules $(\mathbf{Mod}_k, \otimes_k, k)$ for a commutative ring k, is closed symmetric monoidal. The internal hom is $\mathrm{Hom}_k(X, Y)$, the set of k-linear maps from X to Y, made into a module by adding and scaling maps pointwise (on each input separately). We get the usual tensor-hom adjunction

$$X \otimes_k Y \to Z \qquad \longleftrightarrow \qquad X \to \mathrm{Hom}_k(Y, Z).$$

- The category $(\mathbf{Set}, \times, *)$ is closed symmetric monoidal using the set of maps $X \to Y$.
- The category $(\mathbf{Top}, \times, *)$ of (CGWH) topological spaces is closed symmetric monoidal using the space $\mathrm{Map}(X, Y)$ of continuous maps from X to Y.[3]
- The category $(\mathbf{Top}_*, \wedge, S^0)$ is closed symmetric monoidal using the space $\mathrm{Map}_*(X, Y)$ of continuous based maps from X to Y.
- The category $(\mathbf{GrMod}_k, \otimes_k, k[0])$ is closed symmetric monoidal using the set of graded maps $\mathrm{Hom}_k(X, Y)$, whose dth level consists of maps from X to Y that shifts the grading up by d, in other words $X_n \to Y_{n+d}$. The categories of chain complexes have similarly-defined internal homs.
- We will see that $\mathbf{Sp}$ and $\mathrm{Ho}\,\mathbf{Sp}$ are also closed symmetric monoidal. The internal hom is called the **function spectrum** $F(X, Y)$. So we have bijections of maps of spectra

$$X \wedge Y \to Z \qquad \longleftrightarrow \qquad X \to F(Y, Z)$$

  and of maps in the homotopy category

$$QX \wedge QY \to RZ \qquad \longleftrightarrow \qquad QX \to F(QY, RZ).$$

In a closed symmetric monoidal category $\mathbf{C}$, the internal hom satisfies identities

$$\mathrm{Hom}(I, X) \cong X, \qquad \mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

See exercise 17. We also get natural composition maps

$$\mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z)$$

that are associative, and have "identity" maps $I \to \mathrm{Hom}(X, X)$. All together this makes $\mathbf{C}$ into a **$\mathbf{C}$-enriched category**. Basically, this is the definition of a category, except that the mapping objects $\mathbf{C}(X, Y)$ are not sets, but objects of $\mathbf{C}$. Taking the set of maps in from the unit I recovers the original category:

$$\mathbf{C}(I, \mathrm{Hom}(X, Y)) \cong \mathbf{C}(X, Y).$$

[3] This is not true for "plain vanilla" topological spaces – we need at least the compactly generated hypothesis to get the adjunction between the product $\times$ and mapping space Map.

Finally, we also get an "external" tensor product on the internal homs,

$$\mathrm{Hom}(X,Y)\otimes \mathrm{Hom}(W,Z) \longrightarrow \mathrm{Hom}(X\otimes W, Y\otimes Z),$$

that commutes with composition in the expected way,

$$(g\otimes g')\circ(f\otimes f') = (g\circ f)\otimes(g'\circ f').$$

An important special case is the **assembly** map

$$\mathrm{Hom}(X,Y)\otimes Z \longrightarrow \mathrm{Hom}(X, Y\otimes Z).$$

Example 4.1.31. In $\mathbf{Mod}_k$ these operations are the obvious ones on the modules $\mathrm{Hom}_k(X,Y)$, and similarly for **Top** these are the obvious operations on the mapping spaces $\mathrm{Map}(X,Y)$.

Lemma 4.1.32. *Suppose* $\mathbf{C}$ *is a symmetric monoidal category satisfying the assumptions of Lemma 4.1.7. Suppose in addition there is a full subcategory* $\mathbf{B}\subseteq\mathbf{C}$ *such that*

- Hom *preserves weak equivalences on* $\mathbf{A}^{\mathrm{op}}\times\mathbf{B}$*,*
- *if* $X\in\mathbf{A}$ *and* $Y\in\mathbf{B}$ *then* $\mathrm{Hom}(X,Y)\in\mathbf{B}$*,*
- *there is a functor* $R\colon\mathbf{C}\to\mathbf{C}$ *landing in* $\mathbf{B}$*, and*
- *there is a natural weak equivalence* $r\colon X\xrightarrow{\sim} RX$*.*

Then the right-derived hom $\mathbb{R}\mathrm{Hom}(X,Y)=\mathrm{Hom}(QX,RY)$ *makes the homotopy category* $\mathrm{Ho}\,\mathbf{C}$ *into a closed symmetric monoidal category.*

For instance, these conditions are satisfied in $(\mathbf{Top}_*,\wedge,S^0)$ and $(\mathbf{Ch}_{\geq 0}(\mathbb{Z}),\otimes,\mathbb{Z}[0])$, where $\mathbf{B}$ is the entire category.

Proof. We only have to show that for each X separately, $\mathbb{R}\mathrm{Hom}(X,-)$ is a right adjoint to $X\otimes^{\mathbb{L}}(-)$. This follows from Proposition 3.4.20 applied to the functors $QX\otimes(-)$ and $\mathrm{Hom}(QX,-)$. $\square$

Example 4.1.33. This makes the derived category of chain complexes $\mathscr{D}(k)_{\geq 0}$ or $\mathscr{D}(k)$ into a closed symmetric monoidal category. The tensor product is computed by taking projective resolutions and tensoring, exactly the kind of manipulation that computes Tor groups. The internal hom takes projective resolution of the source and then takes the chain complex of maps, exactly the kind of manipulation that computes Ext groups.

Example 4.1.34 (Smash product as a black box, part 3)**.** Continuing from Example 4.1.9 and Example 4.1.27, we assert that

- **Sp** has an internal hom called the **function spectrum** $F(X,Y)$. So maps $X \to F(Y,Z)$ correspond to maps $X \wedge Y \to Z$, and we have natural isomorphisms of spectra (not just equivalences!)

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)), \qquad F(\mathbb{S}, X) \cong X.$$

- More generally, for any based space K, it follows from the isomorphism $(\Sigma^\infty K) \wedge X \cong K \wedge X$ that functions out of a suspension spectrum are the same thing as the cotensor from Definition 2.3.8:

$$F(\Sigma^\infty K, X) \cong F(K, X).$$

- The conditions of Lemma 4.1.32 are satisfied, with **B** the subcategory of Ω-spectra. So we have a right-derived function spectrum

$$\mathbb{R}F(X,Y) = F(QX, RY),$$

  where R is any replacement by an Ω-spectrum, such as the one in Proposition 2.2.9.

It follows from Lemma 4.1.32 that the stable homotopy category Ho **Sp** is closed symmetric monoidal. So there are natural bijections

$$[X \wedge^{\mathbb{L}} Y, Z]_s \cong [X, \mathbb{R}F(Y,Z)]_s. \tag{4.1.35}$$

The reader might want to verify that this is consistent with the isomorphisms we got in (3.4.22) when K is a based space:

$$[X, \mathbb{R}F(K,Y)]_s \cong [K \wedge^{\mathbb{L}} X, Y]_s \cong [K, \mathbb{R}\mathrm{Map}_*(X,Y)]_*.$$

It also follows with a good deal of additional work (exercise 23) that if $X \to Y \to Z$ is a cofiber/fiber sequence and W is a spectrum, then we have cofiber/fiber sequences

$$\mathbb{R}F(Z,W) \to \mathbb{R}F(Y,W) \to \mathbb{R}F(X,W), \qquad \mathbb{R}F(W,X) \to \mathbb{R}F(W,Y) \to \mathbb{R}F(W,Z).$$

4.2 Duality and the Spanier-Whitehead category

In each symmetric monoidal category **C** there is a class of especially well-behaved objects called the dualizable objects. Intuitively, these are the objects that are "finite enough."

Dualizable objects have many nice properties, for instance the assembly map

$$\mathrm{Hom}(X,Y) \otimes Z \longrightarrow \mathrm{Hom}(X, Y \otimes Z)$$

is an isomorphism if either X or Z is dualizable, and the "double dual" map

$$X \longrightarrow \mathrm{Hom}(\mathrm{Hom}(X,I),I)$$

is an isomorphism if X is dualizable. Setting $X^* = \mathrm{Hom}(X,I)$, we therefore have

$$X^* = \mathrm{Hom}(X,I), \qquad X \cong \mathrm{Hom}(X^*,I).$$

Dualizability makes several more manipulations possible. For instance, every map $X \to Y$ induces a dual map $Y^* \to X^*$ going the other way. Any self-map of a dualizable object X has a "trace" that is a morphism $I \to I$.

Because dualizable objects are simpler than general objects, it's also sometimes easier to define the subcategory of $\mathbf{C}$ on the dualizable objects. This is especially true for the homotopy category of spectra $\mathrm{Ho}\,\mathbf{Sp}$ – the subcategory of dualizable objects was first defined by Spanier and Whitehead in the 1950s, before the smash product was defined on the larger category $\mathrm{Ho}\,\mathbf{Sp}$ by Boardman and Adams, and long before the smash product was defined on $\mathbf{Sp}$ by Hovey-Shipley-Smith and Elmendorff-Kriz-Mandell-May in the 1990s.

4.2.1 Duality in a symmetric monoidal category

Definition 4.2.1. Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category. An object X is **dualizable** if the functor $X \otimes (-)$ has a right adoint of the form $X^* \otimes (-)$.

In other words, X is dualizable if there exists a dual object X^*, and evaluation and coevaluation maps

$$e : X \otimes X^* \longrightarrow I, \qquad c : I \longrightarrow X^* \otimes X,$$

such that the following two composites are identity maps:

$$\begin{gathered} X \cong X \otimes I \xrightarrow{1\otimes c} X \otimes X^* \otimes X \xrightarrow{e\otimes 1} I \otimes X \cong X \\ X^* \cong I \otimes X^* \xrightarrow{c\otimes 1} X^* \otimes X \otimes X^* \xrightarrow{1\otimes e} X^* \otimes I \cong X^* \end{gathered} \tag{4.2.2}$$

The conditions in (4.2.2) are called the **triangle identitites** for c and e.

Example 4.2.3. In k-vector spaces $\mathbf{C} = \mathbf{Vect}_k$, a vector space V is dualizable iff it is finite-dimensional. Its dual $V^* = \mathrm{Hom}_k(V,k)$ is the usual k-linear dual.

The evaluation map $V \otimes V^* \to k$ is the usual map that plugs a vector v into a linear functional $f(-)$ to get a scalar $f(v)$.

The coevaluation map $k \to V^* \otimes V$ is the map that sends 1 to the sum $\sum_i e_i^* \otimes e_i$, where $\{e_1, \dots, e_n\}$ is any basis for V. Along the isomorphism $V^* \otimes V \cong \mathrm{Hom}_k(V,V)$, this corresponds to the identity transformation $V \to V$.

The previous example is typical.

Proposition 4.2.4. *In any closed symmetric monoidal category* $\mathbf{C}$*, the dual of* X*, if it exists, must be the mapping object to the unit,*

$$DX = \mathrm{Hom}(X, I).$$

Proof. The dual X^* is defined so that $X^* \otimes -$ is the right adjoint of $X \otimes -$. But right adjoints are unique up to isomorphism, and we already have a right adjoint $\mathrm{Hom}(X, -)$. So the two adjoints must be isomorphic:

$$X^* \otimes Y \cong \mathrm{Hom}(X, Y).$$

Plugging in $Y = I$ gives $X^* \cong \mathrm{Hom}(X, I)$. □

Definition 4.2.5. In any closed symmetric monoidal category $\mathbf{C}$, we call $DX = \mathrm{Hom}(X, I)$ the **functional dual** of X. It always exists, but it only satisfies the properties of the dual if X is dualizable.

Example 4.2.6. If V is an infinite-dimensional vector space, it has a functional dual $DV = \mathrm{Hom}_k(V, k)$, but $D(DV)$ is not canonically isomorphic to V again.

Proposition 4.2.7. *If* X *is an object in a closed symmetric monoidal category* $\mathbf{C}$*, the following are equivalent:*

1. X *is dualizable.*
2. *The assembly map* $DX \otimes X \longrightarrow \mathrm{Hom}(X, X)$ *is an isomorphism.*
3. *The assembly map* $DX \otimes Y \longrightarrow \mathrm{Hom}(X, Y)$ *is an isomorphism for all* Y*.*

Proof. (1) ⇒ (3) If X is dualizable, the operation $\mathrm{Hom}(X, -)$ is naturally isomorphic to $DX \otimes (-)$. The assembly map is exactly this isomorphism, so it is an isomorphism for all Y.

(3) ⇒ (2) Obvious.

(2) ⇒ (1) We define the coevaluation map c to be the composite

$$I \xrightarrow{\mathrm{id}} \mathrm{Hom}(X, X) \cong DX \otimes X$$

and the evaluation map e to be the map

$$X \otimes DX \cong \mathrm{Hom}(I, X) \otimes \mathrm{Hom}(X, I) \xrightarrow{\circ} \mathrm{Hom}(I, I) \cong I.$$

It is then an exercise to verify the triangle identities for c and e. □

Example 4.2.8. In vector spaces $\mathbf{C} = (\mathbf{Vect}_k, \otimes_k, k)$, the map $DV \otimes_k V \to \mathrm{Hom}_k(V,V)$ is injective, and its image is the endomorphisms of finite rank. It is an isomorphism iff V is finite-dimensional. Hence by Proposition 4.2.7, V is dualizable iff it is finite-dimensional.

Example 4.2.9. If k is instead a commutative ring and $\mathbf{C} = \mathbf{Mod}_k$ is the category of k-modules with tensor product $\otimes_k$, a k-module M is dualizable iff it is finitely generated and projective over k.

Indeed, assume that M is dualizable. Then $\mathrm{Hom}_k(M,-) \cong DM \otimes_k (-)$ is right-exact, and hence M is projective. Since $DM \otimes_k M \to \mathrm{Hom}_k(M,M)$ is an isomorphism, it hits the identity map of M, expressing this identity map as a finite sum of maps, each of which has image in the span of a single element of M. It follows that M is finitely generated.

Conversely, if M is finitely generated and projective then it is a retract of $k^{\oplus n}$. We can show the assembly map $DM \otimes_k M \to \mathrm{Hom}_k(M,M)$ is a retract of the same assembly map for $k^{\oplus n}$, and is therefore an isomorphism.

Example 4.2.10. Passing the previous example to bounded or unbounded chain complexes $\mathbf{C} = \mathbf{Ch}_k$, a chain complex is dualizable iff it is bounded (only finitely many nonzero terms) and finitely generated projective in each degree. Note that the internal hom is given by the product

$$\mathrm{Map}(A_*, B_*)_n = \prod_k \mathrm{Hom}_R(A_n, B_{n+k})$$

with boundary map

$$\partial_{\mathrm{Map}(A,B)}(f) = \partial_B \circ f - (-1)^{|f|} f \circ \partial_A,$$

so that

$$(DA)_n = \mathrm{Map}(A, \underline{R})_n = \mathrm{Hom}(A_{-n}, R).$$

Essentially, the dual flips the chain complex over and dualizes each level separately.

If we pass to the homotopy category $\mathscr{D}(k) = \mathrm{Ho}\,\mathbf{Ch}_k$, a chain complex is dualizable iff it is **perfect**, meaning that it is quasi-isomorphic to a bounded complex of finitely generated projective modules. The dual is computed by making the complex into a bounded complex of finitely generated projective modules and then doing the above procedure.

Example 4.2.11 (Smash product as a black box, part 4)**.** We add to our black box the claim that a spectrum is dualizable if and only if it is stably equivalent to a finite spectrum. That is, a spectrum with finitely many stable cells.

4.2.2 The Spanier-Whitehead category

It's much easier to define the smash product on the subcategory of dualizable spectra in $\mathrm{Ho}\,\mathbf{Sp}$ than the entire category. It builds intuition for what happens in general, so let's start there first.

Recall from Remark 3.2.14 that the **Spanier-Whitehead category** is the full subcategory $\operatorname{Ho}\mathbf{Sp}_{\mathrm{fin}} \subseteq \operatorname{Ho}\mathbf{Sp}$ on the finite spectra. A finite spectrum is one that has finitely many stable cells – equivalently, it is a shift of a suspension spectrum $\Sigma^\infty A$ of a finite CW complex A. See Section 2.7, exercise 41.

So, we can describe the Spanier-Whitehead category as having an object (A,k) for every finite based CW complex A and integer $k \in \mathbb{Z}$. This corresponds to the spectrum

$$\operatorname{sh}^k \Sigma^\infty A \cong \begin{cases} F_0 \Sigma^k A & \text{when } k \geq 0, \\ F_{|k|} A & \text{when } k \leq 0. \end{cases}$$

In particular, (S^0, k) corresponds to the shifted sphere spectrum $\mathbb{S}^k = \operatorname{sh}^k \mathbb{S}$.

By Corollary 3.2.13, the morphisms are computed as a colimit of homotopy classes of based maps,

$$\operatorname*{colim}_{n\to\infty} [\Sigma^{k+n} A, \Sigma^{\ell+n} B]_*.$$

Here the colimit system is defined as soon as n is large enough that both $k+n$ and $\ell+n$ are nonnegative. When $k=\ell$, it is classical to refer to this colimit as $\{A,B\}$, the abelian group of "stable maps" from the CW complex A to the CW complex B. Collecting this all together:

Proposition 4.2.12. *The Spanier-Whitehead category* $\operatorname{Ho}\mathbf{Sp}_{\mathrm{fin}}$ *is equivalent to the category whose objects are pairs* (A,k) *with* A *a finite based CW complex and* $k \in \mathbb{Z}$*. The morphisms from* (A,k) *to* (B,ℓ) *are the colimit*

$$\operatorname*{colim}_{n\to\infty} [\Sigma^{k+n} A, \Sigma^{\ell+n} B]_*$$

Definition 4.2.13. We define the symmetric monoidal structure on $\operatorname{Ho}\mathbf{Sp}_{\mathrm{fin}}$ by taking smash product of the CW complexes and adding the integers:

$$(A,k) \wedge (C,m) := (A \wedge C, k+m).$$

This extends to the morphisms by taking each pair of maps

$$f\colon \Sigma^{k+n_1} A \longrightarrow \Sigma^{\ell+n_1} B, \qquad g\colon \Sigma^{m+n_2} C \longrightarrow \Sigma^{j+n_2} D$$

to the map

$$\begin{tikzcd}
\Sigma^{k+m+n_1+n_2} A \wedge C \arrow[d, "\cong"', "(-1)^{mn_1}"] & \Sigma^{\ell+j+n_1+n_2} B \wedge D \\
\Sigma^{k+n_1} A \wedge \Sigma^{m+n_2} C \arrow[r, "f\wedge g"] & \Sigma^{\ell+n_1} B \wedge \Sigma^{j+n_2} D \arrow[u, "\cong"', "(-1)^{jn_1}"]
\end{tikzcd}$$

The minus signs indicate that we should apply a map of the specified degree to the sphere $S^{k+m+n_1+n_2}$ or $S^{\ell+j+n_1+n_2}$. This follows the Koszul sign rule, and guarantees that this rule is independent of the choice of n_1 and n_2. See exercise 24.

We define the associativity and unit isomorphisms

$$((A,k)\wedge(B,\ell))\wedge(C,m)\cong(A,k)\wedge((B,\ell)\wedge(C,m)), \qquad (A,k)\wedge(S^0,0)\cong(A,k)$$

in the obvious way. For the symmetry isomorphism $(A,k)\wedge(C,m)\cong(C,m)\wedge(A,k)$, we use another Koszul sign rule

$$\Sigma^{k+m}A\wedge C \xrightarrow[\cong]{(-1)^{km}} \Sigma^{m+k}C\wedge A.$$

Proposition 4.2.14. *This defines a symmetric monoidal structure on the Spanier-Whitehead category* $\mathrm{Ho}\,\mathbf{Sp}_{\mathrm{fin}}$.

Proof. It is a straightforward exercise to check that the isomorphisms we defined are *natural*, meaning they commute with the morphisms in the category, see exercise 25. The diagrams above Theorem 4.1.4 all commute because the signs we introduce are the signs of permutations for the sphere coordinates, and when we go around each diagram, the composite permutation is the identity, so the product of the signs is $+1$. □

It is a consequence of Theorem 3.2.5 that the Spanier-Whitehead category is additive: the morphisms $(A,k)\to(B,\ell)$ are abelian groups, there is a zero object, and finite coproducts and products exist and are isomorphic to each other. It follows from exercise 5 that the smash product distributes over the sums of maps, so that $f\wedge(g_1+g_2)=(f\wedge g_1)+(f\wedge g_2)$.

Lemma 4.2.15. *Dualization commutes with finite sums and shifts:*

$$D(X\vee Y)\simeq DX\vee DY, \qquad D(\Sigma^k X)=\Sigma^{-k}DX.$$

This is left to exercise 26. As a special case, the dual of the n-sphere is the $(-n)$-sphere:

$$D(\mathbb{S}^n)=D(S^0,n)\cong(S^0,-n)=\mathbb{S}^{-n}.$$

It turns out that much more is true: every finite spectrum is dualizable.

Proposition 4.2.16 (Spanier-Whitehead duality)**.** *Every object (A,k) in the Spanier-Whitehead category is dualizable.*

Intuitively, the dual of a finite cellular spectrum X is a spectrum DX with a stable $(-k)$-cell for every stable k-cell of X. To give a more specific construction in the Spanier-Whitehead category, the dual of (A,k) is computed by embedding A into a sphere S^{n+1}, letting B be a finite CW complex equivalent to the complement $S^{n+1}\setminus A$, and taking the dual to be $(B,n-k)$. The proof of Proposition 4.2.16 is pleasantly geometric, but goes outside the scope of our immediate focus, so it will be deferred to **??**.

Corollary 4.2.17. *The Spanier-Whitehead category is closed symmetric monoidal – there is an internal hom that is right adjoint to the smash product.*

Proof. If X and Y are objects of the Spanier-Whitehead category and X^* is the dual of X, we *define* $\mathrm{Hom}(X,Y)$ to be $X^* \wedge Y$. It is then immediate that $\mathrm{Hom}(X,-)$ is a right adjoint to $X \wedge -$ as required. □

In the special case that $A = M_+$ where M is a closed smooth manifold, if we embed M_+ into S^n and take its complement B, the suspension of B becomes identified with $N/\partial N$, a tubular neighborhood of M in $\mathbb{R}^n$ modulo its boundary. This result is commonly called Atiyah duality.

Theorem 4.2.18 (Atiyah duality)**.** *If M is a closed smooth manifold, then the dual of $(M_+, 0)$ in the Spanier-Whitehead category is $(N/\partial N, -n)$, for N any tubular neighborhood of a smooth embedding $M \to \mathbb{R}^n$.*

Example 4.2.19. If $M = S^1$, we can embed it into $\mathbb{R}^2$ in the standard way, and its neighborhood N is an annulus. It is easy to see that $N/\partial N$ is homotopy equivalent to $S^1 \vee S^2$. We therefore get a duality

$$D(S^1_+, 0) \simeq (S^1 \vee S^2, -2).$$

This could have also been calculated directly. By Section 2.7, exercise 33, we have an isomorphism in the Spanier-Whitehead category

$$(S^1_+, 0) \cong \mathbb{S}^1 \vee \mathbb{S}^0$$

where $\mathbb{S}^k = (S^0, k)$ is the shifted sphere spectrum. Therefore by Lemma 4.2.15,

$$D(S^1_+, 0) \cong D(\mathbb{S}^1 \vee \mathbb{S}^0) \cong \mathbb{S}^{-1} \vee \mathbb{S}^0 \cong (S^1 \vee S^2, -2).$$

4.2.3 Consequences of duality

Let us now return to the larger stable homotopy category $\mathrm{Ho}\,\mathbf{Sp}$. Recall that in Example 4.1.9, Example 4.1.27, Example 4.1.34, and Example 4.2.11 we gave some black-boxed properties of the smash product on this larger category. In particular, $\mathrm{Ho}\,\mathbf{Sp}$ is a closed symmetric monoidal category, under the left-derived smash product $X \wedge^{\mathbb{L}} Y = QX \wedge QY$ and right-derived function spectrum $\mathbb{R}F(X,Y) = F(QX, RY)$, and a spectrum is dualizable if and only if it is finite.

Lemma 4.2.20. *If X is a finite (dualizable) spectrum, then the homology of DX is isomorphic to the cohomology of X,*

$$E_q(DX) \cong E^{-q}(X).$$

Proof. The left-hand side is defined to be $[\mathbb{S}^q, DX \wedge^{\mathbb{L}} E]_s$, maps in the stable category from $\mathbb{S}^q$ to the left-derived smash product $DX \wedge^{\mathbb{L}} E$. Since $DX \wedge^{\mathbb{L}} (-)$ is the right adjoint of $X \wedge^{\mathbb{L}} (-)$, by uniqueness of right adjoints, we must have

$$\begin{aligned}[\mathbb{S}^q, DX \wedge^{\mathbb{L}} E]_s &\cong [\mathbb{S}^q, \mathbb{R}F(X,E)]_s \\ &\cong [\mathbb{S}^q \wedge^{\mathbb{L}} X, E]_s \\ &\cong [\mathrm{sh}^q X, E]_s \\ &\cong E^{-q}(X).\end{aligned}$$

The second-to-last isomorphism uses exercise 18. □

Of course, since DX is also dualizable and $DDX \simeq X$, we also get

$$E^q(DX) \cong E_{-q}(X).$$

We can now use this observation to give the stable-homotopy proof of Poincaré duality.

Theorem 4.2.21 (Poincaré duality, cf. Theorem 1.3.18)**.** *If M is a closed smooth d-dimensional manifold, there is an isomorphism*

$$H^i(M;\mathbb{Z}) \cong H_{d-i}(M;\widetilde{\mathbb{Z}})$$

where $\widetilde{\mathbb{Z}}$ is the local coefficient system whose fiber is $\mathbb{Z}$ and that twists according to the orientation of M. In case M is orientable, this simplifies to

$$H^i(M;\mathbb{Z}) \cong H_{d-i}(M;\mathbb{Z}).$$

In the general, non-orientable case we also get isomorphism with $\mathbb{Z}/2$ coefficients

$$H^i(M;\mathbb{Z}/2) \cong H_{d-i}(M;\mathbb{Z}/2).$$

Proof. By Lemma 4.2.20, the cohomology of M is the homology of its Spanier-Whitehead dual:

$$H^i(M;\mathbb{Z}) \cong H_{-i}(DM;\mathbb{Z}).$$

By Theorem 4.2.18, this Spanier-Whitehead dual is a shift of $N/\partial N$, where N is a tubular neighborhood of M in $\mathbb{R}^n$:

$$H_{-i}(DM;\mathbb{Z}) \cong H_{n-i}(N/\partial N;\mathbb{Z}).$$

The space $N/\partial N$ can be identified with the Thom space $\mathrm{Th}(\nu)$ of the normal bundle ν of M in $\mathbb{R}^n$, which has dimension $(n-d)$. By the Thom isomorphism of Example 2.6.38, if this bundle is orientable then

$$H_{n-i}(N/\partial N;\mathbb{Z}) \cong H_{n-i}(\mathrm{Th}(\nu);\mathbb{Z}) \cong H_{d-i}(M;\mathbb{Z}),$$

and more generally it gives $H_{d-i}(M;\widetilde{\mathbb{Z}})$ where $\widetilde{\mathbb{Z}}$ twists according to the orientation of the normal bundle of M. This is the same as the orientation of the tangent bundle of M, and the conclusion follows. The isomorphism with $\mathbb{Z}/2$ coefficients follows in the same way. □

This proof has the advantage that it generalizes to coefficients in any *extraordinary* homology theory E. We say that the normal bundle of $M \to \mathbb{R}^n$ is E**-orientable** if there is an isomorphism

$$E_{(n-d)+q}(\mathrm{Th}(\nu)) \cong E_q(M).[4]$$

Theorem 4.2.22. *If the normal bundle of M is E-orientable,*

$$E^i(M) \cong E_{d-i}(M).$$

The proof is the same as that of Theorem 4.2.21.

We can also now give the proof of Theorem 2.5.13:

Theorem 4.2.23 (Whitehead representability)**.** *If h_* is any extraordinary homology theory then there is a spectrum E and a natural isomorphism of homology theories $E_* \cong h_*$, where E_* is defined as in Proposition 2.5.7. Furthermore E is unique up to stable equivalence.*

Proof. Given an extraordinary homology theory h_*, define

$$h^n(X) := h_{-n}(F(X,\mathbb{S})).$$

This is a cohomology theory on finite spaces, not all spaces, because h_* doesn't take products to products. By Theorem 2.5.26, this is represented by a (unique) spectrum E. Then we get for finite spaces

$$\begin{aligned} h_n(X) &\cong h_n(DDX) \\ &\cong h^{-n}(DX) \\ &= [S^n, \mathbb{R}F(DX,E)] \\ &\cong [S^n, DDX \wedge^{\mathbb{L}} E] \\ &\cong [S^n, X \wedge^{\mathbb{L}} E] \\ &\cong E_n(X). \end{aligned}$$

These isomorphisms are natural in the finite CW complex X, so $h_* \cong E_*$ on all finite CW complexes. By the direct limit axiom (Remark 2.5.6), we therefore get $h_* \cong E_*$ on the larger category of all CW complexes. □

[4] Actually, the definition of orientability should be a little stricter than this – it should say that the bundle ν looks trivial after taking a smash product with E. We would need parametrized spectra to state this precisely, see [ABG+14]. In the end, this gives the same isomorphism that we want on the homology $E_*(\mathrm{Th}(\nu))$.

4.2.4 Traces

Definition 4.2.24. Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category. Let X be any dualizable object, with coevaluation and evaluation maps c and e as in Definition 4.2.1, and $f\colon X \to X$ any endomorphism of X. The **trace** of f is the map $\mathrm{tr}(f)\colon I \to I$ given by the composite

$$I \xrightarrow{c} X^* \otimes X \xrightarrow{1 \otimes f} X^* \otimes X \xleftrightarrow{\cong} X \otimes X^* \xrightarrow{e} I. \tag{4.2.25}$$

The **Euler characteristic** $\chi(X)\colon I \to I$ is the trace of the identity map of X.

Example 4.2.26. In k-vector spaces $\mathbf{C} = \mathbf{Vect}_k$, the trace of a map $f\colon V \to V$ is a k-linear map $k \to k$, which we can interpret simply as an element of k. Under this interpretation, $\mathrm{tr}(f) \in k$ is exactly the usual definition of trace, the sum of the diagonal entries of any matrix that represents f. The Euler characteristic of V is its dimension $\dim V \in \mathbb{Z}$, mapped forward to k.

Example 4.2.27. Suppose $\mathbf{C} = \mathbf{Mod}_k$ is the category of modules over the commutative ring k. By Example 4.2.9, a module M is dualizable iff it is finitely generated and projective, so we have $M \oplus N \cong k^n$ for some other module N.

If $f\colon M \to M$ is a k-linear map, its trace is computed by forming the map $(f, 0)\colon k^n \to k^n$ that is f on M and 0 on N. Then we take the matrix for $(f, 0)$ in any basis of k^n and add up its diagonal entries. The result does not depend on n, N, or the choice of basis!

In particular, the Euler characteristic of M is the trace of the idempotent map $k^n \to k^n$ that projects to M and includes back into k^n. This trace is the "dimension" of M, but it is not always an integer!

Example 4.2.28. Suppose $\mathbf{C} = \mathbf{Ch}_k$ is unbounded chain complexes over k. If $C_\cdot$ is a bounded complex of finitely generated projective modules and $f\colon C_\cdot \to C_\cdot$ is a self-map, its trace is the alternating sum

$$\mathrm{tr}(f) = \sum_n (-1)^n \mathrm{tr}(f_n\colon C_n \to C_n),$$

where the trace of f_n is computed as in the previous exercise. This expression comes from writing out (4.2.25) explicitly. The Koszul sign rule in the definition of $\gamma\colon X \otimes Y \cong Y \otimes X$ is responsible for the presence of signs here. See exercise 27.

In particular, the Euler characteristic of $C_\cdot$ is the alternating sum of the ranks of its levels. If $C_\cdot$ is the cellular chain complex of some finite CW complex X, this agrees with the Euler characteristic $\chi(X)$.

Example 4.2.29. Suppose $\mathbf{C} = \mathrm{Ho}\,\mathbf{Ch}_k$ is the derived category. Then the trace of a map is computed as in Example 4.2.28, only we change $C_\cdot$ up to quasi-isomorphism first into a bounded complex of finitely generated projective modules.

Remark 4.2.30. If k is a principal ideal domain (PID), it is a theorem in algebra that the trace of a map of complexes $f: C. \to C.$ is also computed as the alternating sum of traces on homology,

$$\mathrm{tr}(f) = \sum_n (-1)^n \mathrm{tr}(f_n : H_n(C.) \to H_n(C.))$$

For chain complexes over $\mathbb{Z}$, we can further pass to $\mathbb{Q}$ when computing this trace without changing the answer. This captures the familiar theorem that the Euler characteristic of a space $\chi(X)$ is the alternating sum of the ranks of the rational homology groups $H_n(X;\mathbb{Q})$.

Example 4.2.31. If X is a finite CW complex and $f: X \to X$ is a self-map, the trace of $C.(f): C.(X) \to C.(X)$ is equal to the alternating sum of traces on rational homology

$$L(f) = \sum_n (-1)^n \mathrm{tr}(f_n : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q})),$$

and is always an integer. We call this number the **Lefschetz number** of f.

It is a theorem of Lefschetz and Hopf that this number equals the number of fixed points of f, added together with appropriate weights. In particular, if there are no fixed points, the number must vanish:

Theorem 4.2.32 (Lefschetz Fixed Point Theorem)**.** *If $f: X \to X$ is a self-map of a finite CW complex, with no fixed points, then $L(f) = 0$.*

Equivalently, if $L(f) \neq 0$ then f *must* have a fixed point. This is an important classical theorem in topology that greatly generalizes the Brouwer fixed point theorem (any self-map of D^n must have a fixed point).

Example 4.2.33. If X is a finite CW complex then its suspension spectrum $\Sigma^\infty_+ X$ is a finite spectrum, and is therefore dualizable in the stable homotopy category Ho **Sp**. If $f: X \to X$ is a self-map, the trace of $\Sigma^\infty_+ f : \Sigma^\infty_+ X \to \Sigma^\infty_+ X$ in the stable homotopy category is a zig-zag $\mathbb{S} \to \mathbb{S}$, equivalently, an integer $\mathrm{tr}(f) \in \mathbb{Z} \cong [\mathbb{S},\mathbb{S}]_s$. The integer we get is exactly the Lefschetz number $L(f)$ – see exercise 28.

For much more information about traces in symmetric monoidal categories, see [PS14], or [DP80] for a more classical reference.

4.3 Exercises

1. We define the tensor product of two chain complexes $A \otimes B$ by the formula

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j, \quad d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db).$$

The differential is defined as above on each homogeneous piece $A_i \otimes B_j$. It extends uniquely to $(A \otimes B)_n$ since it is a homomorphism.

Note that the differential resembles the Leibniz rule from calculus, but with a sign that's introduced when we "switch d past a," consistent with the Koszul sign rule.

(a) Verify that this is indeed a chain complex. Explain what would go wrong without the $(-1)^{|a|}$ term.

(b) Define a symmetry isomorphism $\gamma\colon A \otimes B \to B \otimes A$ by

$$\gamma(a \otimes b) = (-1)^{|a||b|}(b \otimes a).$$

Verify that this is indeed a map of chain complexes, commuting with the differential. Again, explain what would happen without the $(-1)^{|a||b|}$ term.

2. We say a symmetric monoidal category $\mathbf{C}$ is **cartesian monoidal** if its symmetric monoidal structure uses the categorical product $\times$. Show that any category $\mathbf{C}$ with finite products is cartesian monoidal. (Note that we are also assuming that the empty product exists, i.e. the terminal object $* \in \mathbf{C}$.)

3. Dualize the previous exercise to cocartesian monoidal categories, in which the structure uses the coproduct $\amalg$.

4. Use Lemma 4.1.7 to show that the stable homotopy category $\operatorname{Ho}\mathbf{Sp}$ is a symmetric monoidal category under the left-derived wedge sum $\vee^{\mathbb{L}}$. Then explain how we could get this more easily from the previous exercise and Proposition 3.2.2.

   (This is *not* the symmetric monoidal structure on spectra that comes from the smash product $\wedge$. This one is *additive*, the other one is *multiplicative*.)

5. Recall from Theorem 3.2.5 that an **additive** category $\mathbf{C}$ is one in which

   - the sets $\mathbf{C}(X, Y)$ are abelian groups,
   - the composition maps $\mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$ are bilinear,
   - $\mathbf{C}$ has all finite coproducts and products, and a zero object $*$, and
   - the canonical map $X \amalg Y \to X \times Y$ is an isomorphism.

   In such a category we let $\oplus$ refer to the coproduct. An **additive symmetric monoidal category** is a category $\mathbf{C}$ that is both additive and symmetric monoidal, whose symmetric monoidal structure $\otimes$ preserves finite coproducts in each slot.

   Prove that in any such category, the tensor $\otimes$ induces bilinear maps

$$\mathbf{C}(X, Y) \times \mathbf{C}(W, Z) \longrightarrow \mathbf{C}(X \otimes W, Y \otimes Z).$$

   (You might want to use the definition of the addition in $\mathbf{C}(X, Y)$ given in the proof of Theorem 3.2.5.)

6. In this exercise we prove that there is no natural transformation in the stable homotopy category
$$X \xrightarrow{\eta} X \wedge X$$
that on suspension spectra agrees with the diagonal map
$$\Sigma^\infty K \xrightarrow{\Delta} \Sigma^\infty(K \wedge K) \cong (\Sigma^\infty K) \wedge (\Sigma^\infty K).$$
Suppose there were such a transformation η. Let $2\colon \mathbb{S} \to \mathbb{S}$ be the map that is two times the identity. Use exercise 5 to argue that the map $2 \wedge 2\colon \mathbb{S} \wedge \mathbb{S} \to \mathbb{S} \wedge \mathbb{S}$ is isomorphic to $4\colon \mathbb{S} \to \mathbb{S}$. If η is natural, it induces a commuting square
$$\begin{tikzcd} \mathbb{S} \arrow[r, "2"] \arrow[d, "\eta"'] & \mathbb{S} \arrow[d, "\eta"] \\ \mathbb{S} \wedge \mathbb{S} \arrow[r, "4"] & \mathbb{S} \wedge \mathbb{S}. \end{tikzcd}$$
Derive a contradiction from this.

7. Suppose that E is a ring spectrum. Show that the cohomology of any *space* X inherits a "cup product" by taking any two maps in the stable homotopy category
$$\Sigma^\infty X \xrightarrow{f} \Sigma^m E, \qquad \Sigma^\infty X \xrightarrow{g} \Sigma^n E$$
to the composite
$$\Sigma^\infty X \xrightarrow{\Delta} \Sigma^\infty(X \wedge X) \cong (\Sigma^\infty X) \wedge (\Sigma^\infty X) \xrightarrow{f \wedge g} \Sigma^m E \wedge \Sigma^n E \xrightarrow{\mu} \Sigma^{m+n} E.$$
The cohomology of a spectrum X does not have such a cup product – why?

8. Suppose **C** is a symmetric monoidal category under its coproduct $\amalg$.

   - Show that every object X of **C** is a monoid object (with respect to $\amalg$) in a canonical way.
   - For each pair of objects X and Y, show that the structure of an X-module on Y (with respect to $\amalg$) is the same thing as a morphism $X \to Y$.

   This is why we only consider ring and module objects in the more "multiplicative" examples.

9. Suppose **C** is a *cartesian* monoidal category, in other words a symmetric monoidal category but the product is the categorical product $\times$. We say that G is a **group**

object in **C** if it is a monoid and it also has an inversion map $i\colon G \to G$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "{(i,1) \text{ or } (1,i)}"] \arrow[d] & G \times G \arrow[d, "\mu"] \\ * \arrow[r, "\eta"] & G. \end{tikzcd}$$

Informally, this means that $gg^{-1} = 1 = g^{-1}g$.

- Identify the group objects in **Set**, **Top**, and the category **Diff** of smooth manifolds and smooth maps.
- Explain why this definition doesn't make sense in a general symmetric monoidal category **C**. (Instead it gets replaced with the notion of a "Hopf algebra.")

10. Let $(\mathbf{C}, \otimes, I)$ be any symmetric monoidal category.

    (a) Prove that the unit object I is always a commutative monoid in **C**. As a result, the sphere spectrum $\mathbb{S}$ must be a commutative ring spectrum.

    (b) Prove that every object $X \in \mathbf{C}$ is an I-module in a canonical way. Therefore, every spectrum is a $\mathbb{S}$-module in a canonical way.

11. Let $(\mathbf{C}, \otimes, I)$ be any symmetric monoidal category. Define the correct notion of a map of monoids $R \to S$ in **C**. When you are done, compare your definition against Definition 6.2.32. (No peeking!)

12. Suppose $(F \dashv G)$ is an adjoint pair of functors. Show that maps
$$F(X \otimes Y) \to F(X) \otimes F(Y)$$
correspond to maps
$$G(X) \otimes G(Y) \to G(X \otimes Y),$$
and that maps $F(I) \to I$ correspond to maps $I \to G(I)$. It turns out that one set of maps is coherent iff the other is. So, G is lax symmetric monoidal iff F is "oplax" symmetric monoidal. In practice, when this happens, usually F is strong symmetric monoidal.

13. Suppose $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal category and $\otimes'$ is another functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ that is naturally isomorphic to $\otimes$. Explain how there is a canonical symmetric monoidal structure on **C** that uses $\otimes'$ instead of $\otimes$.

14. Suppose (F, G) is an equivalence of categories $\mathbf{C} \simeq \mathbf{D}$. Show that if $(\mathbf{C}, \otimes, I)$ is a symmetric monoidal category, we can define a symmetric monoidal structure on **D** by
$$X \otimes_{\mathbf{D}} Y := F(G(X) \otimes_{\mathbf{C}} G(Y)).$$

What is the unit object?

15. Finish the proof of Proposition 4.1.23 by showing that the diagrams in Definition 4.1.11 commute for $F(R)$ if they commuted for R.

16. Suppose $F: \mathbf{C} \to \mathbf{D}$ and $G: \mathbf{D} \to \mathbf{E}$ are both lax monoidal functors. Explain why the composite $G \circ F: \mathbf{C} \to \mathbf{E}$ is also a lax monoidal functor.

17. Prove that in any closed symmetric monoidal category $\mathbf{C}$, we have isomorphisms

$$\mathrm{Hom}(I, X) \cong X, \qquad \mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)).$$

(Hint: Use the Yoneda Lemma. If $\mathbf{C}(-, A)$ and $\mathbf{C}(-, B)$ are isomorphic as functors then necessarily $A \cong B$.)

18. Using the claimed properties in Example 4.1.9, prove that for any integer $d \in \mathbb{Z}$, the derived smash product $\mathbb{S}^d \wedge^{\mathbb{L}} X$ is equivalent to the shift $\mathrm{sh}^d X$. You might want to prove this for nonnegative d first, where $\mathrm{sh}^d X \simeq \Sigma^d X$, and then use the uniqueness of adjoints to argue the result for negative d.

19. Using the claimed properties in Example 4.1.9, prove that the smash product preserves wedge sums, homotopy pushouts, and mapping telescopes in each variable.

20. Using the claimed properties in Example 4.1.9, prove that the smash product of cellular spectra is a cellular spectrum, as illustrated at the beginning of Chapter 4.

21. Using the claimed properties in Example 4.1.9, prove that for any cofiber/fiber sequence $X \to Y \to Z$ and spectrum W, the derived smash products

$$W \wedge^{\mathbb{L}} X \to W \wedge^{\mathbb{L}} Y \to W \wedge^{\mathbb{L}} Z$$

form a cofiber/fiber sequence as well.

22. Using exercise 18 and the claimed properties in Example 4.1.34, prove that for any integer $d \in \mathbb{Z}$, the derived mapping space $\mathbb{R}F(\mathbb{S}^d, X)$ is equivalent to the negative shift $\mathrm{sh}^{-d} X$.

23. In this exercise we use the claimed properties of the function spectrum in Example 4.1.34 to show that the function spectrum preserves cofiber and fiber sequences in each slot.

    (a) Using that right adjoints preserve limits (Theorem 1.6.9), prove that the function spectrum F sends colimits in the first slot, or limits in the second slot, to limits.

(b) For a map of spectra $f\colon X \to Y$ with homotopy cofiber Cf and homotopy fiber Ff, prove that $F(Cf,W)$ is the homotopy fiber of $F(f,W)$, and that $F(W,Ff)$ is the homotopy fiber of $F(W,f)$.

(c) Prove that for any cofiber/fiber sequence $X \to Y \to Z$ and spectrum W, the derived maps into W

$$\mathbb{R}F(Z,W) \to \mathbb{R}F(Y,W) \to \mathbb{R}F(X,W)$$

and derived maps out of W

$$\mathbb{R}F(W,X) \to \mathbb{R}F(W,Y) \to \mathbb{R}F(W,Z)$$

form cofiber/fiber sequences as well.

24. Show that the rule for the smash product of two morphisms in Definition 4.2.13 is independent of n_1 and n_2, so that it gives a well-defined map

$$\operatorname*{colim}_{n_1\to\infty}[\Sigma^{k+n_1}A,\Sigma^{\ell+n_1}C]_* \times \operatorname*{colim}_{n_2\to\infty}[\Sigma^{m+n_2}B,\Sigma^{j+n_2}D]_* \longrightarrow \operatorname*{colim}_{n\to\infty}[\Sigma^{k+m+n}A\wedge C,\Sigma^{\ell+j+n}B\wedge D]_*.$$

25. Prove that the map $(A,k)\wedge(C,m)\cong(C,m)\wedge(A,k)$ from Definition 4.2.13 is natural, i.e. every diagram of the following form commutes:

$$\begin{tikzcd}
(A,k)\wedge(C,m) \arrow[r, "f\wedge g"] \arrow[d, "\cong"'] & (B,\ell)\wedge(D,j) \arrow[d, "\cong"] \\
(C,m)\wedge(A,k) \arrow[r, "g\wedge f"] & (D,j)\wedge(B,\ell)
\end{tikzcd}$$

Optionally, do the same for the associativity and unit isomorphisms.

26. Prove Lemma 4.2.15. Essentially, you just have to check that sums and shifts can be pulled through the definition of dual pair from Definition 4.2.1.

27. Verify Example 4.2.28 by checking that the trace of a self-map of a complex is the alternating sum of traces on each level.

28. (a) Prove that a strong symmetric monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ preserves dualizable objects: if X is dualizable in $\mathbf{C}$ then $F(X)$ is dualizable in $\mathbf{D}$.

(b) Prove that F also preserves traces: along the isomorphism $I_{\mathbf{D}} \cong F(I_{\mathbf{C}})$, the map $F(\operatorname{tr}(f))$ agrees with the map $\operatorname{tr}(F(f))$, for any self-map $f\colon X \to X$.

(c) The rational homology functor $H_*(-;\mathbb{Q})\colon \operatorname{Ho}\mathbf{Sp} \to \mathbf{GrMod}_k$ is strong symmetric monoidal. Use this to show that the trace of a finite complex in the stable homotopy category $\operatorname{Ho}\mathbf{Sp}$ agrees with the Lefschetz number $L(f)$.

Chapter 5

Spectra as a model category

What do spaces and spectra have in common? They are both categories with weak equivalences, so we can form a homotopy category $\mathrm{Ho}\,\mathbf{C}$, and talk about derived functors, as in Chapter 3.

There is another thing in common – they both have a notion of a "cell complex." A cell complex of spaces is built by taking successive pushouts along maps of the form $S^{n-1} \to D^n$. A cell complex of spectra is built in the same way from maps of the form $F_n S^{k-1}_+ \to F_n D^k_+$.

There is a third thing that they have in common – they both have a notion of "fibrant object." A fibrant topological space is... just a topological space. In other words, every space is fibrant. A fibrant spectrum, on the other hand, is an Ω-spectrum.

Finally, in both settings, maps in the homotopy category $\mathrm{Ho}\,\mathbf{C}(X, Y)$ are the same thing as homotopy classes of maps $X \to Y$, if X is a cell complex and Y is fibrant. So the cell complexes and the fibrant objects are the "nice" objects, for the purpose of counting maps in the homotopy category.

In this chapter, we introduce a formal framework that generalizes this: cofibrantly generated model categories. A **model category** is a category with three classes of maps, cofibrations, weak equivalences, and fibrations, that satisfy a few axioms. A **cofibrantly generated model category** also has a notion of "cell complex." The cofibrations are the maps that are retracts of cell complexes, and the fibrations are the maps that have a lifting property along a certain class of cell complexes.

Why do we need this terminology? Soon, we will introduce two new kinds of spectra called **symmetric spectra** and **orthogonal spectra**. We sometimes refer to both of these as "diagram spectra."

In order to work with diagram spectra effectively, we have to build cell complexes of diagram spectra, and show that they have the same behavior as cell complexes of ordinary (sequential) spectra. We also want to define the smash product of diagram spectra $X \wedge Y$, and prove that it preserves stable equivalences when X and Y are cellular diagram spectra. Both of these tasks are difficult to do without model categories.

How do model categories help us? One of the big things they do is break up the weak equivalences into "chunks." So, we can prove that a functor like $X \wedge Y$ preserves weak equivalences by checking it on one chunk at a time. In topological spaces, the chunks are maps of the form $D^n \times \{0\} \to D^n \times I$. It is very easy to show that maps like this go to weak equivalences. For spectra, the chunks are more complicated, because they also involve the truncation maps of the form $F_n S^n \to F_0 S^0$, which are stable equivalences. Still, this is a powerful technique. As we already mentioned, it's pretty difficult to do diagram spectra without it.

5.1 Generalized cell complexes and fibrations

Our treatment of model categories will be somewhat backwards. We'll start with the definition of a "cell complex" in an arbitrary category **C**. Then we'll show how to factor every map in **C** into a cell complex, followed by a map that has a lifting property with respect to cell complexes. This is the famous **small-object argument**. It was developed by Quillen when he invented model categories, and it's really the core idea that powers the whole theory.

5.1.1 I-cell complexes and I-injective maps

Let **C** be any category. Assume that every diagram in **C** has a colimit.[1] In particular, we're assuming that **C** has all coproducts, pushouts, and sequential colimits.

Let I be any set of maps in **C**. For example, I could be the set $\{S^{n-1} \to D^n\}_{n \geq 0}$ in the category **Top**. We think of the maps in I as the "cells" in **C**.

Definition 5.1.1. An I**-cell complex** is a sequential composition of pushouts of coproducts of maps in I. In other words, it is a composite

$$A = X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \ldots \longrightarrow X^{(n-1)} \longrightarrow X^{(n)} \longrightarrow \ldots \longrightarrow X = \operatorname*{colim}_{n \to \infty} X^{(n)}.$$

where $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a collection of cells $K_\alpha \to L_\alpha$ from the set

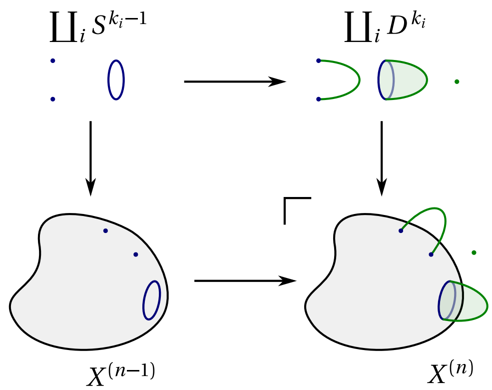
[1] Recall from Section 1.1 that a diagram is defined to be a functor $\mathbf{I} \to \mathbf{C}$ where $\mathbf{I}$ is a small category. In other words, the objects and morphisms of $\mathbf{I}$ form a set, not a proper class.

I:

$$\begin{tikzcd}
\coprod_\alpha K_\alpha \arrow[r] \arrow[d] & \coprod_\alpha L_\alpha \arrow[d] \\
X^{(n-1)} \arrow[r] & X^{(n)}.
\end{tikzcd}$$

Example 5.1.2.

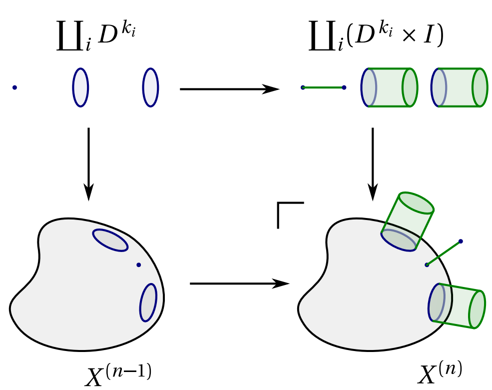
- If $\mathbf{C} = \mathbf{Top}$ and $I = \{S^{n-1} \to D^n\}_{n\geq 0}$, an I-cell complex is the same thing as a relative cell complex in the usual sense (Definition 1.1.8).



- If $\mathbf{C} = \mathbf{Sp}$ and $I = \{F_n S^{k-1}_+ \to F_n D^k_+\}_{n,k\geq 0}$, an I-cell complex is the same thing as a relative cellular spectrum (Definition 2.6.7 and Proposition 2.6.11).

- If $\mathbf{C} = \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ is nonnegatively-graded chain complexes of abelian groups, and I consists of all shifts of the map of chain complexes $(0 \to \mathbb{Z}) \to (\mathbb{Z} \to \mathbb{Z})$ (see Example 5.2.10), then an I-cell complex in $\mathbf{Ch}_{\geq 0}$ is a map $C_\bullet \to D_\bullet$ such that each $C_n \to D_n$ is injective, with quotient a free abelian group $\bigoplus \mathbb{Z}$.

- If $\mathbf{C} = \mathbf{Top}$ and $J = \{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$, a J-cell complex is a map built by successive **elementary expansions**. In other words, we select a collection of discs in the space $D^n \to A$, extrude each one to form a cylinder $D^n \times I$, then repeat the process countably many times, to get a larger space X. Clearly the inclusion $A \to X$ is both a relative cell complex and a weak equivalence.



Remark 5.1.3. It is common to use a more general definition of cell complex that involves transfinite compositions, see e.g. [Hov99, §2.1]. However, this extra generality is not necessary for the examples we consider in this book. In fact, in the examples we consider, cell complexes in the more general sense *coincide* with cell complexes in our sense. So we are talking about the same cell complexes as everyone else.

The set I also determines a class of generalized "fibrations" in $\mathbf{C}$ – those maps that have lifts along every map in I.

Definition 5.1.4. An **I-injective map** is a map $p\colon X \to Y$ such that, for any commuting square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{5.1.5}$$

with i a map in I, a dashed lift exists, making both triangles commute. We call this condition the **right lifting property** with respect to I.

Example 5.1.6. If $\mathbf{C} = \mathbf{Top}$ and $I = \{S^{n-1} \to D^n\}_{n\geq 0}$, an I-injective map is a map of topological spaces $X \to Y$ such that every square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ D^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{5.1.7}$$

has a dashed lift. Clearly this implies that $\pi_n(Y, X) = 0$, so the map $X \to Y$ is a weak equivalence. Furthermore, for any cell complex $A \to B$, any commuting square of the form (5.1.5) has a lift, by defining the lift inductively, one cell at a time. Therefore, in particular, every square of the form

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a lift. In other words, $X \to Y$ is a Serre fibration as well.

Conversely, if $X \to Y$ is both a weak equivalence and a Serre fibration, then it is I-injective. You show this by first showing that the square (5.1.7) has a lift up to homotopy, then using the homotopy lifting property of the fibration to modify this to a strict lift. (Section 1.7, exercise 19.)

In conclusion, the I-injective maps in **Top** are the maps that are both weak equivalences and Serre fibrations. We call these maps the **acyclic Serre fibrations**.

Example 5.1.8. If $\mathbf{C} = \mathbf{Top}$ and $J = \{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$, then by definition, a J-injective map is the same thing as a Serre fibration.

Example 5.1.9. If $\mathbf{C} = \mathbf{Sp}$ and $I = \{F_n S_+^{k-1} \to F_n D_+^k\}_{n,k\geq 0}$, an I-injective map is a map of spectra $X \to Y$ such that every square

$$\begin{tikzcd} F_n S_+^{k-1} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ F_n D_+^k \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a dashed lift. Using the adjunction $(F_n(-)_+, \mathrm{ev}_n)$ between spectra and unbased spaces, this rearranges to finding a lift

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & X_n \arrow[d, "p_n"] \\ D^k \arrow[r] \arrow[ur, dashed] & Y_n. \end{tikzcd} \tag{5.1.10}$$

Therefore, $X \to Y$ is I-injective iff it is an acyclic Serre fibration (a weak equivalence and a Serre fibration) at each spectrum level.

5.1.2 The small-object argument

Recall that $\mathbf{C}$ is a category with all (small) colimits, and I is any set of maps in $\mathbf{C}$.

Definition 5.1.11. We say that I has the **countable smallness condition** if for each map $K \to L$ in the set I, and each I-cell complex

$$X^{(0)} \longrightarrow X^{(1)} \longrightarrow \ldots X^{(\infty)} = \operatorname*{colim}_{n\to\infty} X^{(n)},$$

every map $K \to X^{(\infty)}$ factors through some $X^{(n)}$.

For instance, the set $I = \{S^{n-1} \to D^n\}_{n\geq 0}$ in **Top** has the countable smallness condition, because S^{n-1} is compact (see Section 1.7, exercise 23).

Theorem 5.1.12 (Small-object argument)**.** *Suppose I satisfies the countable smallness condition. Then every map in $\mathbf{C}$ can be factored into an I-cell complex followed by an I-injective map:*

$$X \xrightarrow{I\text{-cell complex}} X^{(\infty)} \xrightarrow{I\text{-injective}} Y.$$

Furthermore, this factorization is functorial.

Proof. Let $f : X \to Y$ be any map. Define $f_n : X^{(n)} \to Y$ for $n \geq 1$ by induction on n. When $n = 0$, we take $X^{(0)} = X$ and $f_0 = f$. Given f_{n-1}, take the set of all possible commuting squares of the form

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d] & L \arrow[d] \\ X^{(n-1)} \arrow[r, "f_{n-1}"] & Y \end{tikzcd} \tag{5.1.13}$$

with $i \in I$. Note that I is assumed to be a *set*, not a proper class, so the collection of all such squares forms a set as well.

Take the coproduct of all the maps i, one for each such square, and then define $X^{(n)}$ by taking the pushout of $X^{(n-1)}$ along this coproduct:

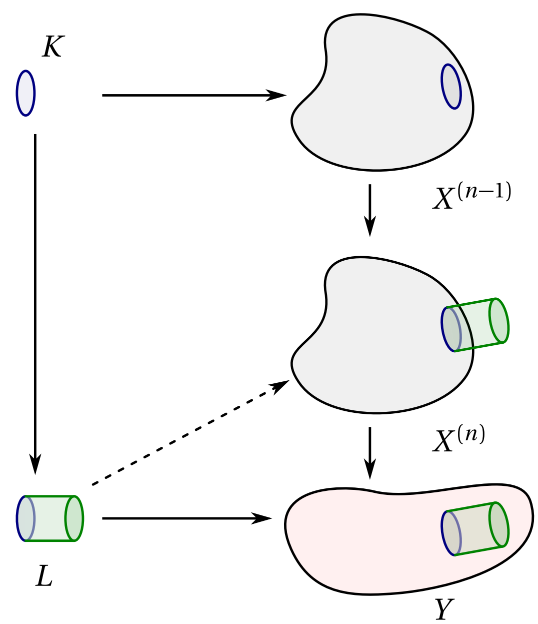
$$\begin{tikzcd}
\coprod_\alpha K_\alpha \arrow[r, "i_\alpha"] \arrow[d] & \coprod_\alpha L_\alpha \arrow[d] \arrow[dr, bend left] & \\
X^{(n-1)} \arrow[r] \arrow[rr, bend right, "f_{n-1}"'] & X^{(n)} \arrow[r, dashed, "f_n"] & Y
\end{tikzcd} \tag{5.1.14}$$

Finally, let $X^{(\infty)} = \operatorname*{colim}_{n\to\infty} X^{(n)}$ and f_∞ the induced map to Y.

We have now factored f into the I-cell complex $X \to X^{(\infty)}$, followed by f_∞, so it remains to prove that f_∞ is I-injective. Given any commuting square

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "i"'] & X^{(\infty)} \arrow[d, "f_\infty"] \\
L \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd} \tag{5.1.15}$$

with $i \in I$, by the countable smallness condition, the map $K \to X^{(\infty)}$ factors through some finite stage $X^{(n-1)}$. By the construction of $X^{(n)}$, this map i is one of the maps we included in the coproduct in (5.1.14), so we get a map $L \to X^{(n)}$. Composing this with $X^{(n)} \to X^{(\infty)}$, gives the dashed map making (5.1.15) commute.



The functoriality is a simple check – given a commuting square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\xi"'] & Y \arrow[d, "\upsilon"] \\
X' \arrow[r, "f'"] & Y',
\end{tikzcd}$$

composing with ξ and υ turns each of the squares in (5.1.13) to a corresponding square for f'. This gives a rule for how to map each $X^{(n)}$ to $(X')^{(n)}$, and therefore a map of their colimits over n, commuting with everything else. This makes the factorization functorial. □

The small-object argument is amazing. It miraculously produces a factorization that has *both* a cell complex and a fibration, even though intuitively, such a factorization does not seem possible. How could a complex built in this way possibly act like a fiber bundle?

The magic happens because we solved all possible lifting problems for $X^{(\infty)} \to Y$, by explicitly attaching a cell for every possible lifting problem. This doesn't work if we only do the cell attachments once, because for every lifting problem that we *solve*, many more new lifting problems are *created*. But, if we perform this procedure countably many times and take the colimit, every lifting problem that we create is solved at the next stage of the construction. That is the miracle of the small-object argument.

Example 5.1.16. If $\mathbf{C} = \mathbf{Top}$ and $I = \{S^{n-1} \to D^n\}_{n\geq 0}$, the small-object argument factors every map $X \to Y$ into a cell complex followed by a map that is *both* a weak equivalence and a Serre fibration:

$$X \xrightarrow{\text{cell complex}} X' \xrightarrow[\text{weak equivalence}]{\text{Serre fibration}} Y.$$

This is surprising, and much stronger than Corollary 1.4.11.

Example 5.1.17. If $\mathbf{C} = \mathbf{Top}$ and $J = \{D^n \times \{0\} \to D^n \times I\}_{n\geq 0}$, the small-object argument factors every map $X \to Y$ into a map that is both a cell complex and a weak equivalence, followed by a Serre fibration:

$$X \xrightarrow[\text{weak equivalence}]{\text{cell complex}} X' \xrightarrow{\text{Serre fibration}} Y.$$

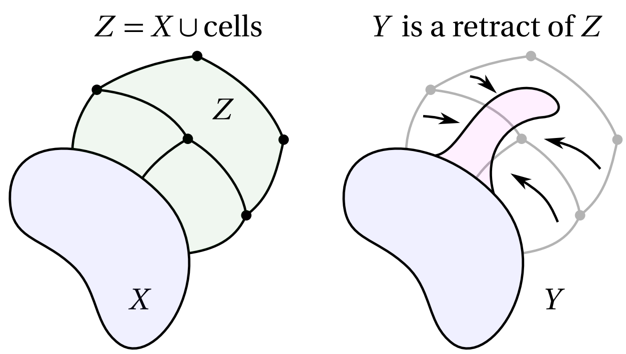
Again, this is stronger than the construction that replaces the map $E \to B$ by the fibration $E \times_B B^I \to B$. The inclusion $E \to E \times_B B^I$ is a weak equivalence, but is not a cell complex in general.

5.1.3 Quillen cofibrations

Definition 5.1.18. A map of spaces $X \to Y$ is a **Quillen cofibration** if it is a retract in the arrow category $\mathbf{Top}^{\{\bullet\to\bullet\}}$ of a relative cell complex.

More generally, in any category $\mathbf{C}$ and set of maps I, an I**-cofibration** is any map that is a retract of an I-cell complex.

It is an exercise to show that, without loss of generality, the cell complex has the same domain X (exercise 4). So $X \to Y$ is a Quillen cofibration if it is a topological inclusion, Y sits inside a larger space Z as a retract, and $X \to Z$ is a relative cell complex.



It follows from Lemma 1.2.2 that every Quillen cofibration is also a Hurewicz cofibration, in other words a map with the homotopy extension property.

Proposition 5.1.19. *Suppose I satisfies Definition 5.1.11. Then a map $A \to B$ is an I-cofibration iff it has the **left lifting property** with respect to every I-injective map. In other words, there is a lift for every commuting square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in which $p\colon X \to Y$ is I-injective.

Proof. We first prove that every I-cofibration has this property. Clearly if the map $A \to B$ is actually in I, then the lift exists, by the definition of I-injective (Definition 5.1.4). By exercise 6 below, the maps with the left lifting property are preserved by coproducts, pushouts, sequential compositions, and retracts. Therefore every retract of an I-cell complex has the left lifting property as well.

Conversely, suppose that $f\colon A \to B$ has the left lifting property. By the small object argument, f factors into an I-cell complex followed by an I-injective map. This allows us to form the following square.

$$\begin{tikzcd} A \arrow[r, "I\text{-cell complex}"] \arrow[d, "f"'] & A' \arrow[d, "I\text{-injective}"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

It follows that f is a retract of the I-cell complex $A \to A'$. □

Corollary 5.1.20. *I-cofibrations are preserved by coproducts, pushouts, sequential compositions, and retracts.*

Proof. Left to exercise 6. □

Example 5.1.21. If $\mathbf{C} = \mathbf{Top}$ and $J = \{D^n \times \{0\} \to D^n \times I\}_{n \geq 0}$, a J-cofibration is a retract of a complex built out of elementary expansions, see Example 5.1.2. These coincide with the maps that have lifts along every Serre fibration.

Proposition 5.1.22. *In* **Top***, with the sets of maps I and J used above, a map $A \to B$ is a J-cofibration iff it is both an I-cofibration and a weak equivalence.*

Proof. It is easy to see that a J-cofibration is both an I-cofibration and a weak equivalence, because the maps in J are all relative cell complexes and weak equivalences.

Conversely, suppose that $f : A \to B$ is both an I-cofibration and a weak equivalence. Then, as in the proof of Proposition 5.1.19, we can factor f into a J-cell complex and a J-injective map (a Serre fibration):

$$\begin{tikzcd} A \arrow[r, "J\text{-cell complex}"] \arrow[d, "f"'] & A' \arrow[d, "J\text{-injective}"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

By 2 out of 3, since both f and the J-cell complex are weak equivalences, the Serre fibration on the right is also a weak equivalence. By Example 5.1.9, the map on the right is therefore I-injective. By Proposition 5.1.19, therefore, a lift exists:

$$\begin{tikzcd} A \arrow[r, "J\text{-cell complex}"] \arrow[d, "f"'] & A' \arrow[d, "I\text{-injective}"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B. \end{tikzcd}$$

Therefore f is a retract of a J-cell complex, in other words, a J-cofibration. □

Definition 5.1.23. An **acyclic Quillen cofibration** is a map in **Top** that satisfies either of the above two equivalent conditions:

- it is both a weak equivalence, and a retract of a cell complex, or
- it is a retract of a complex of elementary expansions.

In summary, Proposition 5.1.22 shows that equivalent subcomplexes can be expressed in terms of elementary expansions. For spaces, this is not such a big deal, because we don't have much difficulty in proving that things like the smash product $X \wedge Y$ preserve weak equivalences. However, the arguments we just performed were very formal, and can be applied to spectra as well. In that setting, this is a powerful technique.

5.2 Model categories

5.2.1 Definition and examples

Let us formalize what we have proven so far about **Top**. Let **C** be any category.

Definition 5.2.1. A **model structure** on **C** is a choice of three subcategories C, W, and F whose maps are called the "cofibrations," "weak equivalences," and "fibrations," respectively. We refer to maps in $W \cap C$ as "acyclic cofibrations," and maps in $W \cap F$ as "acyclic fibrations." These must satisfy four axioms:

- The weak equivalences W are closed under 2-out-of-3 (see Definition 3.3.13).
- Each of the three classes W, C, and F is closed under retracts, in the category of arrows $\mathbf{C}^{\{\bullet\to\bullet\}}$. In other words, for any commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ X \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

  in which the composite maps $A \longrightarrow A$ and $X \longrightarrow X$ are identity maps, if $g \in C$ then $f \in C$. The same applies to W and to F.

- C has the left lifting property with respect to $W \cap F$, and $W \cap C$ has the left lifting property with respect to F.

  In other words, given a commuting square in **C**

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd} \tag{5.2.2}$$

  if $i \in C$ and $p \in W \cap F$, or $i \in W \cap C$ and $p \in F$, a dashed lift exists.

- There is a functorial factorization of each map f into $f = p \circ i$, where i is a cofibration and p is an acyclic fibration. There is another functorial factorization in which i is an acyclic cofibration and p is a fibration.

The last two axioms can be summarized pictorially as follows.

$$\bullet \xrightarrow{C} \bullet \xrightarrow{W\cap F} \bullet \qquad\qquad \bullet \xrightarrow{W\cap C} \bullet \xrightarrow{F} \bullet$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "C"'] & \bullet \arrow[d, "W\cap F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "W\cap C"'] & \bullet \arrow[d, "F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

Definition 5.2.3. We say **C** has all small colimits and limits if for every diagram in **C**, i.e. a functor from a small category **I** to **C**, the colimit and the limit of the diagram both exist.

Definition 5.2.4. A **model category** is a category **C** that has all small colimits and limits, plus a choice of model structure on **C**.

We can neatly organize the conclusions of Theorem 5.1.12, Proposition 5.1.19, and Proposition 5.1.22 by saying that **Top** is a model category:

Theorem 5.2.5 (Quillen)**.** *The category of unbased topological spaces* **Top** *has a model structure in which*

- *the cofibrations are the Quillen cofibrations,*
- *the weak equivalences are the weak homotopy equivalences, and*
- *the fibrations are the Serre fibrations.*

We would like to produce more examples. We can use the same technique: start with a category **C**, and pick a subcategory W, and two sets of maps I and J. We then declare that

- the cofibrations are the I-cofibrations,
- the weak equivalences are the maps in W, and
- the fibrations are the J-injective maps.

Proposition 5.2.6. *This defines a model structure on* **C***, if the following list of conditions is satisfied.*

1. W is closed under 2-out-of-3 and retracts.
2. I satisfies the countable smallness condition.
3. J satisfies the countable smallness condition.
4. J-cell complexes are in $W \cap I$-cof.
5. I-inj $\subseteq W \cap J$-inj.
6. Either $W \cap I$-cof $\subseteq J$-cof or $W \cap J$-inj $\subseteq I$-inj.

Proof. If in the last point if we know that $W \cap J$-inj $\subseteq I$-inj, then the proofs of Theorem 5.1.12, Proposition 5.1.19, and Proposition 5.1.22 all apply, proving that we have a model structure.

If in the last point we have $W \cap I$-cof $\subseteq J$-cof, then we don't need Proposition 5.1.22 to prove that $W \cap I$-cof $\subseteq J$-cof. However, we do need to know that $W \cap J$-inj $\subseteq I$-inj. We show this by dualizing the proof of Proposition 5.1.22, see exercise 5. The rest of the argument then proceeds as before. See also [Hov99, 2.1.19]. □

Definition 5.2.7. A **cofibrantly generated** model structure on **C** is one produced by the method of Proposition 5.2.6.[2] In such a model structure, we always have

- $C = I$-cof, $\quad W \cap C = J$-cof,
- $F = J$-inj, $\quad W \cap F = I$-inj.

So the cofibrations, acyclic cofibrations, fibrations, and acyclic fibrations are all neatly described in terms of the generators in I and J. For this reason, we call I the **generating cofibrations** and J the **generating acyclic cofibrations**.

Think of a cofibrantly generated model category as a category with "generalized cell complexes and fibrations." All of the nice properties of cell complexes and Serre fibrations in **Top** carry over to any cofibrantly generated model category.

Example 5.2.8. The category of based topological spaces $\mathbf{Top}_*$ has a cofibrantly-generated model structure with

- C the Quillen cofibrations,
- W the weak homotopy equivalences,
- F the Serre fibrations,
- $I = \{S^{n-1}_+ \to D^n_+\}_{n\geq 0}$, and
- $J = \{(D^n \times \{0\})_+ \to (D^n \times I)_+\}_{n\geq 0}$.

Essentially, a map is a cofibration, weak equivalence, or fibration if it is such after forgetting about the basepoint. The generators have disjoint basepoints added to them. See exercise 9.

Example 5.2.9. The categories **Top** and $\mathbf{Top}_*$ in this book have objects the compactly generated weak Hausdorff spaces. It turns out that if we drop the weak Hausdorff condition, or both conditions, then a model structure still exists, with essentially the same definition as above.

[2] Our definition is a little bit more stringent than the standard one. The standard one allows cell complexes with uncountably many levels, which in turn makes the smallness condition a little bit weaker. That extra level of generality is not needed in any of our examples. See [Hov99] for details.

Example 5.2.10. For any ring R, the category $\mathbf{Ch}_{\geq 0}(R)$ of nonnegatively-graded chain complexes of R-modules has a cofibrantly-generated model structure in which

- the cofibrations are the maps that in each degree are injective and with cokernel a projective R-module,
- the weak equivalences are the quasi-isomorphisms, i.e. the maps inducing isomorphisms on homology,
- the fibrations are the maps that are surjective in all positive degrees,
- I consists of all maps of chain complexes of the form $(0 \to R) \to (R \to R)$, and their shifts, and also $(0) \to (R)$ in degree zero, and
- J consists of all maps from the zero complex to a shift of the complex $(R \to R)$.

In the definitions of I and J, every unnamed map $R \to R$ is the identity map of R. Note that attaching a "cell" in I has the effect of adding a summand $\oplus R$ to one level of the chain complex, and also specifying its image under the boundary map, if we added it above level 0. The verification of the conditions in Proposition 5.2.6 is left to exercise 11.

The larger category of unbounded chain complexes has a similar model structure with essentially the same generators I and J, but with cofibrations and fibrations a little different – see exercise 12.

The following is the main theorem of this chapter.

Theorem 5.2.11 (Bousfield-Friedlander)**.** *The category of (sequential) spectra* **Sp** *has the following two model structures.*

In the **level model structure**,

- the cofibrations are the retracts of the relative cell complex spectra,
- the weak equivalences are the level equivalences $X_n \xrightarrow{\sim} Y_n$, and
- the fibrations are the level fibrations: the maps $X \to Y$ in which each level $X_n \to Y_n$ is a Serre fibration.

We establish this model structure in Theorem 5.3.37.

In the **stable model structure**,

- the cofibrations are the same, the retracts of the relative cell complex spectra,
- the weak equivalences are the stable equivalences, and

- the fibrations are the stable fibrations: the maps $X \to Y$ in which each level $X_n \to Y_n$ is a Serre fibration and each square

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \Omega X_{n+1} \arrow[d] \\ Y_n \arrow[r] & \Omega Y_{n+1} \end{tikzcd}$$

  is a homotopy pullback square.

We establish this model structure in Section 5.6.

For the level model structure, the generating cofibrations are the maps

$$I = \{F_n S^{k-1}_+ \to F_n D^k_+\}_{n,k\geq 0}$$

from Example 5.1.9, and the generating acyclic cofibrations are the maps

$$J = \{F_n(D^k \times \{0\})_+ \to F_n(D^k \times I)_+\}_{n,k\geq 0}.$$

In the stable model structure, we use the same set I (Definition 5.6.1), but a larger set J (Definition 5.6.3).

Remark 5.2.12. Not every model structure is cofibrantly generated. There is a **Strøm model structure** on **Top** in which

- the cofibrations are the classical ones, i.e. the maps with the homotopy extension property,
- the weak equivalences are the homotopy equivalences, and
- the fibrations are the Hurewicz fibrations.

This is not (at least in any obvious way) generated by sets of maps I and J, so it cannot be proven by the method we've developed thus far. Fortunately, all of the examples we actually care about are cofibrantly generated, so we won't spend much time thinking about model structures that are not cofibrantly generated.

Our goal in the remainder of the chapter is to develop the properties of model categories in general, and to prove Theorem 5.2.11, giving two model structures on spectra.

5.2.2 Cofibrant and fibrant objects

Let **C** be a model category. So it has a model structure, and also, it has all small colimits and limits. In particular, **C** has an initial object $\emptyset$, and a terminal object $*$. For any object X, there is a unique map $\emptyset \to X$, and a unique map $X \to *$.

Definition 5.2.13. We say that

- X is **cofibrant** if $\emptyset \to X$ is a cofibration.
- X is **fibrant** if $X \to *$ is a fibration.
- X is **bifibrant** if it is both cofibrant and fibrant.

You should think of the cofibrant and fibrant objects as the "good" or "well-behaved" objects.

Example 5.2.14. In **Top**, the initial object is the empty set, so a space X is cofibrant iff it is a retract of a cell complex.

The terminal object is the one-point space. It is easy to see that $X \to *$ is always a Serre fibration, so every space is fibrant.

Example 5.2.15. In $\mathbf{Ch}_{\geq 0}(R)$, the initial and terminal object are both the zero complex. A complex is cofibrant if it is a retract of a levelwise free complex. Equivalently, if it is levelwise projective. Every complex is fibrant.

Example 5.2.16. In **Sp**, the initial and terminal object are both the zero spectrum. In both model structures, a spectrum is cofibrant if it is a retract of a cellular spectrum. In the level model structure, every spectrum is fibrant. In the stable model structure, the fibrant spectra are precisely the Ω-spectra!

Definition 5.2.17. The **cofibrant replacement** of an object X is the object QX and map $QX \to X$ obtained by factoring the unique map $\emptyset \to X$ into a cofibration followed by an acyclic fibration:

$$\emptyset \longrightarrow QX \xrightarrow{\sim} X$$

Note that Q is a functor, and $QX \to X$ is a natural transformation.

Similarly, the **fibrant replacement** of an object X is the object RX and map $X \to RX$ obtained by factoring the unique map $X \to *$ into an acyclic cofibration followed by a fibration:

$$X \xrightarrow{\sim} RX \longrightarrow *.$$

Note that R is a functor, and $X \to RX$ is a natural transformation.

Lemma 5.2.18. *The cofibrant replacement and fibrant replacement functors are homotopical – they preserve weak equivalences.*

Proof. This follows quickly from the 2 out of 3 property. □

Lemma 5.2.19. *Q preserves fibrant objects and R preserves cofibrant objects. So QRX and RQX are both bifibrant.*

Proof. If X is fibrant, then $X \to *$ is a fibration, and $QX \to X$ is an acyclic fibration. Composing them, we conclude that $QX \to *$ is a fibration. The argument for R is similar. □

Remark 5.2.20. In **Sp** with the stable model structure, these operations behave like the Q and R that we defined in Theorem 2.6.12 and Proposition 2.2.9. The difference is that:

- They have *better properties*: $QX \to X$ is now a fibration and $X \to RX$ is now a cofibration, in addition to these maps being weak equivalences.
- However, they are *less explicit*, especially compared to the construction in Proposition 2.2.9.

5.2.3 Fundamental properties

Lemma 5.2.21. *In any model category* **C** *the following statements hold:*

- f is an cofibration iff it has the left-lifting property with respect to all acyclic fibrations.
- f is an acyclic cofibration iff it has the left-lifting property with respect to all fibrations.
- f is an fibration iff it has the right-lifting property with respect to all acyclic cofibrations.
- f is an acyclic fibration iff it has the right-lifting property with respect to all cofibrations.
- Every isomorphism in **C** is a cofibration, a fibration, and a weak equivalence.
- The cofibrations are closed under pushouts, coproducts, sequential compositions, and retracts. The same applies to acyclic cofibrations.
- The fibrations are closed under pullbacks, products, sequential inverse compositions, and retracts. The same applies to acyclic fibrations.

Proof. The first four points are similar to the proof of Proposition 5.1.19. For example, if $f : A \to B$ is a map that has the left lifting property with respect to all fibrations, we factor f into an acyclic cofibration followed by a fibration. This gives a commuting square

$$\begin{tikzcd} A \arrow[r, "\in W\cap C"] \arrow[d, "f"'] & A' \arrow[d, "\in F"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B. \end{tikzcd}$$

The lift exists because f has the left lifting property. But this shows that f is a retract of a map in $W \cap C$, and therefore $f \in W \cap C$.

An isomorphism clearly has the left lifting property and right lifting property with respect to every map. Therefore it is a cofibration, a fibration, and a weak equivalence.

The last two points are left as exercises, see exercises 6 and 7. □

In any model category **C**, it is possible to talk about homotopies. We just have to define a suitable replacement for the cylinder $X \times I$ in topological spaces.

Definition 5.2.22. For any cofibrant object X, a **cylinder** on X is any object Z fitting into a diagram of the following form.

$$\begin{tikzcd} & Z \arrow[dr, "\in W\cap F"] & \\ X \amalg X \arrow[ur, "\in C"] \arrow[rr, "(\mathrm{id}_X, \mathrm{id}_X)"'] & & X. \end{tikzcd}$$

Example 5.2.23. In **Top**, the product $X \times I$ is an example of a cylinder, because if X is a cell complex, the inclusion of $X \times \{0\}$ and $X \times \{1\}$ into $X \times I$ is also a relative cell complex. The same conclusion follows for retracts of cell complexes. The projection $X \times I \to X$ is clearly a fibration as well.

We can similarly define a mapping cylinder for a map $f : X \to Y$ by factoring the map $(f, \mathrm{id}_Y) : X \amalg Y \to Y$ into a cofibration and an acyclic fibration. This idea is used to prove the following.

Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor from a model category **C** to a category **D**. Suppose that **D** has a class of weak equivalences, satisfying 2-out-of-3.

Lemma 5.2.24. *(Ken Brown's Lemma) Suppose F takes acyclic cofibrations $X \to Y$ between cofibrant objects $X, Y \in \mathbf{C}$ to weak equivalences $F(X) \xrightarrow{\sim} F(Y)$ in* **D**. *Then F takes* all *weak equivalences of cofibrant objects to weak equivalences.*

Proof. The idea is to use the mapping cylinder. Given any weak equivalence of cofibrant objects $f : X \to Y$, factor $(f, \mathrm{id}_Y) : X \amalg Y \to Y$ into a cofibration and an acyclic fibration, and let M_f be the intermediate space. Since f is a weak equivalence and Y is cofibrant, the composition $X \to X \amalg Y \to M_f$ is both a weak equivalence and a cofibration. Similarly $Y \to M_f$ is an acyclic cofibration. So F sends both of these maps to weak equivalences.

The projection $M_f \to Y$ is a left inverse to $Y \to M_f$, and is therefore F sends it to a weak equivalence as well. The map f is now the composition $X \to M_f \to Y$ of maps that F sends to weak equivalences, so F sends $X \to Y$ to a weak equivalence. □

Lemma 5.2.25. *Suppose F takes acyclic fibrations $X \to Y$ between fibrant objects $X, Y \in$ $\mathbf{C}$ to weak equivalences $F(X) \xrightarrow{\sim} F(Y)$ in $\mathbf{D}$. Then F takes* all *weak equivalences of fibrant objects to weak equivalences.*

Proof. This is left to exercise 21. □

Returning to ordinary cylinders, we say that two maps $f, g\colon X \rightrightarrows Y$ are **left homotopic** if there is some extension of $(f, g)\colon X \amalg X \to Y$ to $Z \to Y$, for some cylinder object $X \amalg X \to Z \to X$. We will only consider this relation when X is cofibrant and Y is fibrant, in which case it does not depend on the choice of Z.

We skip the proof of the following, as it is rather long and introduces several more definitions that we will not need. See [Hov99, §1.2] for a full proof. Recall that $\mathrm{Ho}\,\mathbf{C} = \mathbf{C}[W^{-1}]$ refers to the category in which the weak equivalences of $\mathbf{C}$ have been inverted.

Theorem 5.2.26 (Fundamental Theorem of Model Categories)**.** *If X is cofibrant and Y is fibrant, then the maps in the homotopy category $\mathrm{Ho}\,\mathbf{C}(X, Y)$ are the maps in the original category $\mathbf{C}(X, Y)$, modulo the relation of left homotopy.*

As in Proposition 3.1.28, since Q and R are functors that are weakly equivalent to the identity, we get equivalences of categories

$$\begin{tikzcd}
\mathrm{Ho}\,\mathbf{C}^{cf} \arrow[r, "\sim"'] \arrow[d, "\sim"] & \mathrm{Ho}\,\mathbf{C}^{f} \arrow[l, bend right, "Q"'] \arrow[d, "\sim"'] \\
\mathrm{Ho}\,\mathbf{C}^{c} \arrow[r, "\sim"] \arrow[u, bend left, "R"] & \mathrm{Ho}\,\mathbf{C} \arrow[l, bend left, "Q"] \arrow[u, bend right, "R"']
\end{tikzcd}$$

where $\mathbf{C}^{cf} \subseteq \mathbf{C}$ denotes the bifibrant objects, $\mathbf{C}^{c}$ the cofibrant objects, and $\mathbf{C}^{f}$ the fibrant objects. As in Remark 3.1.29, we conclude that for any pair of objects $X, Y \in \mathbf{C}$, the maps in the homotopy category can be computed as

$$\begin{aligned}
\mathrm{Ho}\,\mathbf{C}(X, Y) &\cong \mathbf{C}(QX, RY)/(\text{left homotopy}) \\
&\cong \mathbf{C}(QRX, QRY)/(\text{left homotopy}) \\
&\cong \mathbf{C}(RQX, RQY)/(\text{left homotopy}).
\end{aligned}$$

Corollary 5.2.27 (Whitehead's Theorem for Model Categories)**.** *Between bifibrant objects, a map $f\colon X \to Y$ is a weak equivalence iff it is a homotopy equivalence.*

In other words, model categories are a formal setting where we get the same results as those in Section 3.1. Note that we have already proven the Fundamental Theorem and Whitehead's Theorem in $\mathbf{Sp}$. (Proposition 3.1.40 and Corollary 2.6.17)

Remark 5.2.28. For the Quillen model structure on spaces, this recovers Proposition 3.1.26, that the homotopy category is equivalent to CW complexes and homotopy classes of maps. For Quillen's model structure on chain complexes, this recovers the fact that the derived category $\mathscr{D}_{\geq 0}(R) = \operatorname{Ho}\mathbf{Ch}_{\geq 0}(R)$ is equivalent to projective chain complexes and chain homotopy classes of maps.

5.2.4 Properness

Definition 5.2.29. A model category $\mathbf{C}$ is **left proper** if for every pushout square as shown, with i a cofibration and f a weak equivalence, the map g is also a weak equivalence.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & B \cup_A C \end{tikzcd}$$

Dually, $\mathbf{C}$ is **right proper** if for every pullback square as shown, with j a fibration and g a weak equivalence, the map f is also a weak equivalence.

$$\begin{tikzcd} B \times_D C \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$

We say $\mathbf{C}$ is **proper** if both of these conditions hold.

Example 5.2.30. The Quillen model structure on **Top** is proper – this is one of the basic facts about spaces that we recalled in Theorem 1.5.10 and Theorem 1.5.24.

Example 5.2.31. Both the level and stable model structures on **Sp** are proper – this can be deduced from the stability theorems of Section 2.4, see exercise 8.

Lemma 5.2.32 (Gluing lemma)**.** *If $\mathbf{C}$ is left proper, any weak equivalence of pushout diagrams*

$$\begin{tikzcd} X \arrow[d, "\sim"] & A \arrow[l, "i"'] \arrow[r] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ X' & A' \arrow[l, "i'"] \arrow[r] & Y', \end{tikzcd}$$

each of which the maps i and i' are cofibrations, induces a weak equivalence of pushouts

$$X \cup_A Y \xrightarrow{\sim} X' \cup_{A'} Y'.$$

In fact, the gluing lemma is equivalent to $\mathbf{C}$ being left proper. We omit the proof because it is formal but quite long. The dual result for pullbacks along fibrations holds if $\mathbf{C}$ is right proper.

Remark 5.2.33. In *any* model category, the pushout preserves weak equivalences if $A \to X$ is a cofibration *and* all three objects A, X, and Y are cofibrant. The proof is left to exercise 13. Dually, the pullback preserves equivalences if one of the maps is a fibration *and* all three objects are fibrant. We can therefore talk about the left-derived pushout, and right-derived pullback, in any model category.

5.2.5 In summary

A model structure on **C** is three classes of maps C, W, F, such that W has 2 out of 3, all three have retracts, and we have factorizations and lifts as pictured.

$$\bullet \xrightarrow{C} \bullet \xrightarrow{W\cap F} \bullet \qquad\qquad \bullet \xrightarrow{W\cap C} \bullet \xrightarrow{F} \bullet$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "C"'] & \bullet \arrow[d, "W\cap F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "W\cap C"'] & \bullet \arrow[d, "F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

A cofibrantly generated model structure arises from sets of maps I and J so that

- C = retracts of I-cell complexes, $\quad W \cap C$ = retracts of J-cell complexes,
- $F = J$-injective maps, $\quad W \cap F = I$-injective maps.

This structure makes it easier to study objects in **C** up to weak equivalence, because:

- the Fundamental Theorem and Whitehead's Theorem make it feasible to count maps in the homotopy category,
- there is a built-in notion of which objects are "nice" (cofibrant and/or fibrant), and everything is weakly equivalent to a nice object, and
- Ken Brown's lemma and the generating acyclics J make it easier to check that a given functor F preserves equivalences of nice objects. (The weak equivalences break up into "chunks.")

5.3 Diagrams of spaces

In this section we give many, many examples of model categories. For any small topological category **I**, the category of **I**-shaped diagrams $\mathbf{Top}^{\mathbf{I}}$ has a model structure.

In particular, we show that spectra are the same thing as diagrams on some category **I**. We use this to prove the existence of the level model structure for spectra.

5.3.1 Diagrams of unbased spaces

Definition 5.3.1. A **topological category** or **category enriched in spaces** is

- a collection of objects $\mathrm{ob}\,\mathbf{C}$,
- for each pair of objects a, b, a topological space of morphisms $\mathbf{C}(a, b)$,
- continuous composition maps $\mathbf{C}(a, b) \times \mathbf{C}(b, c) \to \mathbf{C}(a, c)$, and
- for each object a a unit map $\{\mathrm{id}_a\} \to \mathbf{C}(a, a)$,

such that the composition in $\mathbf{C}$ is associative and each id_a acts as an identity element. A **functor** of topological categories $F\colon \mathbf{C} \to \mathbf{D}$ is just a functor such that the maps $\mathbf{C}(a, b) \to \mathbf{D}(F(a), F(b))$ are continuous.

Example 5.3.2. Every ordinary category is a topological category, by giving the mapping sets the discrete topology.

Example 5.3.3. The categories **Top** and $\mathbf{Top}_*$ are topological in the obvious way. Recall from Definition 2.3.12 that **Sp** is also a topological category, the space of maps $\mathbf{Sp}(X, Y)$ being defined as the product of the mapping spaces $\mathrm{Map}(X_n, Y_n)$, restricted to the subspace of those maps that commute with the bonding maps of X and Y.

Example 5.3.4. If $\mathbf{I}$ is a topological category, its opposite category $\mathbf{I}^{\mathrm{op}}$ is also a topological category in a canonical way.

Recall that a category or topological category $\mathbf{I}$ is **small** if its objects form a set, rather than a proper class.

Definition 5.3.5. Let $\mathbf{I}$ be any small category, or more generally, any small topological category. A **diagram** (of spaces) is a continuous functor $X\colon \mathbf{I} \to \mathbf{Top}$. Equivalently, it consists of

- a space $X(i)$ for each $i \in \mathrm{ob}\,\mathbf{I}$, and
- a continuous composition map $X(i) \times \mathbf{I}(i, j) \to X(j)$,

such that the identity map of i acts by the identity on $X(i)$, and $g \circ f$ acts by the composite of g and f. A **map of diagrams** is a natural transformation $X \to Y$. Equivalently, it consists of continuous maps $X(i) \to Y(i)$ for all $i \in \mathrm{ob}\,\mathbf{I}$, commuting with the action of every map $f\colon i \to j$. We let $\mathbf{Top}^{\mathbf{I}}$ refer to the category of diagrams.

Remark 5.3.6. The above definition implicitly uses the fact that a continuous map $\mathbf{I}(i, j) \to \mathrm{Map}(X(i), X(j))$ is the same data as a continuous map $X(i) \times \mathbf{I}(i, j) \to X(j)$.

Example 5.3.7.
- If $\mathbf{I} = \{\bullet\}$ has a single object and only the identity morphism, an $\mathbf{I}$-diagram is just a space. A map of diagrams is a map of spaces.
- If $\mathbf{I} = \{\bullet, \bullet\}$ has two objects but only identity morphisms, an $\mathbf{I}$-diagram is a pair of spaces (X, Y), with no relationship. (In other words, we are not asking for Y to be a subspace of X.) A map of diagrams is a pair of maps of spaces.
- If $\mathbf{I} = \{0 \to 1\}$ has a single nontrivial morphism, an $\mathbf{I}$-diagram is an arrow $X(0) \to X(1)$. A map of diagrams is a commuting square.
- If $\mathbf{I} = \{1 \leftarrow 0 \to 2\}$ is the span category, an $\mathbf{I}$-diagram is a span $X(1) \leftarrow X(0) \to X(2)$. A map of diagrams is a map of spans. As remarked in Example 3.3.26, there is a functor $\mathbf{Top}^{\{1\leftarrow 0 \to 2\}} \to \mathbf{Top}$ taking each span to its pushout.
- If $\mathbf{I}$ is the commuting square category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

  then an $\mathbf{I}$-diagram is a commuting square. A map of diagrams is a commuting cube.

Definition 5.3.8. A **topological monoid** is a space G with a multiplication map $G \times G \to G$ that is continuous, associative, and has unit $1 \in G$:

$$g \cdot 1 = g = 1 \cdot g, \qquad g_1(g_2 g_3) = (g_1 g_2) g_3.$$

Note that this is the same thing as a topological category $\mathbf{I}$ with one object.

A **topological group** is a topological monoid with inverses, such that the map $g \mapsto g^{-1}$ is continuous.

A **right action** of G on a space X is a continuous map $X \times G \to G$ that is associative and unital in the sense that

$$x \cdot 1 = x, \qquad (x g_1) g_2 = x(g_1 g_2).$$

A **left action** of G on Y is a continuous map $G \times Y \to Y$ such that

$$1 \cdot y = y, \qquad g_2(g_1 y) = (g_2 g_1) y.$$

In this case we call X a **left G-space** and Y a **left G-space**.

A map G-spaces $f\colon Y \to Z$ is **equivariant** if it commutes with the G-action. For left actions, this means that $f(g y) = g f(y)$ for all $y \in Y$ and $g \in G$.

Example 5.3.9. If G is any topological monoid, then as noted above, it also defines a category with one object. A G-diagram is the same thing as a space with right G-action, and a map of G-diagrams is the same thing as an equivariant map. Similarly, an G^{op}-diagram is a space with a left G-action.[3]

Definition 5.3.10. For each object $i \in \mathrm{ob}\,\mathbf{I}$ and space A, the **free diagram** $F_i A$ is the $\mathbf{I}$-diagram whose value at j is the product

$$(F_i A)(j) = A \times \mathbf{I}(i, j),$$

with $\mathbf{I}$ acting by composition on the $\mathbf{I}(i, j)$ term.

Lemma 5.3.11. *This is the left adjoint of the forgetful functor* $\mathrm{ev}_i : \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ *sending the diagram X to the space $X(i)$.*

Proof. It is straightforward to see that a map of *diagrams* $F_i A \to X$ is the same data as a map of *spaces* $A \to X(i)$. Once you know where $A \times \{\mathrm{id}_i\}$ goes in X, the rest of the maps $A \times \{f\}$ are determined by the fact that $F_i A \to X$ has to be a map of diagrams. □

Example 5.3.12.

- If $\mathbf{I} = \{\bullet\}$, a free diagram is any space (A).
- If $\mathbf{I} = \{\bullet, \bullet\}$, a free diagram is any pair in which one of the spaces is empty: $(A, \emptyset)$ or $(\emptyset, A)$.
- If $\mathbf{I} = \{\bullet \to \bullet\}$, a free diagram is either an arrow of the form $A = A$, or $\emptyset \to A$.
- If $\mathbf{I} = \{\bullet \leftarrow \bullet \to \bullet\}$ is the span category, a free diagram is one of the following three forms:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \\ A \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \\ A \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \\ \emptyset \end{tikzcd}$$

- If $\mathbf{I}$ is the commuting square category, a free diagram is one of the following four forms:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \end{tikzcd} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ A \arrow[r] & A \end{tikzcd} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ \emptyset \arrow[r] & A \end{tikzcd} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ \emptyset \arrow[r] & A. \end{tikzcd}$$

- If G is a topological monoid, a free G-diagram is a space of the form $A \times G$, with G acting on the right by $(a, g_0) \cdot g_1 = (a, g_0 g_1)$.

[3] Of course, which one is left or right is an artifact of the conventions we choose to use. If the opposite convention feels more natural to you, I insist that you use that one instead.

5.3.2 The projective model structure

Our next goal is to prove that **I**-diagrams are a model category.

Lemma 5.3.13. *The category* $\mathbf{Top}^{\mathbf{I}}$ *has all colimits and limits. The colimit or limit of a diagram of diagrams is computed at each object* $i \in \mathbf{I}$ *separately.*

Definition 5.3.14. A map of diagrams $X \to Y$ is a **level equivalence** if $X(i) \to Y(i)$ is a weak equivalence for every $i \in \operatorname{ob} \mathbf{I}$. Similarly $X \to Y$ is a **level fibration** if $X(i) \to Y(i)$ is a Serre fibration for every i.

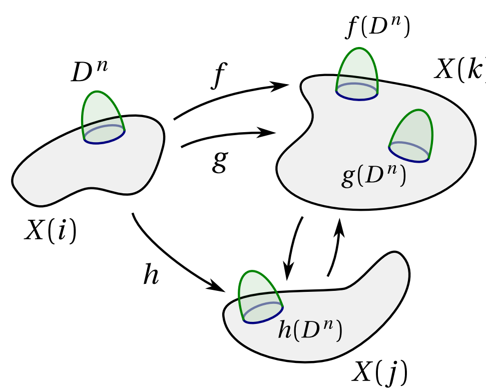
Example 5.3.15. For spaces with G-action, a level equivalence is an equivariant map $X \xrightarrow{\sim} Y$, that is a weak equivalence if we forget the G-action. These are often called **Borel equivalences**, or coarse equivalences, and the homotopy category $\operatorname{Ho} \mathbf{Top}^G$ formed using these equivalences is the **Borel homotopy category**. We have this special notation because there is also a more sophisticated kind of weak equivalence, that keeps track of the subspaces of H-fixed points X^H for $H \leq G$. See **??**.

Let $I = \{S^{n-1} \to D^n\}_{n \geq 0}$ be the class of cells from Example 5.1.2. Let FI be the set of all maps obtained by applying the free diagram functors F_i of Definition 5.3.10 to the maps in I:

$$FI = \{ F_i S^{n-1} \to F_i D^n : n \geq 0,\ i \in \operatorname{ob} \mathbf{I} \}.$$

A **relative cell complex of diagrams** is an FI-cell complex. Note that if $X \to Y$ is a cell complex of diagrams then $X(j) \to Y(j)$ is not necessarily a relative cell complex in the traditional sense. It is built out of the "cells" of the form $\mathbf{I}(i,j) \times (S^{n-1} \to D^n)$.

Intuitively, an FI-cell complex is obtained by starting with a diagram X, and attaching cells to the spaces $X(i)$. But, every time you attach a cell to one space $X(i)$, you also have to attach a cell to $X(j)$ for *every* map $f : i \to j$. That way, your new cell in $X(i)$ has somewhere to go under the map f, so that you still have a diagram.



As in Example 5.1.9, a square of diagrams

$$\begin{tikzcd} F_i S^{n-1} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ F_i D^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

is equivalent to a square of spaces

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X(i) \arrow[d, "p_i"] \\ D^n \arrow[r] \arrow[ur, dashed] & Y(i), \end{tikzcd}$$

using the adjunction between F_i and ev_i. Therefore, a map of diagrams $X \to Y$ is FI-injective iff it is a weak equivalence and Serre fibration at each level. In other words, if it is a level equivalence and a level fibration.

We define FJ similarly:

$$FJ = \{\, F_i(D^n \times \{0\}) \to F_i(D^n \times I) : n \geq 0,\ i \in \mathrm{ob}\,\mathbf{I}\,\}.$$

A map of diagrams $X \to Y$ is FJ-injective iff it is a level fibration.

A **free cofibration** of diagrams is any retract of an FI-cell complex.

Theorem 5.3.16. *The category* $\mathbf{Top}^{\mathbf{I}}$ *has a* ***projective model structure*** *in which*

- *the cofibrations are free cofibrations,*
- *the weak equivalences are level equivalences, and*
- *the fibrations are the level fibrations.*

Proof. We check that the level equivalences, and the sets of maps FI and FJ defined just above, satisfy the conditions of Proposition 5.2.6.

1. W **is closed under 2-out-of-3 and retracts.** Can be checked at each $i \in \mathrm{ob}\,\mathbf{I}$ separately.

2. FI **satisfies the countable smallness condition.** By the free-forget adjunction, a map $F_i S^{n-1} \to X$ is the same as a map $S^{n-1} \to X(i)$. It is easy to see that $\mathbf{I}(i,j) \times (S^{n-1} \to D^n)$ is a closed inclusion. It follows that the skeleta of any FI-cell complex of diagrams are related by closed inclusions. So, by Section 1.7, exercise 23, the map from S^{n-1} factors through some finite stage.

3. FJ **satisfies the countable smallness condition.** Same as the previous point.

4. FJ**-cell complexes are in** $W \cap FI$**-cof.** The proof is essentially the same as when we proved Theorem 5.2.5.

5 & 6. FI**-inj** $= W \cap FJ$**-inj.** As discussed above, by the free-forget adjunction, a map of diagrams is FI-injective iff it is a level equivalence and a level fibration, iff it is FJ-injective.

□

The weak equivalences and fibrations in this model category are pretty boring, but the cofibrations are interesting.

Example 5.3.17. We say a diagram X is **cellular** if the map from the empty digram $\emptyset \to X$ is a cell complex of diagrams, and **cofibrant** if $\emptyset \to X$ is a free cofibration, in other words, a retract of a cell complex of diagrams.

- If $\mathbf{I} = \{\bullet, \bullet\}$, a diagram (X_0, X_1) is cellular iff both X_0 and X_1 are cell complexes.
- If $\mathbf{I} = \{\bullet \to \bullet\}$, a diagram $X_0 \to X_1$ is cellular iff X_0 is a cell complex and $X_0 \to X_1$ is a relative cell complex.
- For $\mathbf{I} = G$, a cell complex of diagrams is called a **free G-cell complex**. It is a complex built out of cells of the form $D^n \times G$, attached along the boundary $S^{n-1} \times G$.

5.3.3 Homotopy colimits and homotopy orbits

We give a short example to illustrate the power of this theory, by proving that the colimit has a left-derived functor. Consider the functor $\mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ that takes every diagram X to its colimit $\operatorname{colim} X$.

Proposition 5.3.18. *This colimit functor preserves all equivalences between cofibrant diagrams.*

Proof. We first show that each map in FJ goes to a weak equivalence. The colimit of a free diagram is the same space back:

$$\operatorname*{colim}_{\mathbf{I}}(F_i A) \cong A,$$

so each map in FJ goes to the map of spaces $D^n \times \{0\} \to D^n \times I$, which is clearly a weak equivalence.

Similarly, a coproduct of maps in FJ goes to a coproduct of maps of the form $D^n \times \{0\} \to D^n \times I$, which is a weak equivalence. A pushout of maps in FJ goes to a pushout in spaces, because colim is a left adjoint. A pushout of maps of the form $D^n \times \{0\} \to D^n \times I$ is a collection of elementary expansions, which we have already noted is a weak equivalence. Finally, a sequential composition of such maps is a weak equivalence, because it is a colimit along a sequence of closed inclusions, using Section 1.7, exercise 23.

In conclusion, every FJ-cell complex goes to a weak equivalence. Every retract of an FJ-cell complex goes to a retract of a weak equivalence, which is therefore a weak equivalence. In conclusion, colim sends every acyclic cofibration (FJ-cofibration, or retract of an FJ-cell complex) to a weak equivalence. By Ken Brown's Lemma (Lemma 5.2.24), we conclude that all weak equivalences of cofibrant diagrams go to weak equivalences. □

Remark 5.3.19. There is a much faster proof of Proposition 5.3.18 using Quillen adjunctions – see exercise 20. We give this version of the proof here as an illustration, before we apply the technique to diagram spectra in the next chapter.

Definition 5.3.20. The **homotopy colimit** is the left-derived functor of the colimit. It exists because, by Theorem 5.3.16, we can replace each diagram X by an equivalent cofibrant diagram QX, and by Proposition 5.3.18, the colimit functor preserves weak equivalences between cofibrant diagrams.

Note that we define homotopy colimits twice in this book – the first definition is here, the second is in **??**. They are always equivalent to each other, so it does not matter which one you use.

Example 5.3.21. If $\mathbf{I}$ is discrete, meaning it has no non-identity morphisms, then the homotopy colimit is obtained by replacing each of the spaces $X(i)$ by a cell complex $QX(i)$, then taking their coproduct.

If $\mathbf{I}$ is the span category $\{\bullet \leftarrow \bullet \rightarrow \bullet\}$ the homotopy colimit is obtained by making all the spaces in the span cell complexes and the maps inclusions of subcomplexes, then taking the pushout.

If $\mathbf{I}$ is a sequential colimit category

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots,$$

the homotopy colimit is obtained by making each space X_n into a cell complex, and the maps $X_n \to X_{n+1}$ into subcomplexes, then taking the colimit.

In each of these three cases, we can see that the homotopy colimit as defined in Definition 5.3.20 is equivalent (though not isomorphic) to the homotopy colimit as defined in Section 1.5. Of course, since they're both examples of left-derived functors of the colimit, they *must* be equivalent, by Corollary 3.4.3.

Recall that for a space with G-action X, the orbit space X_G is obtained by identifying every point x to all of its images under the G-action, $x \sim xg$. This is the same thing as taking the colimit of X as a diagram over the category G. Recall also that a Borel equivalence is a map that is both G-equivariant, and a weak equivalence after forgetting the G-action.

Corollary 5.3.22. *Any Borel equivalence of free G-cell complexes $X \xrightarrow{\sim} Y$ induces a weak equivalence of orbit spaces $X_G \xrightarrow{\sim} Y_G$.*

This is not true if we drop the "free" condition. For instance, the infinite dimensional sphere S^∞ has a $\mathbb{Z}/2$-action by the antipodal map. The one-point space $*$ has a trivial action by $\mathbb{Z}/2$. The map $S^\infty \to *$ is a Borel equivalence, but on orbits, it induces the map $\mathbb{RP}^\infty \to *$, which is decidedly not a weak equivalence.

Definition 5.3.23. The **homotopy orbit space** X_{hG} is the is the left-derived functor of the orbits, in other words, the homotopy colimit of X as a diagram over G. Concretely, we replace X by an equivalent free G-cell complex, then take its G-orbits.

This is a left-derived functor, so it preserves all equivalences. Therefore every Borel equivalence $X \xrightarrow{\sim} Y$ induces an equivalence of homotopy orbits $X_{hG} \to Y_{hG}$.

Example 5.3.24. The homotopy orbits of both S^∞ and $*$ under the $\mathbb{Z}/2$-action are homotopy equivalent to $\mathbb{RP}^\infty$.

The homotopy colimit generalizes to the **homotopy coend**, see exercise 15.

Remark 5.3.25. The constructions of the homotopy colimit and homotopy orbits in this section are not very explicit – you use the small-object argument to replace the diagram X by a huge diagram QX with an absurd number of cells, then take its colimit. In exercise 14 and **??**, we give weakly equivalent, and much more explicit, models for the homotopy colimit.

Remark 5.3.26. One might hope that we can define homotopy limits in an analogous way, by showing that the limit functor $\mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$ preserves equivalences of fibrant diagrams. Unfortunately, it does not! We would have to change the model structure to one in which the *cofibrations* are the level cofibrations, and the fibrations are defined by a right lifting property. This would be called the **injective model structure** on $\mathbf{Top}^{\mathbf{I}}$. Unfortunately, it does not exist!

There are a few standard ways around this: we could pass to the setting of simplicial sets where the injective model structure does exist, or we could argue abstractly that a right-derived functor of the limit exists, as in [DHKS04]. We'll follow the approach of [Shu06] and use bar constructions to derive the limit instead.

5.3.4 Based diagrams

Now we consider the category $\mathbf{Top}_*^{\mathbf{I}}$ of diagrams of based spaces. This requires a few changes to the definitions, but the proofs do not change.

Definition 5.3.27. A **based topological category** or **category enriched in based spaces** is

- a collection of objects $\operatorname{ob}\mathbf{C}$,

- for each pair of objects a, b, a based space of morphisms $\mathbf{C}(a, b)$,
- continuous composition maps $\mathbf{C}(a, b) \wedge \mathbf{C}(b, c) \to \mathbf{C}(a, c)$, and
- for each object a a unit map $\{\mathrm{id}_a\} \to \mathbf{C}(a, a)$,

with the same conditions as in Definition 5.3.1. For a functor of based categories, we require that the maps $\mathbf{C}(a, b) \to \mathbf{D}(F(a), F(b))$ preserve the basepoint.

Remark 5.3.28. A based topological category the same thing as a topological category in which there is a "zero map" $*: a \to b$ for every pair of objects a and b, such that the composition of any morphism with zero is always equal to zero.

Example 5.3.29. The categories $\mathbf{Top}_*$ and $\mathbf{Sp}$ are based topological categories.

Definition 5.3.30. Suppose $\mathbf{I}$ is a based category. A **based diagram** is a continuous functor $X : \mathbf{I} \to \mathbf{Top}_*$. Equivalently, it consists of

- a based space $X(i)$ for each $i \in \mathrm{ob}\,\mathbf{I}$, and
- a continuous composition map $X(i) \wedge \mathbf{I}(i, j) \to X(j)$,

with the same conditions as in Definition 5.3.5. By abuse of notation, we let $\mathbf{Top}_*^{\mathbf{I}}$ refer to the category of based diagrams.

Definition 5.3.31. The **sphere category S** is a based topological category, with one object for each nonnegative integer $n \geq 0$. The morphism spaces are spheres

$$\mathbf{S}(m, n) = \begin{cases} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

The nontrivial compositions are all homeomorphisms

$$S^{n-m} \wedge S^{p-n} \cong S^{p-m}.$$

To be specific, we use the canonical homeomorphism of Definition 2.1.4. This is the one-point compacitification of the the linear isomorphism $\mathbb{R}^{n-m} \times \mathbb{R}^{p-n} \cong \mathbb{R}^{p-m}$ that concatenates the coordinates.

Lemma 5.3.32. *There is an isomorphism of categories* $\mathbf{Top}_*^{\mathbf{S}} \cong \mathbf{Sp}$.

In other words, spectra are a special case of diagrams!

Proof. Given a spectrum X, form a diagram over $\mathbf{S}$ using the spaces X_n. The maps $X_m \wedge S^{n-m} \to X_n$ are the compositions of the bonding maps. It is easy to check this forms a diagram. Going the other way, for any $\mathbf{S}$-diagram, the action of $\mathbf{S}(n, n+1) \cong S^1$ gives maps $X_n \wedge S^1 \to X_{n+1}$, making the spaces X_n into a spectrum.

These operations are inverses, and along this identification, a map of diagrams corresponds to a map of spectra. We therefore get an isomorphism of categories. □

Definition 5.3.33. For each object $i \in \operatorname{ob} \mathbf{I}$ and based space A, the **free based diagram** $F_i A$ is the based $\mathbf{I}$-diagram whose value at j is the smash product

$$(F_i A)(j) = A \wedge \mathbf{I}(i, j),$$

with $\mathbf{I}$ acting by composition on the $\mathbf{I}(i, j)$ term.

Lemma 5.3.34. *This is the left adjoint of the forgetful functor* $\mathrm{ev}_i \colon \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*$ *sending the based diagram X to the based space $X(i)$.*

The proof is the same as that of Lemma 5.3.11.

Example 5.3.35. For the sphere category $\mathbf{S}$, a free diagram $F_d A$ is precisely the same thing as a shift desuspension spectrum from Example 2.1.8. At level n, if $n \geq d$, we get the suspension $A \wedge S^{n-d} = \Sigma^{n-d} A$. When $n < d$, we get the zero space $*$, because $A \wedge * \cong *$.

Definition 5.3.36. As in Definition 5.3.14, a map of based diagrams is **level equivalence** if $X(i) \to Y(i)$ is a weak equivalence for every $i \in \operatorname{ob} \mathbf{I}$. Similarly $X \to Y$ is a **level fibration** if $X(i) \to Y(i)$ is a Serre fibration for every i.

As before, we define the sets

$$\begin{aligned} F I_+ &= \{\, F_i S_+^{n-1} \to F_i D_+^n : n \geq 0,\ i \in \operatorname{ob} \mathbf{I} \,\}, \\ F J_+ &= \{\, F_i (D^n \times \{0\})_+ \to F_i (D^n \times I)_+ : n \geq 0,\ i \in \operatorname{ob} \mathbf{I} \,\}. \end{aligned}$$

A free cofibration in $\mathbf{Top}_*^{\mathbf{I}}$ is a retract of an $F I_+$-cell complex.

Theorem 5.3.37. *The category of based diagrams $\mathbf{Top}_*^{\mathbf{I}}$ has a* ***projective model structure*** *in which*

- *the cofibrations are free cofibrations,*
- *the weak equivalences are level equivalences, and*
- *the fibrations are the level fibrations.*

Proof. The proof is identical to that of Theorem 5.3.16, using the sets $F I_+$ and $F J_+$. □

Example 5.3.38. For the sphere category $\mathbf{S}$, this gives the level model structure on spectra from Theorem 5.2.11.

5.3.5 Based diagrams on unbased categories

A large class of examples of based diagrams comes from the following construction.

Example 5.3.39. For any unbased category $\mathbf{I}$, we form a based category $\mathbf{I}_+$ by adding a disjoint basepoint to each of the mapping spaces. The composition becomes

$$\mathbf{I}(a,b)_+ \wedge \mathbf{I}(b,c)_+ \cong (\mathbf{I}(a,b) \times \mathbf{I}(b,c))_+ \longrightarrow \mathbf{I}(a,c)_+.$$

Lemma 5.3.40. *A based diagram $\mathbf{I}_+ \to \mathbf{Top}_*$ is the same thing as a functor $\mathbf{I} \to \mathbf{Top}_*$. It is also the same thing as an unbased diagram $\mathbf{I} \to \mathbf{Top}$ that contains the constant diagram $*$ as a retract.*

Proof. These all describe the same thing: a based space $X(i)$ for each $i \in \operatorname{ob}\mathbf{I}$, and a based map $X(i) \to X(j)$ for each $i \to j$ in $\mathbf{I}$. □

Example 5.3.41. If G is a topological monoid, a **based G-space** is a based space X with basepoint-preserving action by G. By Lemma 5.3.40, this could also be described as a based space with action maps of the form

$$G_+ \wedge X \longrightarrow X,$$

or as a G-space that contains the trivial G-space $*$ as a retract.

We write the category of based diagrams on $\mathbf{I}_+$ as $\mathbf{Top}_*^{\mathbf{I}}$, because by Lemma 5.3.40, this is the same as a functor from $\mathbf{I}$ to based spaces. Consider the forgetful map $\mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{I}}$ that forgets the basepoints.

Lemma 5.3.42. *A map in $\mathbf{Top}_*^{\mathbf{I}}$ is a cofibration, weak equivalence, or fibration in the model structure of Theorem 5.3.37 iff its image in $\mathbf{Top}^{\mathbf{I}}$ is respectively a cofibration, weak equivalence, or fibration in the model structure of Theorem 5.3.16.*

Proof. For weak equivalences and fibrations, this is immediate. For cofibrations, we have to observe that attaching a cell of the form

$$S_+^{n-1} \wedge \mathbf{I}(i,-)_+ \longrightarrow D_+^n \wedge \mathbf{I}(i,-)_+$$

gives a homeomorphic result if we instead write the cell in the form

$$S^{n-1} \times \mathbf{I}(i,-) \longrightarrow D^n \times \mathbf{I}(i,-).$$

□

Remark 5.3.43. One might expect that you could take the colimit or homotopy colimit of a based diagram X on a based category $\mathbf{I}$. However, this colimit is always the one-point space $*$, because the zero morphisms in $\mathbf{I}$ identify everything to zero. As a result, you can't meaningfully take a "colimit of a spectrum along itself."

You *can* use Lemma 5.3.40, and consider a diagram of based spaces indexed by an unbased category $\mathbf{I}$. Then you take the colimit along $\mathbf{I}$, *not* the larger category $\mathbf{I}_+$. This works, and the corresponding **based homotopy colimit** turns out to be the quotient of unbased homotopy colimits

$$\operatorname*{hocolim}_{i\in\mathbf{I}}{}^{(b)} X(i) \simeq \big(\operatorname*{hocolim}_{i\in\mathbf{I}}{}^{(u)} X(i)\big)\big/\big(\operatorname*{hocolim}_{i\in\mathbf{I}}{}^{(u)} (*)\big).$$

See **??**. You can also take a more general notion of a weighted colimit or a left Kan extension for based diagrams – see Section 5.4.3.

5.4 Quillen adjunctions and equivalences

Now that we have many, many examples of model categories, we can dig more into the relationships between different model categories. The two basic kinds of relationships are Quillen adjunctions and Quillen tensors. We start with Quillen adjunctions in this section.

5.4.1 Quillen adjunctions

Suppose $\mathbf{C}$ and $\mathbf{D}$ are model categories. In particular, they are categories with weak equivalences, so we can talk about derived functors.

Recall from Definition 3.3.15 that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is left-deformable if we can define a functor QX, equivalent to X by a map $QX \xrightarrow{\sim} X$, and landing in a full subcategory $\mathbf{A} \subseteq \mathbf{C}$ on which F preserves all weak equivalences. The left-derived functor is then $\mathbb{L}F = F \circ Q$.

If $\mathbf{C}$ is a model category, there is an obvious choice to make for $\mathbf{A}$: the subcategory of cofibrant objects $\mathbf{C}^c \subseteq \mathbf{C}$. By Ken Brown's Lemma (Lemma 5.2.24), to check that F preserves equivalences on $\mathbf{A}$, it is enough to show that F of any acyclic cofibration is a weak equivalence. For instance, we used this technique to show that the colimit has a left-derived functor in Proposition 5.3.18.

Definition 5.4.1. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is **left Quillen** if it is a left adjoint, and preserves both the cofibrations and the acyclic cofibrations:

$$F(C) \subseteq C, \qquad F(W \cap C) \subseteq W \cap C.$$

Similarly, $G\colon \mathbf{D} \to \mathbf{C}$ is **right Quillen** if it is a right adjoint, and preserves both the fibrations and the acyclic fibrations.

A **Quillen adjunction**, or Quillen pair, is an adjoint pair $(F \dashv G)$ in which F is left Quillen and G is right Quillen.

From Definition 3.3.15, it follows that any such F has a **left-derived functor** $\mathbb{L}F(X) = F(QX)$, any such G has a **right-derived functor** $\mathbb{R}G(Y) = G(RY)$. By Lemma 3.3.18, these derived functors preserve weak equivalences, so they define maps on the homotopy category

$$\mathbb{L}F\colon \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}, \qquad \mathbb{R}G\colon \operatorname{Ho}\mathbf{D} \to \operatorname{Ho}\mathbf{C}.$$

By Proposition 3.4.20, these derived functors are again adjoint to each other, $(\mathbb{L}F \dashv \mathbb{R}G)$.

To give examples, it will be helpful to first make the definition a little easier to check.

Lemma 5.4.2. *If $(F \dashv G)$ is an adjoint pair of functors on the model categories* $\mathbf{C}$ *and* $\mathbf{D}$*, then F is left Quillen iff G is right Quillen.*

We leave the proof to exercise 18. As a result, we only have to check that F is left Quillen, or G is right Quillen.

This check gets even easier if $\mathbf{C}$ is cofibrantly generated:

Lemma 5.4.3. *If* $\mathbf{C}$ *is cofibrantly generated, $F\colon \mathbf{C} \to \mathbf{D}$ is a left adjoint, and*

$$F(I) \subseteq C, \qquad F(J) \subseteq W \cap C,$$

then F is left Quillen (and therefore its right adjoint G is right Quillen).

Proof. Since F is a left adjoint, it preserves coproducts, pushouts, sequential compositions, and retracts – exactly the moves we use to build an I-cofibration out of the maps in I. Therefore, if $F(I) \subseteq C$, it follows that $F(C) \subseteq C$. The same argument applies to J and $W \cap C$. □

Example 5.4.4. Using Lemma 5.4.2 or Lemma 5.4.3, or both, we can see that the following pairs are Quillen adjunctions:

Left adjoint	**Right adjoint**
Disjoint basepoint $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$	Forget basepoint $U : \mathbf{Top}_* \to \mathbf{Top}$
$A \times (-) : \mathbf{Top} \to \mathbf{Top}$ (if A is a cell complex or cofibrant)	$\mathrm{Map}(A, -) : \mathbf{Top} \to \mathbf{Top}$
$P \otimes (-) : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$ (if P is projective)	$\mathrm{Hom}(P, -) : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{Ch}_{\geq 0}(\mathbb{Z})$
$A \wedge (-) : \mathbf{Top}_* \to \mathbf{Top}_*$ (if A is cofibrant)	$\mathrm{Map}_*(A, -) : \mathbf{Top}_* \to \mathbf{Top}_*$
Tensor $A \wedge (-) : \mathbf{Sp} \to \mathbf{Sp}$ (if A is cofibrant)	Cotensor $F(A, -) : \mathbf{Sp} \to \mathbf{Sp}$
Free diagram $F_i : \mathbf{Top} \to \mathbf{Top}^{\mathbf{I}}$	Evalulation $\mathrm{ev}_i : \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$
Unbased colimit, $\mathrm{colim}^{(u)} : \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}$	Constant diagram $\Delta : \mathbf{Top} \to \mathbf{Top}^{\mathbf{I}}$
Based colimit, $\mathrm{colim}^{(b)} : \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*$	Constant diagram $\Delta : \mathbf{Top}_* \to \mathbf{Top}_*^{\mathbf{I}}$
Free diagram $F_i : \mathbf{Top}_* \to \mathbf{Top}_*^{\mathbf{I}}$	Evalulation $\mathrm{ev}_i : \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*$
$F_i(-)_+ : \mathbf{Top} \to \mathbf{Top}_*^{\mathbf{I}}$	$U \circ \mathrm{ev}_i : \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}$

As a special case we also get:

Left adjoint	**Right adjoint**
Suspension $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$	Loopspace $\Omega : \mathbf{Top}_* \to \mathbf{Top}_*$
Suspension $\Sigma : \mathbf{Sp} \to \mathbf{Sp}$	Loopspace $\Omega : \mathbf{Sp} \to \mathbf{Sp}$
Suspension spectrum $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}$	0th space $\mathrm{ev}_0 : \mathbf{Sp} \to \mathbf{Top}_*$
Suspension spectrum $\Sigma^\infty_+ : \mathbf{Top} \to \mathbf{Sp}$	0th space $U \circ \mathrm{ev}_0 : \mathbf{Sp} \to \mathbf{Top}$
Free spectrum $F_k : \mathbf{Top}_* \to \mathbf{Sp}$	kth space $\mathrm{ev}_k : \mathbf{Sp} \to \mathbf{Top}_*$

Remark 5.4.5. We have already seen that many of these functors can be derived. By Proposition 3.4.2, derived functors are unique up to equivalence, so it doesn't matter whether we use a model structure, or some other technique, to form the derived functor. The result is the same either way.

Remark 5.4.6. If F is left Quillen, the left-derived functor $\mathbb{L}F$ is sampling the behavior of F on the cofibrant objects, and extending that behavior to the rest of $\mathbf{C}$. Of course, $\mathbb{L}F$ is equivalent to F on every cofibrant object. Following Definition 3.4.10, we say that F is correct, or has the correct homotopy type, when it is evaluated on a cofibrant object.

Quillen pairs can be composed. If we have three model categories and two left Quillen functors

$$\mathbf{C}_1 \xrightarrow{F_1} \mathbf{C}_2 \xrightarrow{F_2} \mathbf{C}_3,$$

then the composite $F_2 \circ F_1$ is clearly left Quillen. If the right adjoints are

$$\mathbf{C}_1 \xleftarrow{G_1} \mathbf{C}_2 \xleftarrow{G_2} \mathbf{C}_3,$$

then the composite $G_1 \circ G_2$ is right Quillen, and $(F_2 \circ F_1 \dashv G_1 \circ G_2)$ form a Quillen pair. By Lemma 3.4.13, we have equivalences of derived functors

$$\mathbb{L}(F_2 \circ F_1) \simeq (\mathbb{L}F_2) \circ (\mathbb{L}F_1), \qquad \mathbb{R}(G_1 \circ G_2) \simeq (\mathbb{R}G_1) \circ (\mathbb{R}G_2).$$

Example 5.4.7. Composing the disjoint basepoint and free diagram functors from Example 5.4.4 gives the free based diagram on an unbased space:

$$\mathbf{Top} \xrightarrow{(-)_+} \mathbf{Top}_* \xrightarrow{F_i(-)} \mathbf{Top}_*^{\mathbf{I}}$$

The right adjoints are all forgetful functors, which compose to give a forgetful functor:

$$\mathbf{Top} \xleftarrow{U} \mathbf{Top}_* \xleftarrow{\mathrm{ev}_i} \mathbf{Top}_*^{\mathbf{I}}$$

The composite Quillen pair is $(F_i(-)_+, U \circ \mathrm{ev}_i)$.

5.4.2 Quillen equivalences

Suppose $(F \dashv G)$ is a Quillen adjunction. By Proposition 3.4.20, the functors $(\mathbb{L}F \dashv \mathbb{R}G)$ form an adjoint pair on the homotopy categories $\operatorname{Ho}\mathbf{C}$ and $\operatorname{Ho}\mathbf{D}$. However, this adjunction is much easier to describe, using the fundamental theorem of model categories, Theorem 5.2.26.

It suffices to restrict attention to cofibrant $X \in \mathbf{C}$, so that $\mathbb{L}F(X) \simeq F(X)$, and fibrant $Y \in \mathbf{D}$, so that $\mathbb{R}G(Y) \simeq G(Y)$. Since F preserves cofibrations and the initial object,

$F(X) \in \mathbf{D}$ is cofibrant. Since G preserves fibrations and the terminal object, $G(Y) \in \mathbf{C}$ is fibrant. By Theorem 5.2.26 we therefore get

$$\begin{aligned} \operatorname{Ho}\mathbf{C}(X, G(Y)) &\cong \mathbf{C}(X, G(Y))/(\text{homotopy}) \\ &\cong \mathbf{D}(F(X), Y)/(\text{homotopy}) \\ &\cong \operatorname{Ho}\mathbf{D}(F(X), Y), \end{aligned}$$

giving the adjunction between $\mathbb{L}F$ and $\mathbb{R}G$.[4]

Definition 5.4.8. A **Quillen equivalence** is a Quillen adjunction $(F \dashv G)$ such that $(\mathbb{L}F \dashv \mathbb{R}G)$ is an *equivalence* of homotopy categories $\operatorname{Ho}\mathbf{C} \simeq \operatorname{Ho}\mathbf{D}$.

Recall that an adjunction is an equivalence if its counit and unit maps are isomorphisms. Equivalently, if isomorphisms $F(X) \to Y$ correspond to isomorphisms $X \to G(Y)$. Therefore:

Lemma 5.4.9. *A Quillen adjunction $(F \dashv G)$ is a Quillen equivalence if, for all cofibrant X in $\mathbf{C}$ and fibrant Y in $\mathbf{D}$, a map $FX \to Y$ is a weak equivalence in $\mathbf{D}$ iff its adjunct $X \to GY$ is a weak equivalence in $\mathbf{C}$.*

Example 5.4.10. Σ and Ω form a Quillen equivalence between **Sp** and itself. The other examples in Example 5.4.4 are not Quillen equivalences.

Example 5.4.11. For any object X in a model category X, the slice category or comma category $(X \downarrow \mathbf{C})$ is the category of maps $X \to Z$, with morphisms the commuting triangles. This is a model category with essentially the same cofibrations, weak equivalences, and fibrations as $\mathbf{C}$, see exercise 9.

Any map $f: X \to Y$ induces a map of comma categories $f^*: (Y \downarrow \mathbf{C}) \to (X \downarrow \mathbf{C})$ by precomposing with f. This operation has a left adjoint $f_!: (X \downarrow \mathbf{C}) \to (Y \downarrow \mathbf{C})$ that takes every $X \to Z$ to the pushout $Y \to Y \cup_X Z$. It is very easy to see that f^* preserves weak equivalences and fibrations, so $(f_! \dashv f^*)$ forms a Quillen pair.

Furthermore, if f is a weak equivalence and $\mathbf{C}$ is left proper (Definition 5.2.29), then $(f_! \dashv f^*)$ is a Quillen equivalence. (This is also true if f is a weak equivalence and both X and Y are cofibrant.)

Example 5.4.12. The previous example can be dualized. The slice category or comma category $(\mathbf{C} \downarrow X)$ is the category of maps $Z \to X$, with morphisms the commuting triangles. This is also model category with essentially the same cofibrations, weak equivalences, and fibrations as $\mathbf{C}$, see exercise 10.

[4] We are skipping some details here about why the adjunction sends homotopies of maps $F(X) \to Y$ to homotopies of maps $X \to G(Y)$. In examples, this is usually obvious. It's also true in general, but you have to use the notions of left and right homotopy in a model category to prove it – see [Hov99, Lem 1.3.10].

Any map $f: X \to Y$ induces a map of comma categories $f^*: (\mathbf{C} \downarrow Y) \to (\mathbf{C} \downarrow \mathbf{X})$ by sending $Z \to Y$ to the pullback $X \times_Y Z$. This operation has a left adjoint $f_!: (\mathbf{C} \downarrow X) \to (\mathbf{C} \downarrow Y)$ that post-composes with f. Again, it is very easy to see that $f_!$ preserves weak equivalences and cofibrations, so $(f_! \dashv f^*)$ forms a Quillen pair.

Furthermore, if f is a weak equivalence and $\mathbf{C}$ is right proper (Definition 5.2.29), then $(f_! \dashv f^*)$ is a Quillen equivalence. (This is also true if f is a weak equivalence and both X and Y are fibrant.)

5.4.3 Restriction and left Kan extension

Perhaps the most important Quillen adjunction is the one between "restriction" and "left Kan extension." This can be done in a great deal of generality, but we focus here on diagrams of spaces.

Let $\mathbf{I}$ and $\mathbf{J}$ be any small topological categories, and $F: \mathbf{I} \to \mathbf{J}$ a topological functor. First, let's consider the case that the categories are unbased.

Definition 5.4.13. The **restriction** $F^*: \mathbf{Top}^{\mathbf{J}} \to \mathbf{Top}^{\mathbf{I}}$ composes every diagram $\mathbf{J} \to \mathbf{Top}$ with the functor F. In other words, it produces the diagram whose value at $i \in \mathbf{I}$ is the original diagram at $F(i) \in \mathbf{J}$:

$$(F^*X)(i) = X(F(i)).$$

The **left Kan extension** $F_!: \mathbf{Top}^{\mathbf{I}} \to \mathbf{Top}^{\mathbf{J}}$ is the left adjoint of F^*. By analogy with the theory of rings, this is also sometimes called **extension of scalars**.

To spell it out explicitly, if the morphism spaces of $\mathbf{I}$ are discrete, then $F_!$ takes the colimit of all of the terms $X(i)$ for each morphism $F(i) \to j$ in $\mathbf{J}$:

$$(F_!X)(j) = \operatorname*{colim}_{F(i) \to j} X(i)$$

This colimit is indexed by the comma category:

Definition 5.4.14. For each $j \in \mathbf{J}$, the comma category $(F \downarrow j)$ has an object for each $i \in \mathbf{I}$ and morphism $p: F(i) \to j$, and a morphism for each morphism $f: i_0 \to i_1$ in $\mathbf{I}$ that forms a commuting triangle

$$\begin{tikzcd} F(i_0) \arrow[rr, "F(f)"] \arrow[dr, "p_0"'] & & F(i_1) \arrow[dl, "p_1"] \\ & j & \end{tikzcd}$$

If F is the inclusion of a single object, this recovers the earlier notion of comma category $(j \downarrow \mathbf{J})$.

More generally, if $\mathbf{I}$ has a topology, then $F_!X$ is defined by taking the *spaces* $X(i)\times\mathbf{J}(F(i),j)$ and modding out by a relation for each commuting triangle in Definition 5.4.14:

$$(F_!X)(j) = \left(\coprod_{i\in\mathbf{I}} X(i)\times\mathbf{J}(F(i),j)\right)\Big/\big(\,(x, F(f)\circ p)\sim(F(f)(x), p)\,\big)$$

We could also express this another way as a coequalizer, i.e. a colimit of a diagram that just has two maps between the same pair of objects:

$$\left(\coprod_{i_0,i_1\in\mathbf{I}} X(i_0)\times\mathbf{I}(i_0,i_1)\times\mathbf{J}(F(i_1),j)\right)\rightrightarrows\left(\coprod_{i\in\mathbf{I}} X(i)\times\mathbf{J}(F(i),j)\right)\longrightarrow(F_!X)(j) \tag{5.4.15}$$

See exercise 15. The following is also left as an exercise.

Lemma 5.4.16. *The functor $F_!$ thus described is the left adjoint of F^*.*

Proposition 5.4.17. *The functors $(F_!\dashv F^*)$ form a Quillen adjunction.*

Proof. By Lemma 5.4.2, it suffices to check that F^* preserves level equivalences and level fibrations of diagrams. This is obviously true, so we are done! □

Of course F^* is homotopical, so its right-derived functor $\mathbb{R}F^*$ is equivalent to F^* itself. The left-derived functor $\mathbb{L}F_!$ is called the **homotopy left Kan extension**. It is formed by first making the diagram X cofibrant, then taking its left Kan extension.

Example 5.4.18. If $\mathbf{J}=*$ is the one-point category with only the identity map, a diagram over $\mathbf{J}$ is just a single space. There is a unique functor $F:\mathbf{I}\to *$. The restriction F^* is the functor that takes each space to the constant diagram on that space. The left Kan extension $F_!$ is the colimit. The homotopy left Kan extension $\mathbb{L}F_!$ is therefore the homotopy colimit.

Example 5.4.19. If $\mathbf{I}=*$ is the one-point category, there is a functor $F:*\to\mathbf{J}$ for every object $j\in\mathbf{J}$. The restriction F^* is the evaluation ev_j that takes the jth level of the diagram $X(j)$ and forgets everything else. The left Kan extension $F_!$ is the free diagram functor $F_j(-)$ from Definition 5.3.10.

Example 5.4.20. Every topological group G is a category with one object, and a functor $F: H\to G$ is a group homomorphism. Suppose the homomorphism is injective. Then F^* is the functor from G-spaces to H-spaces that forgets the action of those elements of G not in the subgroup H. The left Kan extension $F_!$ is the extension or balanced product

$$F_!X = G\times_H X = \big(G\times X\big)\big/(g, hx)\sim(gh, x),$$

see exercise 14. This is like extension of scalars in ring theory.

The case of based diagrams is entirely similar. Let **I** and **J** be any small based topological categories, and $F\colon \mathbf{I} \to \mathbf{J}$ a based topological functor. The restriction $F^*\colon \mathbf{Top}_*^{\mathbf{J}} \to \mathbf{Top}_*^{\mathbf{I}}$ composes every based diagram $\mathbf{J} \to \mathbf{Top}_*$ with the functor F, just as before. The left Kan extension $F_!\colon \mathbf{Top}_*^{\mathbf{I}} \to \mathbf{Top}_*^{\mathbf{J}}$ is the left adjoint of F^*. It is defined just as before but using smash products:

$$(F_!X)(j) = \left(\bigvee_{i\in\mathbf{I}} X(i) \wedge \mathbf{J}(F(i), j)\right) \Big/ \big((x, F(f)\circ p) \sim (F(f)(x), p) \big). \tag{5.4.21}$$

Example 5.4.22. Again let $\mathbf{I} = *$ be the one-point category, and $S^0 = \mathbf{I}_+$ the based category obtained by adding a disjoint basepoint. It has one object, and morphism set S^0, and the composition is the isomorphism $S^0 \wedge S^0 \cong S^0$. A based functor $F\colon S^0 \to \mathbf{J}$ is the same thing as an object $j \in \mathbf{J}$. As before, the restriction F^* is the evaluation ev_j, and the left Kan extension $F_!$ is the free diagram functor $F_j(-)$ from Definition 5.3.33.

Example 5.4.23. If instead we let $\mathbf{J}_+ = S^0$, then there isn't a unique functor $\mathbf{I} \to \mathbf{J}_+$ anymore for an arbitrary based category **I**. This is consistent with the idea that you can't really take colimits of diagrams on based categories (Remark 5.3.43).

5.5 Quillen tensors*

Quillen tensors are essentially Quillen adjunctions that have two inputs instead of one. So they are functors from a pair of model categories **C** and **D** to a third model category **E**:

$$\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}.$$

We develop their properties here, and use them to finish the proof of the stable model structure for spectra.

5.5.1 Two-variable adjunctions

Definition 5.5.1. A **two-variable left adjoint** is a functor $\otimes\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ that has a right adjoint in each slot. In other words, each of the functors $X \otimes -$ and $- \otimes Y$ has a right adjoint.

It follows from the abstract properties of adjoints (exercise 23) that the right adjoints assemble together into functors

$$\mathrm{Hom}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{D}, \qquad \mathrm{Hom}_{\mathbf{D}}\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{C},$$

and we have bijections

$$\mathbf{C}(X, \mathrm{Hom}_{\mathbf{D}}(Y, Z)) \cong \mathbf{E}(X \otimes Y, Z) \cong \mathbf{D}(Y, \mathrm{Hom}_{\mathbf{C}}(X, Z))$$

that are natural in X, Y, and Z. All together this is called a **two-variable adjunction**. (cf. Remark 3.4.25)

Example 5.5.2. The Cartesian product and unbased mapping space

$$\begin{aligned}\times\colon \mathbf{Top}\times\mathbf{Top} &\to \mathbf{Top}\\ \mathrm{Map}\colon \mathbf{Top}^{\mathrm{op}}\times\mathbf{Top} &\to \mathbf{Top}\\ \mathrm{Map}\colon \mathbf{Top}^{\mathrm{op}}\times\mathbf{Top} &\to \mathbf{Top}\end{aligned}$$

form a two-variable adjunction. So we have natural bijections

$$\mathbf{Top}(X,\mathrm{Map}(Y,Z))\cong\mathbf{Top}(X\times Y,Z)\cong\mathbf{Top}(Y,\mathrm{Map}(X,Z))$$

for any three unbased spaces X, Y, and Z.

Example 5.5.3. The smash product and based mapping space

$$\begin{aligned}\wedge\colon \mathbf{Top}_*\times\mathbf{Top}_* &\to \mathbf{Top}_*\\ \mathrm{Map}_*\colon \mathbf{Top}_*^{\mathrm{op}}\times\mathbf{Top}_* &\to \mathbf{Top}_*\\ \mathrm{Map}_*\colon \mathbf{Top}_*^{\mathrm{op}}\times\mathbf{Top}_* &\to \mathbf{Top}_*\end{aligned}$$

form a two-variable adjunction. So we have natural bijections

$$\mathbf{Top}_*(X,\mathrm{Map}_*(Y,Z))\cong\mathbf{Top}_*(X\wedge Y,Z)\cong\mathbf{Top}_*(Y,\mathrm{Map}_*(X,Z))$$

for any three based spaces X, Y, and Z.

Example 5.5.4. Consider the functors

$$\begin{aligned}\times\colon \mathbf{Top}\times\mathbf{Top}^{\mathbf{I}} &\to \mathbf{Top}^{\mathbf{I}}\\ \mathrm{Map}\colon \mathbf{Top}^{\mathrm{op}}\times\mathbf{Top}^{\mathbf{I}} &\to \mathbf{Top}^{\mathbf{I}}\\ \mathrm{Map}_{\mathbf{I}}\colon (\mathbf{Top}^{\mathbf{I}})^{\mathrm{op}}\times\mathbf{Top}^{\mathbf{I}} &\to \mathbf{Top}.\end{aligned}$$

The first functor takes a space X and applies $X\times -$ to every level of a diagram Y. The second functor similarly applies $\mathrm{Map}(X,-)$ to every level. The third functor takes the space of maps of diagrams $\mathrm{Map}_{\mathbf{I}}(Y,Z)$. It is the product of mapping spaces $\mathrm{Map}(Y(i),Z(i))$, restricted to the subspace of those maps that form a natural transformation of diagrams.

These form a two-variable adjunction. So we have natural bijections

$$\mathbf{Top}(X,\mathrm{Map}_{\mathbf{I}}(Y,Z))\cong\mathbf{Top}^{\mathbf{I}}(X\times Y,Z)\cong\mathbf{Top}^{\mathbf{I}}(Y,\mathrm{Map}(X,Z))$$

for any space X and diagrams Y and Z.

Example 5.5.5. The previous example has a variant where we use based spaces and the smash product instead. Taking $\mathbf{I}$ to be the sphere category $\mathbf{S}$ of Definition 5.3.31, we get the tensor, cotensor, and the mapping space

$$\begin{gathered} \wedge\colon \mathbf{Top}_* \times \mathbf{Sp} \to \mathbf{Sp} \\ F\colon \mathbf{Top}_*^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Sp} \\ \mathrm{Map}_*\colon \mathbf{Sp}^{\mathrm{op}} \times \mathbf{Sp} \to \mathbf{Top}_* \end{gathered}$$

So we have natural bijections

$$\mathbf{Top}_*(X, \mathrm{Map}_*(Y,Z)) \cong \mathbf{Sp}(X \wedge Y, Z) \cong \mathbf{Sp}(Y, F(X,Z))$$

for any space X and spectra Y and Z.

5.5.2 Pushout-products and pullback-homs

We would like to say that the product of two topological spaces

$$\times\colon \mathbf{Top} \times \mathbf{Top} \to \mathbf{Top}$$

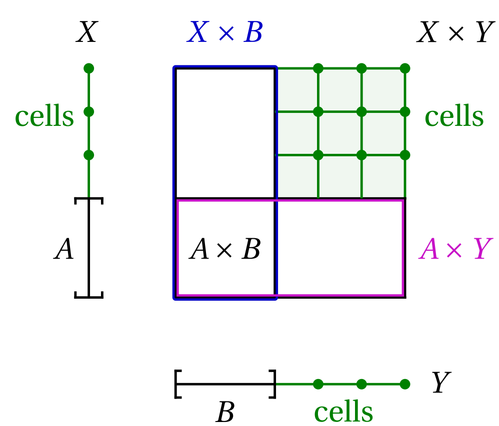
is a left Quillen functor, in some sense. The problem is that, if $A \to X$ and $B \to Y$ are both Quillen cofibrations (retracts of relative cell complexes), the product map

$$A \times B \to X \times Y$$

will not be a Quillen cofibration.

But products of spaces are nice, right? So what's going on? The problem is that we are asking for the wrong condition.

To illustrate the problem, assume further that both maps are cell complexes. So the complement $X \setminus A$ is composed of cells, and the complement $Y \setminus B$ is also composed of cells. The complement $(X \times Y) \setminus (A \times B)$ has all the products of these cells, but it also has products of A with all the cells in $Y \setminus B$, and also the product of B with all the cells in $X \setminus A$.



What is the correct statement? The cells in $X \times Y$ are located away from the points in which *either* the first coordinate is in A or the second coordinate is in B. So we have to consider $X \times Y$ relative to a larger subspace.

Given two maps of spaces $f: A \to X$ and $g: B \to Y$, we define the pushout-product $f \square g$ to be the inclusion

$$(A \times Y) \cup_{(A \times B)} (X \times B) \longrightarrow X \times Y.$$

More generally, suppose **C**, **D**, and **E** are categories, **E** has all pushouts, and

$$\otimes: \mathbf{C} \times \mathbf{D} \to \mathbf{E}$$

is a functor that preserves colimits in each slot.

Definition 5.5.6. If $f: A \to X$ is a map in **C** and $g: B \to Y$ is a map in **D**, we use $\otimes$ to form the following commuting square.

$$\begin{tikzcd} A \otimes B \arrow[r, "\mathrm{id} \otimes g"] \arrow[d, "f \otimes \mathrm{id}"'] & A \otimes Y \arrow[d, "f \otimes \mathrm{id}"] \\ X \otimes B \arrow[r, "\mathrm{id} \otimes g"'] & X \otimes Y \end{tikzcd}$$

The **pushout-product** $f \square g$ is the map from the pushout of the first three terms to the final vertex:

$$f \square g : (X \otimes B) \cup_{A \otimes B} (A \otimes Y) \longrightarrow X \otimes Y.$$

Note that $f \square g$ is a morphism in the category **E**.

In particular, if $\otimes$ is a functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the pushout-product of any two maps in **C** is another map in **C**.

Example 5.5.7. Since $\otimes$ preserves colimits in each slot, we have $\emptyset \otimes X \cong \emptyset$. Therefore:

$$(\emptyset \longrightarrow X) \square (B \xrightarrow{g} Y) = (X \otimes B \xrightarrow{\mathrm{id} \otimes g} X \otimes Y)$$

$$(A \xrightarrow{f} X) \square (\emptyset \longrightarrow Y) = (A \otimes Y \xrightarrow{f \otimes \mathrm{id}} X \otimes Y)$$

In other words, pushout-product with a map of the form $(\emptyset \to X)$ is the same thing as tensoring the map with the object X.

As a further special case, we get

$$(\emptyset \longrightarrow X) \square (\emptyset \longrightarrow Y) = (\emptyset \longrightarrow X \otimes Y).$$

Example 5.5.8. In based spaces $\mathbf{Top}_*$, we have two products, $\times$ and $\wedge$. We can take pushout-product with respect to either one, to make another map of based spaces. If there is any risk of confusion, we use a subscript, $f \square_\times g$ or $f \square_\wedge g$, to denote which one we are taking.

Example 5.5.9. There are isomorphisms of maps

$$(S^{m-1} \to D^m) \square (S^{n-1} \to D^n) \cong (S^{m+n-1} \to D^{m+n})$$

$$(S^{m-1} \to D^m) \square (D^n \times \{0\} \to D^n \times I) \cong (D^{m+n} \times \{0\} \to D^{m+n} \times I).$$

Once we have checked this, we observe that $A \times (-)$ commutes with the pushout-product in unbased spaces, so we get

$$(A \times S^{m-1} \to A \times D^m) \square (S^{n-1} \to D^n) \cong (A \times S^{m+n-1} \to A \times D^{m+n})$$

$$(A \times S^{m-1} \to A \times D^m) \square (D^n \times \{0\} \to D^n \times I) \cong (A \times D^{m+n} \times \{0\} \to A \times D^{m+n} \times I).$$

We could prove these more elegantly by proving that $\square$ is associative (exercise 24). Then in the above two isomorphisms, each side can be written as a three-fold pushout-product

$$(\emptyset \to A) \square (S^{m-1} \to D^m) \square (S^{n-1} \to D^n),$$

$$(\emptyset \to A) \square (S^{m-1} \to D^m) \square (D^n \times \{0\} \to D^n \times I).$$

Similarly, if we work in based spaces and use the smash product, we have isomorphisms

$$(S^{m-1}_+ \to D^m_+) \square_\wedge (S^{n-1}_+ \to D^n_+) \cong (S^{m+n-1}_+ \to D^{m+n}_+)$$

$$(S^{m-1}_+ \to D^m_+) \square_\wedge ((D^n \times \{0\})_+ \to (D^n \times I)_+) \cong ((D^{m+n} \times \{0\})_+ \to (D^{m+n} \times I)_+)$$

and similar isomorphisms that smash one of the maps with a based space A.

The pushout-product is a functor of arrow categories

$$\square\colon \mathbf{C}^{\{\bullet\to\bullet\}} \times \mathbf{D}^{\{\bullet\to\bullet\}} \longrightarrow \mathbf{E}^{\{\bullet\to\bullet\}}.$$

If $\otimes$ has a right adjoint in each slot, and $\mathbf{C}$ and $\mathbf{D}$ have all limits, then $\square$ has a right adjoint in each slot as well. The right adjoint is called the pullback-hom.

Definition 5.5.10. If $g : B \to Y$ is a map in $\mathbf{D}$ and $h\colon C \to Z$ is a map in $\mathbf{E}$, we can use $\mathrm{Hom}_{\mathbf{D}}(-,-)$ to form the following commuting square.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(Y,C) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{D}}(B,C) \arrow[d] \\
\mathrm{Hom}_{\mathbf{D}}(Y,Z) \arrow[r] & \mathrm{Hom}_{\mathbf{D}}(B,Z)
\end{tikzcd}$$

The **pullback-hom** $\mathrm{Hom}_{\mathbf{D},\square}(g,h)$, or just $\mathrm{Hom}_\square(g,h)$, is the map in $\mathbf{C}$ from the first vertex of this square to the pullback of the remaining three terms:

$$\mathrm{Hom}_\square(g,h)\colon \mathrm{Hom}_{\mathbf{D}}(Y,C) \to \mathrm{Hom}_{\mathbf{D}}(B,C) \times_{\mathrm{Hom}_{\mathbf{D}}(B,Z)} \mathrm{Hom}_{\mathbf{D}}(Y,Z).$$

This can be visualized as the space of all choices of diagonal in the square below, mapping to the space of all choices of two horizontal maps making the square commute.

$$\begin{tikzcd} B \arrow[r, dashed] \arrow[d, "g"'] & C \arrow[d, "h"] \\ Y \arrow[r, dashed] \arrow[ur, dashed] & Z \end{tikzcd}$$

Example 5.5.11. Similarly to Example 5.5.7, if we plug an initial object in for B, the pullback-hom simplifies to $\mathrm{Hom}_{\mathbf{D}}(Y,-)$

$$\mathrm{Hom}_{\square}((\emptyset \longrightarrow Y),\ (C \xrightarrow{h} Z)) = (\mathrm{Hom}_{\mathbf{D}}(Y,C) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(Y,Z)),$$

and if we plug in a terminal object for Z, the pullback-hom simplifies to $\mathrm{Hom}_{\mathbf{D}}(-,C)$:

$$\mathrm{Hom}_{\square}((B \xrightarrow{g} Y),\ (C \longrightarrow *)) = (\mathrm{Hom}_{\mathbf{D}}(Y,C) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(B,C)),$$

If we do both, we get a single mapping space and its map to the terminal object:

$$\mathrm{Hom}_{\square}((\emptyset \longrightarrow Y),\ (C \longrightarrow *)) = (\mathrm{Hom}_{\mathbf{D}}(Y,C) \longrightarrow *).$$

We are now ready to prove that $\mathrm{Hom}_{\square}$ is the right adjoint of $\square$ in each slot.

Proposition 5.5.12. *We have natural bijections between commuting squares of the following forms.*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "\mathrm{Hom}_{\square}(g,h)"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "f\square g"'] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "g"'] & \bullet \arrow[d, "\mathrm{Hom}_{\square}(f,h)"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Moreover, a lift in any one square gives lifts in the others:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "\mathrm{Hom}_{\square}(g,h)"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "f\square g"'] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "g"'] & \bullet \arrow[d, "\mathrm{Hom}_{\square}(f,h)"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

Proof. Pick any three maps

$$f\colon A \to X \text{ in } \mathbf{C}, \qquad g\colon B \to Y \text{ in } \mathbf{D}, \qquad h\colon C \to Z \text{ in } \mathbf{E}.$$

A choice of dotted maps making the square

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d, "f"'] & \mathrm{Hom}_{\mathbf{D}}(Y,C) \arrow[d, "\mathrm{Hom}_{\square}(g,h)"] \\ X \arrow[r, dashed] & \mathrm{Hom}_{\mathbf{D}}(B,C) \times_{\mathrm{Hom}_{\mathbf{D}}(B,Z)} \mathrm{Hom}_{\mathbf{D}}(Y,Z) \end{tikzcd}$$

commute in **C** can be expressed uniquely by the data of three maps $A \otimes Y \to C$, $X \otimes B \to C$, and $X \otimes Y \to Z$ in **E**, subject to three compatibility conditions, two that say the above square commutes, and one that says that the maps to the lower-right land not just in the product but in the fiber product.

These three compatibility conditions correspond to the three regions in the diagram below, and they hold if the maps around these regions commute.

$$\begin{tikzcd}
 & A \otimes B \arrow[dl, "f\otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes g"] & \\
X \otimes B \arrow[r, dashed] \arrow[ddr, "\mathrm{id}\otimes g"'] & C \arrow[d, "h"] & A \otimes Y \arrow[l, dashed] \arrow[ddl, "f\otimes \mathrm{id}"] \\
 & Z & \\
 & X \otimes Y \arrow[u, dashed] &
\end{tikzcd}$$

This rearranges to the statement that the square

$$\begin{tikzcd}
X \otimes B \cup_{A\otimes B} A \otimes Y \arrow[r, dashed] \arrow[d, "f\square g"] & C \arrow[d, "h"] \\
X \otimes Y \arrow[r, dashed] & Z
\end{tikzcd}$$

commutes (two of the conditions) and that the top map is in fact a map out of the pushout and not just the coproduct (the last condition).

Similarly, a choice of lift $X \to \mathrm{Hom}_{\mathbf{D}}(Y, C)$ in the first square corresponds to a map $X \otimes Y \to C$ in the last square, and in each case we need the same three conditions to guarantee that these lifts commute with the other maps in the square. □

In summary, the pushout-product □ is an operation on *maps* in the categories **C** and **D**. And if the tensor $\otimes: \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ has a right adjoint in each slot, then □ has a right adjoint in each slot as well.

5.5.3 Definition of a Quillen tensor

If we have both a tensor product and a model structure, how should they interact? By the previous subsection, the right condition to ask for is that the *pushout-product* of any two cofibrations is a cofibration.

Suppose that **C**, **D**, and **E** are model categories, and $\otimes: \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ a functor.

Definition 5.5.13. We say that $\otimes$ is a **Quillen tensor**, or **left Quillen bifunctor**, if the following conditions hold.

- $\otimes$ is a left adjoint in each slot.
- For each cofibration f in $\mathbf{C}$ and cofibration g in $\mathbf{D}$, the pushout-product $f\square g$, formed using $\otimes$, is a cofibration.
- If, in addition to being a cofibration, one of the maps f or g is a weak equivalence, then the pushout-product $f\square g$ is also a weak equivalence.

We can summarize these conditions as follows.

$$C \square C \subseteq C, \qquad (W\cap C)\square C \subseteq (W\cap C), \qquad C \square (W\cap C) \subseteq (W\cap C).$$

Proposition 5.5.14. *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are cofibrantly generated, then it suffices to check the last two conditions of Definition 5.5.13 on the generating cofibrations* I *and generating acyclic cofibrations* J *in the categories* $\mathbf{C}$ *and* $\mathbf{D}$*. In shorthand,*

$$I \square I \subseteq C, \qquad J \square I \subseteq (W\cap C), \qquad I \square J \subseteq (W\cap C).$$

Proof. One might think this proceeds exactly as in Lemma 5.4.3, and indeed it is possible to prove it that way, but the proof is a bit harder because you have to figure out what $-\square g$ does to pushouts and sequential compositions. It's much faster to use the adjoints of $\square$ and lifting properties.

The argument can be given pictorially as follows.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "I\square I"'] & \bullet \arrow[d, "W\cap F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "I"'] & \bullet \arrow[d, "\mathrm{Hom}_\square(I,W\cap F)"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "C"'] & \bullet \arrow[d, "\mathrm{Hom}_\square(I,W\cap F)"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "C\square I"'] & \bullet \arrow[d, "W\cap F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

In words, since each map in $I \square I$ is a cofibration, lifts in the first square always exist. By Proposition 5.5.12, this is equivalent to the existence of lifts in the second square. Since maps with the left-lifting property are closed under coproducts, pushouts, sequential compositions, and retracts, we can pass from I to C, giving the third square. Finally, rearranging again gives the fourth square. We conclude that $C \square I \subseteq C$.

Repeating the argument on the other side, we conclude that $C \square C \subseteq C$. The other two cases are similar. $\square$

Example 5.5.15. The following are Quillen tensors, using Proposition 5.5.14 and Example 5.5.9.

Functor	Notes
$\times\colon \mathbf{Top} \times \mathbf{Top} \to \mathbf{Top}$	Cartesian product of unbased spaces.
$\wedge\colon \mathbf{Top}_* \times \mathbf{Top}_* \to \mathbf{Top}_*$	Smash product of based spaces.
$\times\colon \mathbf{Top}^{\mathbf{I}} \times \mathbf{Top} \to \mathbf{Top}^{\mathbf{I}}$	Multiply each space in the diagram $X(i)$ by a fixed space A.
$\wedge\colon \mathbf{Top}_*^{\mathbf{I}} \times \mathbf{Top}_* \to \mathbf{Top}_*^{\mathbf{I}}$	Smash each $X(i)$ with a fixed based space A.
$\wedge\colon \mathbf{Sp} \times \mathbf{Top}_* \to \mathbf{Sp}$	Smashing a spectrum with a space.[5]

There is another entry for this list called the "coend" $\mathbf{Top}^{\mathbf{I}} \times \mathbf{Top}^{\mathbf{I}^{\mathrm{op}}} \to \mathbf{Top}$, see exercise 15.

Example 5.5.16. The tensor product of chain complexes

$$\otimes\colon \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \times \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \longrightarrow \mathbf{Ch}_{\geq 0}(\mathbb{Z})$$

is also a Quillen tensor.

Remark 5.5.17. We can use Proposition 5.5.12 to rearrange the conditions of a Quillen tensor. For instance, if $\otimes$ has a right adjoint in each slot, then $\otimes$ is a Quillen tensor iff one of the right adjoints $\mathrm{Hom}(-,-)$ has

$$\mathrm{Hom}_\square(C,F) \subseteq F, \quad \mathrm{Hom}_\square((W\cap C),F) \subseteq (W\cap F), \quad \mathrm{Hom}_\square(C,(W\cap F)) \subseteq (W\cap F).$$

This is often called **Quillen's SM7 axiom**. See exercise 27.

5.5.4 Properties of Quillen tensors

What does the condition of a Quillen tensor give us? For one, it guarantees that $\otimes$ will be well-behaved on all cofibrant objects.

Lemma 5.5.18. *Suppose $\otimes$ is a Quillen tensor.*

- *If X and Y are cofibrant, then $X\otimes Y$ is cofibrant.*
- *If A is cofibrant and $B \to Y$ is an (acyclic) cofibration, then the product $A\otimes B \to A\otimes Y$ is an (acyclic) cofibration.*
- *If B is cofibrant and $A \to X$ is an (acyclic) cofibration, then the product $A\otimes B \to X\otimes B$ is an (acyclic) cofibration.*

Proof. This follows from the definition of a Quillen tensor, and the simplifications described in Example 5.5.7. □

Lemma 5.5.19. *If $\otimes$ is a Quillen tensor, then the functor $(-)\otimes(-)$ preserves all weak equivalences between pairs of cofibrant objects.*

Proof. This follows from Lemma 5.5.18 and Ken Brown's Lemma (Lemma 5.2.24). □

Corollary 5.5.20. *If $\otimes$ is a Quillen tensor and $X \in \mathbf{C}$ is cofibrant, then $X \otimes (-)$ is left Quillen functor $\mathbf{D} \to \mathbf{E}$. Similarly if $Y \in \mathbf{D}$ is cofibrant, then $(-)\otimes Y$ is a left Quillen functor $\mathbf{C} \to \mathbf{E}$.*

We can recover some of the examples of Example 5.4.4 from Example 5.5.15 using this fact. For instance, the smash product of spaces $\wedge$ is a Quillen tensor. Therefore, if X is a cofibrant space, then $X \wedge (-)$ is left Quillen.

We can dualize the above results, and say that the right adjoints $\operatorname{Hom}(-,-)$ are well-behaved whenever the source is cofibrant and the target is fibrant.

Lemma 5.5.21. *Suppose $\operatorname{Hom}(-,-)$ is one of the right adjoints of a Quillen tensor.*

- *If X is cofibrant and Y is fibrant, then $\operatorname{Hom}(X, Y)$ is fibrant.*
- *If A is cofibrant and $X \to Y$ is an (acyclic) fibration, then the induced map $\operatorname{Hom}(A, X) \to \operatorname{Hom}(A, Y)$ is an (acyclic) fibration.*
- *If Y is fibrant and $A \to B$ is an (acyclic) cofibration, then the induced map going the other way $\operatorname{Hom}(B, Y) \to \operatorname{Hom}(A, Y)$ is an (acyclic) fibration.*

Proof. This follows from Example 5.5.11 and Proposition 5.5.12.

To spell out one case in detail, if $i : \emptyset \to A$ is a cofibration and $p : X \to Y$ is a fibration, we use Example 5.5.11 to write the map $\operatorname{Hom}(A, X) \to \operatorname{Hom}(A, Y)$ as $\operatorname{Hom}_{\square}(i, p)$. We show this is a fibration by showing it has lifts along any acyclic cofibration j:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "j"'] & \operatorname{Hom}(A, X) \arrow[d, "\operatorname{Hom}_{\square}(i,p)"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \operatorname{Hom}(A, Y) \end{tikzcd}$$

By Proposition 5.5.12, this is equivalent to finding a lift in

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "j\square i"'] & X \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

But p is a fibration, and $j\square i$ is an acyclic cofibration, because $\otimes$ is a Quillen tensor. Therefore the required lift exists.

The remaining cases are left to exercise 28. □

Lemma 5.5.22. *If* $\mathrm{Hom}(-,-)$ *is one of the right adjoints of a Quillen tensor, then it preserves all weak equivalences of pairs in which the first object is cofibrant and the second object is fibrant.*

Proof. This follows from Lemma 5.5.21 and Ken Brown's Lemma for fibrations, Lemma 5.2.25. $\square$

Corollary 5.5.23. *If* $\mathrm{Hom}(-,-)$ *is one of the right adjoints of a Quillen tensor, and* $X \in \mathbf{C}$ *is cofibrant, then* $\mathrm{Hom}(X,-)$ *is right Quillen.*

Example 5.5.24. The unbased mapping space $\mathrm{Map}(X,Y)$ is the right adjoint of a Quillen tensor. Therefore it preserves equivalences if X is cofibrant and Y is fibrant. You could prove this directly using the Whitehead theorem and cellular arguments, but it would be quite tedious!

Example 5.5.25. When X and Y are spectra, the mapping space $\mathrm{Map}_*(X,Y)$ is the right adjoint of a Quillen tensor. Therefore it preserves equivalences if X is cofibrant and Y is fibrant. This gives a second proof of Lemma 3.3.32.

Example 5.5.26. When K is a based space and X is a spectrum, the cotensor $F(K,X)$ is the right adjoint of a Quillen tensor. So it preserves equivalences if K is a retract of a cell complex, and X is an Ω-spectrum.

To summarize, a Quillen tensor $\otimes$ is well-behaved on pairs of cofibrant objects, and its right adjoints $\mathrm{Hom}_{\mathbf{C}}(-,-)$ and $\mathrm{Hom}_{\mathbf{D}}(-,-)$ are well-behaved when the source is cofibrant and the target is fibrant. We therefore have a left-derived tensor and right-derived homs

$$X \otimes^{\mathbb{L}} Y := QX \otimes QY,$$

$$\mathbb{R}\mathrm{Hom}_{\mathbf{C}}(X,Z) := \mathrm{Hom}_{\mathbf{C}}(QX,RZ), \qquad \mathbb{R}\mathrm{Hom}_{\mathbf{D}}(Y,Z) := \mathrm{Hom}_{\mathbf{D}}(QY,RZ).$$

As in Remark 3.4.25, these form a two-variable adjunction on the homotopy category, so we have natural isomorphisms

$$[X,\mathbb{R}\mathrm{Hom}_{\mathbf{D}}(Y,Z)]_{\mathrm{Ho}\,\mathbf{C}} \cong [X \otimes^{\mathbb{L}} Y, Z]_{\mathrm{Ho}\,\mathbf{E}} \cong [Y,\mathbb{R}\mathrm{Hom}_{\mathbf{C}}(X,Z)]_{\mathrm{Ho}\,\mathbf{D}}.$$

Concretely, these come about by the identifications

$$\begin{aligned} \mathbf{C}(X,\mathrm{Hom}_{\mathbf{D}}(Y,Z))/(\text{homotopy}) &\cong \mathbf{E}(X \otimes Y, Z)/(\text{homotopy}) \\ &\cong \mathbf{D}(Y,\mathrm{Hom}_{\mathbf{C}}(X,Z))/(\text{homotopy}) \end{aligned}$$

where we take the identifications of mapping sets in Definition 5.5.1, and pass to homotopy classes of maps.

Corollary 5.5.27. *If* $\mathbf{C}$ *is both a model category and a closed symmetric monoidal category, if the unit* I *is cofibrant and* $\otimes$ *is a Quillen tensor, then the left-derived tensor* $\otimes^{\mathbb{L}}$ *and right-derived hom* $\mathbb{R}\mathrm{Hom}$ *make the homotopy category* $\operatorname{Ho}\mathbf{C}$ *into a closed symmetric monoidal category.*

Proof. Follows from the above discussion along with Lemma 4.1.7 and Lemma 4.1.32. $\square$

5.6 Proof of the stable model structure*

We are finally ready to establish the stable model structure on **Sp** from Theorem 5.2.11.

Definition 5.6.1. We define the generating cofibrations to be

$$I = \{\, F_i S^{n-1}_+ \longrightarrow F_i D^n_+ \,:\, n, i \geq 0 \,\}.$$

We have already observed that an I-cell complex is the same thing as a cellular spectrum. For each $i, j \geq 0$, let

$$\lambda_{i,j} \colon F_{i+j} S^j \longrightarrow F_i S^0$$

be the map adjoint to the identity $S^j \cong S^j$. Concretely, this is a truncation map that cuts off a few of the levels of $F_i S^0$, and includes the remaining levels back into the spectrum. It follows that $\lambda_{i,j}$ is a stable equivalence.

Let $\mathrm{Cyl}_{i,j}$ denote the based mapping cylinder of $\lambda_{i,j}$, and let $k_{i,j}$ inclusion of the front end of that mapping cylinder. This is summarized in the following diagram.

$$\begin{tikzcd}
F_{i+j}S^j \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\lambda_{i,j}"'] & \{1\}_+ \wedge F_{i+j}S^j \arrow[r, "\sim"] \arrow[d, "\lambda_{i,j}"'] & I_+ \wedge F_{i+j}S^j \arrow[d] & \{0\}_+ \wedge F_{i+j}S^j \arrow[l, "\sim"'] \arrow[dl, "k_{i,j}"] \\
F_i S^0 \arrow[r, leftrightarrow, "\cong"] & \{1\}_+ \wedge F_i S^0 \arrow[r, "\sim"] & \mathrm{Cyl}_{i,j} \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd} \tag{5.6.2}$$

Definition 5.6.3. We define the generating acyclic cofibrations to be

$$\begin{aligned} J = \{\, & F_i(D^n \times \{0\})_+ \longrightarrow F_i(D^n \times I)_+ \,:\, n, i \geq 0 \,\} \\ \cup \{\, & k_{i,j} \,\square\, \left(S^{n-1}_+ \longrightarrow D^n_+\right) \,:\, i, j, n \geq 0 \,\}. \end{aligned}$$

Here the pushout-product $\square$ is being taken with respect to the tensoring functor

$$\wedge \colon \mathbf{Sp} \times \mathbf{Top}_* \to \mathbf{Sp}.$$

We have already established in Example 5.5.15 that this is a Quillen tensor, using the level model structure on spectra.

Lemma 5.6.4. *The map $k_{i,j}$ is a stable equivalence and an I-cofibration.*

Proof. In (5.6.2), the maps marked $\sim$ are all level equivalences. Since $\lambda_{i,j}$ is a stable equivalence, we conclude that $k_{i,j}$ is a stable equivalence. The following diagram shows that $k_{i,j}$ is a composition of two I-cell complexes of spectra, and is therefore an I-cofibration.

$$\begin{tikzcd}
 & \{0,1\}_+ \wedge F_{i+j}S^j \arrow[r] \arrow[d] & I_+ \wedge F_{i+j}S^j \arrow[d] \\
\{0\}_+ \wedge F_{i+j}S^j \arrow[r] \arrow[rr, bend right, "k_{i,j}"'] & (\{0\}_+ \wedge F_{i+j}S^j) \vee (\{1\}_+ \wedge F_i S^0) \arrow[r] & \mathrm{Cyl}_{i,j} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

□

Lemma 5.6.5. *The map $k_{i,j} \square \left(S^{n-1}_+ \to D^n_+\right)$ is a stable equivalence and an I-cofibration.*

Proof. Since $\wedge$ is a Quillen tensor using the level model structure, and this uses the same set of maps I, we know that $k_{i,j} \square \left(S^{n-1}_+ \to D^n_+\right)$ is an I-cofibration.

To show it is a stable equivalence, we replace $k_{i,j}$ by any stable equivalence of cofibrant spectra $k\colon X \to Y$, and show that $k \square \left(S^{n-1}_+ \to D^n_+\right)$ is a stable equivalence. Consider the square

$$\begin{tikzcd}
X \wedge S^{n-1}_+ \arrow[r] \arrow[d, "k\wedge \mathrm{id}"'] & X \wedge D^n_+ \arrow[d, "k\wedge \mathrm{id}"] \\
Y \wedge S^{n-1}_+ \arrow[r] & Y \wedge D^n_+ .
\end{tikzcd}$$

By Corollary 2.4.23, or alternatively Lemma 5.5.19, the vertical maps are both stable equivalences. On the other hand, the horizontal maps are cofibrations at each spectrum level, so the strict pushout

$$(Y \wedge S^{n-1}_+) \cup_{X \wedge S^{n-1}_+} (X \wedge D^n_+)$$

is equivalent to the homotopy pushout. Since $X \wedge S^{n-1}_+ \to Y \wedge S^{n-1}_+$ is an equivalence, this homotopy pushout is equivalent to $X \wedge D^n_+$. Therefore the map from the pushout to $Y \wedge D^n_+$ is an equivalence. □

Lemma 5.6.6. *Let $p\colon X \to Y$ be a map of spectra. Then p is I-injective if and only if each $p_i\colon X_i \to Y_i$ is an acyclic Serre fibration.*

Furthermore p is J-injective if and only if two conditions hold: each $p_i\colon X_i \to Y_i$ is a Serre fibration, and each square (5.6.7) *is a homotopy pullback square.*

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d, "p_i"'] & \Omega^j X_{i+j} \arrow[d, "\Omega^j p_{i+j}"] \\
Y_i \arrow[r] & \Omega^j Y_{i+j}
\end{tikzcd} \tag{5.6.7}$$

Proof. The first claim has already been proven in Theorem 5.3.37, using the free-forget adjunction (F_n, ev_n). See Theorem 5.3.16.

By the same argument, p has the right-lifting property with respect to the first set of maps in J, iff each p_i is a Serre fibration. It remains to prove that, under this assumption, p has the right-lifting property with respect to the second set of maps in J iff each square (5.6.7) is a homotopy pullback square. So we assume that p is a level fibration for the rest of the proof.

By Proposition 5.5.12, p has the right-lifting property with respect to the second set of maps in J iff $\mathrm{Hom}_\square(k_{i,j}, p)$ has the right-lifting property with respect to $S^{n-1}_+ \to D^n_+$, in other words iff $\mathrm{Hom}_\square(k_{i,j}, p)$ is an acyclic Serre fibration.

Note that in the level model structure on spectra, $k_{i,j}$ is a cofibration and p is a fibration. By Remark 5.5.17, therefore $\mathrm{Hom}_\square(k_{i,j}, p)$ is a Serre fibration. So we only have to consider whether it is a weak equialence as well.

Recall that the map $\mathrm{Hom}_\square(k_{i,j}, p)$ is written as

$$\mathrm{Map}_*(\mathrm{Cyl}_{i,j}, X) \longrightarrow \mathrm{Map}_*(F_{i+j}S^j, X) \times_{\mathrm{Map}_*(F_{i+j}S^j, Y)} \mathrm{Map}_*(\mathrm{Cyl}_{i,j}, Y). \tag{5.6.8}$$

Since Map_* is the right adjoint of a Quillen tensor for the level model structure, Lemma 5.5.21 applies. It tells us two things.

First, $\mathrm{Map}_*(F_{i+j}S^j, X) \to \mathrm{Map}_*(F_{i+j}S^j, Y)$ is a Serre fibration. Therefore the pullback on the right-hand side of (5.6.8) is a homotopy pullback.

Second, Map_* preserves equivalences when the source is a cellular spectrum. Therefore we may replace $\mathrm{Cyl}_{i,j}$ by the level equivalent spectrum $F_i S^0$:

$$\mathrm{Map}_*(F_i S^0, X) \longrightarrow \mathrm{Map}_*(F_{i+j}S^j, X) \times^h_{\mathrm{Map}_*(F_{i+j}S^j, Y)} \mathrm{Map}_*(F_i S^0, Y). \tag{5.6.9}$$

This map is weakly equivalent to (5.6.8), because we replaced two of the terms by weakly equivalent spaces, and the pullback is a homotopy pullback.

We can simplify one more time. By composing adjoints, we get an isomorphism $\mathrm{Map}_*(F_m S^n, X) \cong \Omega^n X_m$. This allows us to rewrite (5.6.9) as the map

$$X_i \longrightarrow \Omega^j X_{i+j} \times^h_{(\Omega^j Y_{i+j})} Y_i \tag{5.6.10}$$

arising from the adjunct bonding maps of X and Y.

In summary, we have shown that p has the right-lifting property with respect to the second set of maps in J iff (5.6.10) is a weak equivalence. However, (5.6.10) is just the map from the first vertex to the homotopy pullback of the other three vertices, in the square (5.6.7). So p has the right-lifting property with respect to the second set of maps in J iff (5.6.7) is a homotopy pullback square. □

Definition 5.6.11. A **stable fibration** is a map of spectra $X \to Y$ that is J-injective. By Lemma 5.6.6, this is the same as being a level fibration and (5.6.7) being a homotopy pullback square. Note that $X \to *$ is a stable fibration iff X is an Ω-spectrum.

Remark 5.6.12. To check that the squares Equation 5.6.7 are homotopy pullbacks, it suffices to check the cases in which $j = 1$. For instance, the square for $j = 2$ can be subdivided as follows.

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "p_i"] & \Omega X_{i+1} \arrow[r] \arrow[d, "\Omega p_{i+1}"] & \Omega^2 X_{i+2} \arrow[d, "\Omega^2 p_{i+2}"] \\ Y_i \arrow[r] & \Omega Y_{i+1} \arrow[r] & \Omega^2 Y_{i+2} \end{tikzcd}$$

If the squares with $j = 1$ are all homotopy pullbacks, then the left-hand square above is a homotopy pullback, and applying Ω, we see that the right-hand square is as well. Homotopy pullback squares are preserved under pasting, hence the large rectangle is a homotopy pullback. The same argument works for larger values of j as well.

As a result, we could have been a little more efficient in defining J. We only had to take the maps $k_{i,j} \square \left(S^{n-1}_+ \to D^n_+\right)$ for $j = 1$, not for all $j \geq 0$.

Proof of Theorem 5.2.11. (The stable model structure on **Sp**)

We verify the six conditions from Proposition 5.2.6.

1. W **is closed under 2-out-of-3 and retracts.** This follows because the stable equivalences are exactly those maps that are sent to isomorphisms under π_*.

2. I **satisfies the countable smallness condition.** As in $\mathbf{Top}^I$, this follows because the domain of each map in I is a free spectrum on a compact space, and every I-cell complex $\cdots \to X^{(n)} \to \cdots \to X^{(\infty)}$ is a cell complex at each spectrum level. The factorization therefore exists by Section 2.7, exercise 28.

3. J **satisfies the countable smallness condition.** Every map in J is a composite of I-cell attachments. Therefore any J-cell complex is also I-cell complex. So now we have to show the domains of J factor through some finite stage in any I-cell complex.

   For the first set of maps in J, the proof is the same as the previous point. For the second set of maps, we note that we are factoring a map $K \to X^{(\infty)}$ through a finite stage $K \to X^{(n)}$. The inclusion $X^{(n)} \to X^{(\infty)}$ is a closed inclusion, at each spectrum level. Therefore, if n is large enough that the factorization $K \to X^{(n)}$ exists, the factorization must also be unique.

   If K is a finite union of spectra K_i, and each piece K_i factors through some finite stage of the cell complex, then the factorizations must agree with each other (assuming n is large enough that they are all defined). Therefore K factors through some finite stage as well. In particular, if K can be expressed as a finite union of free spectra on compact spaces, then this factorization exists.

By (5.6.2), the spectrum $\mathrm{Cyl}_{i,j}$ is a union of three free spectra on compact spaces. It therefore factors through some finite stage in any I-cell complex. The same is true if we take the tensor with a sphere, $\mathrm{Cyl}_{i,j} \wedge S^{n-1}_+$, because smash product commutes with free spectrum,

$$(F_i A) \wedge B \cong F_i(A \wedge B).$$

Now we turn to the domain of $k_{i,j} \,\square\, \left(S^{n-1}_+ \to D^n_+\right)$. This is of the form

$$(\mathrm{Cyl}_{i,j} \wedge S^{n-1}_+) \cup_{F_{i+j} S^j \wedge S^{n-1}_+} (F_{i+j} D^n_+).$$

We now know that each of these three pieces factors through some finite stage, and therefore the entire union does as well.

4. **J-cell complexes are in $W \cap I$-cof.** As in the above point, every J-cell complex is also an I-cell complex, and so is an I-cofibration. By Lemma 5.6.5, each map $K_\alpha \to L_\alpha$ in J is also a stable equivalence. A coproduct of such maps $\bigvee_\alpha K_\alpha \to \bigvee_\alpha L_\alpha$ is a stable equivalence by Section 2.7, exercise 29. Any pushout of this coproduct map is also a stable equivalence, because the pushout map has the same homotopy cofiber (Lemma 2.4.14), and this homotopy cofiber is contractible. Therefore in a J-cell complex, the successive stages $X^{(n)} \to X^{(n+1)}$ are all stable equivalences. Finally, the sequential composition these stable equivalences is a stable equivalence, by Section 2.7, exercise 27. We conclude that every J-cell complex is a stable equivalence.

5. **I-inj $\subseteq W \cap J$-inj.** By Lemma 5.6.6, if p is I-injective then it is a both a fibration and a weak equivalence at each spectrum level. Therefore it is a stable equivalence, so $p \in W$. Moreover, since p is a level equivalence, the squares (5.6.7) have both verticals weak equivalences, and therefore they are homotopy pullback squares. Combining this with the fact that p is a level fibration, by Lemma 5.6.6 again we conclude that p is J-injective.

6. **$W \cap J$-inj $\subseteq I$-inj.** If $p \in W \cap J$-inj, then by Lemma 5.6.6, each p_i is a Serre fibration and the squares (5.6.7) are homotopy pullback squares. We just need to prove that each p_i is also a weak equivalence, using the fact that p is a stable equivalence.

   Examine the strict fiber spectrum F of the map p, in other words the pullback of X along the basepoint $* \to Y$. Because p is a level fibration, F equivalent to the homotopy fiber spectrum. Since p is a stable equivalence, by the long exact sequence of Lemma 2.4.8, F is stably equivalent to the zero spectrum.

   Since (5.6.7) is a homotopy pullback square, the induced map of fibers of the vertical maps is a weak equivalence:

$$F_i \xrightarrow{\sim} \Omega^j F_{i+j}.$$

Therefore F is an Ω-spectrum. The map $F \to *$ is a stable equivalence of Ω-spectra, so by Lemma 2.2.5, it is also a level equivalence. Therefore each of the spaces F_i is weakly contractible.

Since F_i is the homotopy fiber of $p_i\colon X_i \to Y_i$ at the basepoint of Y_i, we conclude that p_i is a weak equivalence, at least over the basepoint component of Y_i. Taking the loopspace, $\Omega p_i\colon \Omega X_i \to \Omega Y_i$ is an equivalence on every component. By the homotopy pullback square (5.6.7), this implies that p_{i-1} is a weak equivalence, for every value of i. Therefore p is an acyclic Serre fibration at each level. By Lemma 5.6.6 again we conclude that p is I-injective.

□

In summary, once we developed the theory of Quillen tensors, we were able to show that the category of spectra **Sp** has a model structure with the stable equivalences. All of the properties of model categories from Section 5.2.5 now apply: we get factorizations

$$X \xrightarrow[\text{stable equivalence}]{\text{cellular spectrum}} X' \xrightarrow{\text{stable fibration}} Y$$

$$X \xrightarrow{\text{cellular spectrum}} X' \xrightarrow[\text{stable equivalence}]{\text{stable fibration}} Y,$$

and lifts of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "C"'] & \bullet \arrow[d, "W\cap F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "W\cap C"'] & \bullet \arrow[d, "F"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet. \end{tikzcd}$$

Note that since the stable and level model structures have the same cofibrations C, they also have the same acyclic fibrations $W \cap F$: a map that is both a stable equivalence and a stable fibration must be a level equivalence and level fibration.

Perhaps the biggest upshot is the ability to check that a functor preserves weak equivalences by using Ken Brown's Lemma and checking the functor on one map of J at a time. We will use this heavily as we develop symmetric and orthogonal spectra, and their smash product.

5.7 Bousfield localization*

A localization of a category **C** is when we invert a class of weak equivalences, turning them into isomorphisms, to form a homotopy category Ho **C**.

If **C** is already a model category, with a class of weak equivalences V, we also use the word “localization” to refer to expanding the class V to a larger class W. This also has the effect of inverting morphisms in the homotopy category $\mathbf{C}[V^{-1}]$, turning the maps in W into isomorphisms to form the homotopy category $\mathbf{C}[W^{-1}]$.

If $\mathbf{C}$ is a model category with weak equivalences V, and we want to expand the weak equivalences to W, we might want to do this in a way that gives a compatible model structure. There is a standard framework for this, called left Bousfield localization.

Definition 5.7.1. Suppose $\mathbf{C}$ is a model category with cofibrations C, weak equivalences V, and fibrations F. Then suppose that W is a class of maps containing V. The **left Bousfield localization**, if it exists, is the model structure (C, W, LF) in which

- the cofibrations are the same,
- the weak equivalences are now W, and
- the fibrations LF are the maps with the right-lifting property with respect to $W \cap C$.

Note that this model structure is unique if it exists, because the fibrations *must* be the maps with the RLP with respect to $W \cap C$, by Lemma 5.2.21.

Example 5.7.2. The stable model structure on **Sp** is a left Bousfield localization of the level model structure. They both have the same cofibrations, but the stable model structure has more equivalences.

Example 5.7.3. A map of spectra $X \to Y$ is a **rational equivalence** if it induces isomorphisms on the rational stable homotopy groups $\pi_*(-) \otimes \mathbb{Q}$, see Example 2.5.30. The **rational stable model structure** on **Sp** is defined to be the left Bousfield localization of the stable model structure, using these rational equivalences.

We can directly show that this model structure exists. You take I to be the same as in Definition 5.6.1, and in J you add to Definition 5.6.3 the mapping cylinders of the maps

$$F_i S^n \longrightarrow F_i S^n$$

that apply a degree k map ϕ_k to the n-sphere, for all values of k. These new maps are all rational equivalences, as is any cell complex built out of them. On the other hand, the proof of Lemma 5.6.6 shows that having the right-lifting property with respect to these new maps is equivalent to asking that the following squares are homotopy pullback squares:

$$\begin{tikzcd}
\Omega^n X_i \arrow[r, "-\circ\phi_k"] \arrow[d, "\Omega^n p_i"'] & \Omega^n X_i \arrow[d, "\Omega^n p_i"] \\
\Omega^n Y_i \arrow[r, "-\circ\phi_k"'] & \Omega^n Y_i
\end{tikzcd}$$

This is equivalent to the fiber of $X \to Y$ (which is already an Ω-spectrum) having $k \cdot -$ act by an isomorphism on the stable homotopy groups, in other words, the fiber is rationally contractible. All together, this is everything we need to show that I and J generate a model structure with W equal to the rational equivalences.

Example 5.7.4. Let p be any prime. A map of spectra $X \to Y$ is a p**-local equivalence** if it induces isomorphisms on the p-local stable homotopy groups $\pi_*(-)\otimes\mathbb{Z}_{(p)}$, see Example 2.5.31. The p**-local stable model structure** on **Sp** is defined to be the left Bousfield localization of the stable model structure, using these p-local equivalences. Its construction is nearly identical to the previous example.

Left Bousfield localizations have a few important features:

- We always have $LF \subseteq F$, because $V \cap C \subseteq W \cap C$. Therefore there are fewer fibrations, and therefore fewer fibrant objects. The fibrant objects in the localized model structure are often called **local objects**, and the fibrant replacement is called **localization**
$$X \longrightarrow LX.$$
- We always have $W \cap LF = V \cap F$, because the cofibrations are the same. It follows that $W = V$ when restricted to the local objects. So objects up to W-equivalence are the same thing as local objects up to V-equivalence.
- We get a **reflective subcategory**: the category $\operatorname{Ho}\mathbf{C}[W^{-1}]$ sits inside $\operatorname{Ho}\mathbf{C}[V^{-1}]$ as the subcategory of local objects, and we have a retract onto this subcategory, by inverting the maps in W. Furthermore, this is an adjunction between $\operatorname{Ho}\mathbf{C}[V^{-1}]$ and $\operatorname{Ho}\mathbf{C}[W^{-1}]$: a map in the W-homotopy category $X \to Y$ is the same thing as a map in the V-homotopy category $X \to LY$. Further-furthermore, we have canonical bijections
$$[X,Y]_W \cong [X,LY]_{V \text{ or } W} \cong [LX,LY]_{V \text{ or } W}$$
and if Z is local then
$$[X,Z]_V \cong [X,Z]_W \cong [X,LZ]_{V \text{ or } W} \cong [LX,LZ]_{V \text{ or } W}.$$
So up to W-equivalence, everything is local, and once you're mapping into a local object, the V- and W-homotopy categories look the same, and it doesn't matter if you apply L additional times.
- As a consequence of this last point, an object $Z \in \operatorname{ob}\mathbf{C}$ is local precisely when
  - It is fibrant in the original model structure (C,V,F), and
  - A map of objects $X \to Y$ is in W iff the induced map $F_V(Y,Z) \to F_V(X,Z)$ is a weak equivalence, where F refers to the homotopical function complex (**??**).

In other words, the W-equivalences are the maps that look like isomorphisms in the V-homotopy category, from the point of view of a local object.

There is a general theory that left Bousfield localizations are guaranteed to exist provided **C** is cofibrantly generated, and **C** and W satisfy mild assumptions, see [Hir03]. We won't cover the general theory in detail here, but we will focus on the special case of localization with respect to a homology theory. Most interesting localizations, including the two we explored above, arise in this way.

Example 5.7.5. Let E be any spectrum, so that it defines an extraordinary homology theory $E_*(-)$ on spectra. A map of spectra $X \to Y$ is an E**-local equivalence** if it induces an isomorphism on E_*. The E**-local model structure** on **Sp** is the left Bousfield localization of the stable model structure, using the E-local equivalences.

Example 5.7.6. When X and Y are *spaces* we say similarly that a map $X \to Y$ is an E-local equivalence if it induces an isomorphism on E_*. The E-local model structure on **Top** or $\mathbf{Top}_*$ is the left Bousfield localization of the Quillen model structure on **Top** or $\mathbf{Top}_*$, using the E-local equivalences.

Theorem 5.7.7 (Bousfield)**.** *The E-local model structures on* **Sp**, **Top**, *and* $\mathbf{Top}_*$ *always exist, for any spectrum E.*

Example 5.7.8. We focus first on the case of spectra.

- If we take $E = \mathbb{S}$ then nothing extra is inverted, we just get the stable model structure back.

- If $E = \mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$ then we get the rational equivalences and the rational stable model structure. The localization (fibrant replacement) in this model structure is called **rationalization** $X_{\mathbb{Q}}$. We will prove later that the rationalization is stably equivalent to the smash product $X \wedge H\mathbb{Q}$.

- If $E = \mathbb{S}_{(p)}$ then we get the p-local model structure. The localization in this model structure is also equivalent to the smash product $X \wedge \mathbb{S}_{(p)}$.

- If $E = \mathbb{S}/p$ then we get the p**-adic equivalences** and a corresponding model structure. A map is an p-adic equivalence if it induces an equivalence on homotopy groups mod p. The localization in this model structure is called p**-completion** $X^\wedge_p$. The p-completion is, informally, the homotopy inverse limit
$$X^\wedge_p \simeq \operatorname*{holim}_{n\to\infty} X/(p^n).$$
More precisely, we can define it as the cotensor
$$X^\wedge_p = \mathrm{sh}^n F(M(\mathbb{Z}/(p^\infty), n), X)$$
where $M(\mathbb{Z}/(p^\infty), n)$ is the Moore space for the Prüfer p-group $\mathbb{Z}/(p^\infty)$.

For spaces, we instead use ordinary homology:

- If $E = H\mathbb{Q}$ then we get the rational equivalences of spaces and the rational stable model structure.
- If $E = H\mathbb{Z}_{(p)}$ then we get the p-local model structure.
- If $E = H\mathbb{Z}/p$ then we get the **p-adic equivalences** and a corresponding model structure. A map is an p-adic equivalence if it induces an equivalence on homology groups mod p.

For spaces, we get nice descriptions of the homotopy and homology groups of the localizations $X_{\mathbb{Q}}$, $X_{(p)}$, and $X^{\wedge}_p$, *if* X is simply-connected, or more generally nilpotent [MP12]. For more general spaces X, the localization still exists, but it is harder to calculate anything about it.

Proof sketch. See [MP12, §19.3] for a proof for unbased spaces; the original proofs for based spaces and spectra are in [Bou75] and [Bou79], respectively.

We sketch the proof for spaces first. The task is to construct a set, not a proper class, of generating acyclic cofibrations J, containing the usual ones, but also containing some additional cofibrations that are E-local equivalences, so that a map $X \to Y$ is J-injective iff it is a fibration and an E-local equivalence. The idea is to take the cardinality κ of the set $E_*(*)$, and let K be the set of all CW pairs (X, A) where the number of cells of X is at most κ, and $E_*(X, A) = 0$. Then every map in K is an E-local equivalence and a cofibration, as desired. Also, having the right lifting property with respect to $J \cup K$ rearranges to being a fibration and an E-local equivalence, as desired.

For spectra, the proof is the same, except that we want to take the set FK of free spectra on the maps in K in the previous proof. □

5.8 Exercises

1. (a) Use Lemma 5.2.21 to show that in a model category **C**, any map isomorphic to a cofibration must also be a cofibration. The same is also true for weak equivalences, and fibrations.

   (b) If **C** is a model category and **D** is another category equivalent to **C**, show that **D** has a model structure in which a map is a cofibration, weak equivalence, or fibration iff its image in **C** is such. (The previous part of this exercise is needed to show that this is well-defined.)

2. Let $\mathbf{C}$ be any category. Show that there is always a **trivial model structure** on $\mathbf{C}$ in which the weak equivalences W are the isomorphisms, and both C and F consist of all of the morphisms in $\mathbf{C}$. Show that there is also a model structure in which C is the isomorphisms, and W and F are every map in $\mathbf{C}$.

3. Let $\mathbf{C}$ be any model category. Show that the opposite category $\mathbf{C}^{\mathrm{op}}$ is also a model category, in which we swap the cofibrations with the fibrations and keep the weak equivalences the same.

4. Suppose $\mathbf{C}$ is a category and I is any set of maps in $\mathbf{C}$. Recall from Definition 5.1.18 that we defined an "I-cofibration" to be a retract of an I-cell complex.

   Show that $X \to Y$ is an I-cofibration iff Y is a retract of a third object Z, and the composite $X \to Y \to Z$ is an I-cell complex. In other words, without loss of generality, the $X \to Y$ is a retract of a cell complex with the same domain X.

5. Finish the proof of Proposition 5.2.6 by assuming that in point (6) we have $W \cap I$-cof $\subseteq J$-cof, and proving that $W \cap J$-inj $\subseteq I$-inj. (Hint: Dualize the proof of Proposition 5.1.22.)

6. Let g be any morphism in any category $\mathbf{C}$. Prove that the set of maps with the left lifting property (LLP) with respect to g is closed under coproducts, pushouts, sequential compositions, and retracts. In particular, I-cofibrations in any category, and cofibrations in any model category, are closed under these operations.

7. Dually, prove that the maps with the right lifting property (RLP) with respect to g are closed under products, pullbacks, sequential limits, and retracts. Therefore fibrations in any model category are closed under these operations.

8. Prove that both model structures on spectra are proper (Definition 5.2.29). You may find it helpful to use the facts about homotopy cofibers and fibers that we proved back in Section 2.4.

9. (a) If $\mathbf{C}$ is a model category and X is any object, prove that the comma category $(X \downarrow \mathbf{C})$ is a model category, in which the cofibrations, fibrations, and weak equivalences are determined by forgetting the map from X.

   (b) If $\mathbf{C}$ is cofibrantly generated by I and J, show that $(X \downarrow \mathbf{C})$ is cofibrantly generated by the sets

$$X \amalg I = \{X \amalg K \xrightarrow{\mathrm{id} \amalg f} X \amalg L : f \in I\}$$

$$X \amalg J = \{X \amalg K \xrightarrow{\mathrm{id} \amalg f} X \amalg L : f \in J\}.$$

   (c) Recover the model structure on $\mathbf{Top}_* = (* \downarrow \mathbf{Top})$ as a special case.

10. (a) As in exercise 9, prove that the comma category $(\mathbf{C} \downarrow X)$ has a model structure in which the cofibrations, fibrations, and weak equivalences are determined by forgetting the map from X

    (b) If $\mathbf{C}$ is cofibrantly generated by I and J, show that $(\mathbf{C} \downarrow X)$ is cofibrantly generated by the sets I/X and J/X, where I/X has one morphism for each commuting triangle

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr] & & L \arrow[dl] \\ & X & \end{tikzcd}$$

    with $f \in I$, and similarly J/X has a morphism for each such triangle with $f \in J$.

    (c) Use this to give a model structure on $(\mathbf{Top} \downarrow B)$ for an unbased space B. What do Whitehead's theorem and the fundamental theorem tell us in this setting? (This is very useful to know when studying fibrations and fiber bundles!)

    (d) Combine this exercise and exercise 9 to give a model structure on the category $\mathscr{R}(B)$ of topological spaces containing B as a retract.

11. Use Proposition 5.2.6 to establish Quillen's model structure on chain complexes $\mathbf{Ch}_{\geq 0}(R)$ from Example 5.2.10. (You might want to figure out what the I-cell complexes, I-injective maps, and J-injective maps are.)

12. Generalize Proposition 5.2.6 to give a model structure on unbounded ($\mathbb{Z}$-graded) chain complexes $\mathbf{Ch}(R)$, with weak equivalences the quasi-isomorphisms and fibrations the maps that are surjective in every degree. Use essentially the same generating maps I and J.

13. In any model category $\mathbf{C}$:

    - Show that any span $X \xleftarrow{i} A \xrightarrow{j} Y$ is weakly equivalent to a span in which all three objects are cofibrant, and the map i is a cofibration.
    - If in addition the map i is a weak equivalence, explain why the pushout of i, $Y \to X \cup_A Y$, is also a weak equivalence.
    - If i is a cofibration and j is a weak equivalence, prove that the pushout of j, $X \to X \cup_A Y$, is also a weak equivalence. (Hint: Imitate the proof of Ken Brown's Lemma, Lemma 5.2.24.)

    Therefore the left properness property of Definition 5.2.29 "almost" holds in any model category. It follows that the gluing lemma, Lemma 5.2.32, also holds when all objects are cofibrant.

14. Suppose G is a topological monoid, X is a right G-space, and Y is a left G-space. The **balanced product** $X \times_G Y$ is the quotient of $X \times Y$ by the equivalence relation

$$(xg, y) \sim (x, gy)$$

for all $x \in X$, $y \in Y$, and $g \in G$.

(a) Prove that $X \times_G *$ is the orbit space X_G.

(b) Prove that this is a Quillen tensor

$$\mathbf{Top}^G \times \mathbf{Top}^{G^{\mathrm{op}}} \to \mathbf{Top}.$$

Therefore the balanced product has a left-derived functor $X \times_G^{\mathbb{L}} Y$, and it is computed by making both X and Y into a cofibrant G-spaces,

$$X \times_G^{\mathbb{L}} Y = QX \times_G QY.$$

(c) Let EG be the cofibrant replacement of the point $*$ in right G-spaces. Consider G-spaces of the form $EG \times Y$, with G acting on (a, y) by (ag^{-1}, gy). The orbits under this action are $EG \times_G Y$. Show that any weak equivalence of G-spaces of this form gives a weak equivalence on orbits, and therefore $EG \times_G Y$ is a model for the homotopy orbits Y_{hG}.

(d) Explain why EG is a free G-cell complex. Show by induction on its skeleta that the functor $EG \times_G (-)$ preserves all equivalences. Therefore for all spaces Y we get

$$Y_{hG} \simeq EG \times_G Y.$$

This makes the construction of homotopy orbits much more concrete.

15. Suppose $\mathbf{I}$ is a topological category, $X : \mathbf{I} \to \mathbf{Top}$ is a diagram, and $Y : \mathbf{I}^{\mathrm{op}} \to \mathbf{Top}$ is a contravariant diagram. Define the **coend** $X \times_{\mathbf{I}} Y$ to be the topological space

$$X \times_{\mathbf{C}} Y = \left(\coprod_{c \in \mathbf{C}} X(c) \times Y(c) \right) \Big/ (f(x), y) \sim (x, f(y))$$

where the relation is taken over all $f : c \to d$, $x \in X(c)$, and $y \in Y(d)$. Essentially, X and Y are glued along the category $\mathbf{C}$ to create a single space, in much the same way that we form the balanced product $X \times_G Y$ in the previous exercise.

(a) Prove that $X \times_{\mathbf{I}} *$ is the colimit of X.

(b) Prove that the coend defines a Quillen tensor

$$\mathbf{Top}^{\mathbf{I}} \times \mathbf{Top}^{\mathbf{I}^{\mathrm{op}}} \to \mathbf{Top}.$$

(c) Conclude that the homotopy colimit is the left-derived tensor with $*$.

If we take the left-derived tensor with an arbitrary diagram Y, this is called an **enriched homotopy colimit**, or a homotopy colimit with weights given by Y.

(d) Let $F: \mathbf{I} \to \mathbf{J}$ be a functor, and $j \in \mathbf{J}$. Notice that $\mathbf{J}(F(-), j)$ is a diagram on $\mathbf{I}^{\mathrm{op}}$. Prove that $X \times_{\mathbf{I}} \mathbf{J}(F(-), j)$ is the left Kan extension $(F_! X)(j)$. Deduce that the homotopy left Kan extension is the left-derived tensor with $\mathbf{J}(F(-), j)$.

16. (a) If $\alpha: \mathbf{I} \to \mathbf{J}$ is a functor, and $F_i A$ is a free diagram on $\mathbf{I}$ at the object $i \in \mathbf{I}$, prove that the left Kan extension $\alpha_! F_i A$ is isomorphic to the free diagram $F_{\alpha(i)} A$. (You might find it easiest to prove this by taking the right adjoints, composing them, then taking the left adjoint of the result.)

(b) Verify that the previous part also holds if the categories are based and the diagrams are diagrams of based spaces.

17. Similarly to the previous exercise, show that if F and G are composable functors

$$\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{G} \mathbf{D}$$

then the left Kan extensions compose, in the sense that $(G \circ F)_!$ is naturally isomorphic to $G_! \circ F_!$. Conclude that for a commuting square of functors

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "H"'] & \mathbf{B} \arrow[d, "G"] \\ \mathbf{C} \arrow[r, "K"] & \mathbf{D} \end{tikzcd}$$

we have $K_! \circ H_! \cong G_! \circ F_!$.

18. If $(F \dashv G)$ are a pair of adjoint functors between model categories, prove that F preserves cofibrations and acyclic cofibrations iff G preserves fibrations and acyclic fibrations. (Hint: by Lemma 5.2.21, a map is a fibration *if and only if* it has the right lifting property with respect to acyclic cofibrations.)

19. Verify that the functors listed in Example 5.4.4 are all Quillen adjunctions.

20. Use Lemma 5.4.2 to give a much shorter proof of Proposition 5.3.18.

21. Dualize the proof of Lemma 5.2.24 to prove Lemma 5.2.25.

22. Suppose $F_1, F_2: \mathbf{C} \rightrightarrows \mathbf{D}$ are two functors, with right adjoints $G_1, G_2: \mathbf{D} \rightrightarrows \mathbf{C}$, respectively. Prove that any natural transformation $\phi: F_1 \to F_2$ has a "mate," a natural transformation $\phi^*: G_2 \to G_1$ going the opposite direction. It should be set up so

that the following square commutes.

$$\begin{tikzcd}
\mathbf{D}(F_2(X), Y) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "-\circ\phi"'] & \mathbf{C}(X, G_2(Y)) \arrow[d, "\phi^*\circ-"] \\
\mathbf{D}(F_1(X), Y) \arrow[r, leftrightarrow, "\cong"] & \mathbf{C}(X, G_1(Y))
\end{tikzcd}$$

23. Let $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ be categories. A functor $F\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$ is a **two-variable left adjoint** if

    - $\forall c \in \mathbf{C}$, the functor $F(c,-)\colon \mathbf{D}\to\mathbf{E}$ has a right adjoint $G(c,-)\colon \mathbf{E}\to\mathbf{D}$, and
    - $\forall d \in \mathbf{D}$, the functor $F(-,d)\colon \mathbf{C}\to\mathbf{E}$ has a right adjoint $H(d,-)\colon \mathbf{E}\to\mathbf{C}$.

    Using exercise 22, explain how as c and d varies, the operations G and H give functors of two variables

    $$G(-,-)\colon \mathbf{C}^{\mathrm{op}}\times\mathbf{E}\to\mathbf{D},$$
    $$H(-,-)\colon \mathbf{D}^{\mathrm{op}}\times\mathbf{E}\to\mathbf{C}$$

    and natural isomorphisms of functors $\mathbf{C}^{\mathrm{op}}\times\mathbf{D}^{\mathrm{op}}\times\mathbf{E}\to\mathbf{Set}$,

    $$\mathbf{C}(c, H(d,e)) \cong \mathbf{E}(F(c,d), e) \cong \mathbf{D}(d, G(c,e)).$$

    This is called a **two-variable adjunction**.

24. Define the three-fold pushout-product $f\square g\square h$, and use it to prove that the pushout-product is associative,

    $$(f\square g)\square h \cong f\square g\square h \cong f\square(g\square h).$$

25. Establish the isomorphisms in Example 5.5.9.

26. Prove Example 5.5.16.

27. Prove the claim in Remark 5.5.17, that $\otimes$ is a Quillen tensor iff one of the right adjoints satisfies:

    $$\mathrm{Hom}_{\square}(C,F)\subseteq F, \quad \mathrm{Hom}_{\square}((W\cap C),F)\subseteq (W\cap F), \quad \mathrm{Hom}_{\square}(C,(W\cap F))\subseteq (W\cap F).$$

28. Give more details in the proof of Lemma 5.5.21.

29. Finish the details of Example 5.7.3 and prove that the rational stable model structure exists.

30. If $\mathbf{I}$ and $\mathbf{J}$ are two based topological categories, define their smash product $\mathbf{I} \wedge \mathbf{J}$ to have objects $(\operatorname{ob}\mathbf{I}) \times (\operatorname{ob}\mathbf{J})$, and morphisms

$$(\mathbf{I} \wedge \mathbf{J})((a,b),(c,d)) = \mathbf{I}(a,c) \wedge \mathbf{J}(b,d),$$

with the evident composition.

Let $\mathbf{S}$ be the sphere category $\mathbf{S}(m,n) = S^{n-m}$ from Definition 5.3.31. Define a **bispectrum** to be a diagram of based spaces indexed by $\mathbf{S} \wedge \mathbf{S}$. Explain how this is a grid of based spaces $X_{p,q}$, and bonding maps $X_{p,q} \wedge S^1 \to X_{p+1,q}$ and $X_{p,q} \wedge S^1 \to X_{p,q+1}$ that form the evident commuting squares.

31. (*) Recall from the previous exercise that a bispectrum is a $(\mathbf{S} \wedge \mathbf{S})$-diagram of based spaces. This long exercise develops the properties of bispectra.

    (a) For each spectrum X, define a suspension bispectrum $\Sigma^\infty X$ by defining level (p,q) to be $X_p \wedge S^q$. More generally, if X and Y are spectra, explain how the smash products $X_p \wedge Y_q$ form a bispectrum.

    (b) Define free bispectra and cellular bispectra. Construct the level model structure on bispectra.

    (c) Define stable homotopy groups for bispectra, as the colimit (along both directions!) of the homotopy groups of each level. Define a stable equivalence to be a map inducing isomorphisms on these homotopy groups.

    (d) Verify that the proof of the stable model structure from Section 5.6 establishes a stable model structure on bispectra.

    (e) Prove that the suspension bispectrum functor $\mathbf{Top}_*^{\mathbf{S}} \to \mathbf{Top}_*^{\mathbf{S} \wedge \mathbf{S}}$ is a Quillen left adjoint.

32. (*) Prove that the suspension bispectrum functor $\mathbf{Top}_*^{\mathbf{S}} \to \mathbf{Top}_*^{\mathbf{S} \wedge \mathbf{S}}$ from the previous exercise is actually a Quillen *equivalence*.

Conclude that bispectra are Quillen equivalent to spectra. Intuitively, once we've stabilized once to form spectra, stabilizing a second time doesn't change the theory.

33. Let $\mathbf{C}$ be any cofibrantly generated model category, and $\mathbf{I}$ any discrete category (no topology on its morphisms). Prove that the category of diagrams $\mathbf{C}^{\mathbf{I}}$ has a projective model structure, as in Theorem 5.3.16. You might need to think a little about how to define the free diagrams $F_i A$ here. (Remember, $\mathbf{C}$ has all coproducts!) Recover the model structures on $\mathbf{Top}^{\mathbf{I}}$ and $\mathbf{Top}_*^{\mathbf{I}}$ as special cases.

34. Now let $\mathbf{C} = \mathbf{Sp}$ be the category of spectra, and let $\mathbf{I}$ be any topological category, either based or unbased. Prove that the category of diagrams $\mathbf{Sp}^{\mathbf{I}}$ has a projective model structure, as in Theorem 5.3.16.

Note that a weak equivalence here will be a **pointwise equivalence**: a map of diagrams of spectra $X \to Y$ such that for each i, the map of spectra $X(i) \to Y(i)$ is a stable equivalence.

(Of course, there are many more categories $\mathbf{C}$ for which $\mathbf{C}^{\mathbf{I}}$ has such a model structure. However, it is difficult to give good, clean, general conditions on $\mathbf{C}$ that make the proof work, specifically the smallness condition we need for the small-object argument. See e.g. [MMSS01] for one example of a general framework – but it does not apply to parametrized spectra!)

Chapter 6

Construction of the smash product

In this chapter we define the smash product of spectra $\wedge$ explicitly, and show that it has all of the properties we promised in Chapter 4. To do this, we introduce new kinds of spectra called symmetric spectra and orthogonal spectra. Together with ordinary (sequential) spectra, we refer to these as **diagram spectra**, because they can all be defined as diagrams on some based topological category **I**.

Recall that we defined the smash product in Definition 2.3.23 by picking a sequence of values of p and q and defining $(X \wedge Y)_{p+q} = X_p \wedge Y_q$. As already mentioned, this depends on choices, and different choices give spectra that are not even isomorphic, only equivalent. As a result, it is impossible to prove things like $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$, and define a symmetric monoidal category of spectra.

With symmetric and orthogonal spectra, we can define a smash product that is completely well-defined, and has isomorphisms

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z), \qquad X \wedge Y \cong Y \wedge X, \qquad X \wedge \mathbb{S} \cong X,$$

making spectra into a symmetric monoidal category, just like the category of abelian groups with the tensor product $\otimes$. As we saw in Chapter 4, lots of things we can do with abelian groups then carry over to spectra. For instance, we can define ring spectra as spectra that have multiplication maps $X \wedge X \to X$ that are associative and that have a unit given by a map $\mathbb{S} \to X$.

Moreover, symmetric and orthogonal spectra both form model categories, that are Quillen equivalent to the usual category of sequential spectra. In other words, they model the same homotopy theory. Therefore, we can use them to put a symmetric monoidal structure onto the homotopy theory $\mathrm{Ho}\,\mathbf{Sp}$ we have been studying all along.

Almost all of the material in this chapter comes from the landmark book project [Sch07] and the beautiful and seminal paper [MMSS01], which in turn draws heavily on [HSS00].

6.1 Symmetric and orthogonal spectra

In this section we introduce new kinds of spectra called symmetric and orthogonal spectra. To differentiate them from the old notion we've been studying them up to this point, we'll sometimes call the old notion **sequential spectra** or **prespectra**, denote the category of such by $\mathbf{Sp}^{\mathbb{N}}$, and denote the free spectra by $F_i^{\mathbb{N}}A$.

6.1.1 Definition and examples

Definition 6.1.1. The symmetric group Σ_n is the group of permutations on n letters. Specifically, if we let $\underline{n} = \{1, 2, \dots, n\}$, then Σ_n is the group of bijections $\underline{n} \to \underline{n}$, with composition written from right to left.

The orthogonal group $O(n)$ is the group of linear isometries $\mathbb{R}^n \to \mathbb{R}^n$.

Taking the one-point compactification, we get a basepoint-preserving action of $O(n)$ on S^n. The symmetric group $\Sigma_n \leq O(n)$ is the subgroup of permutation matrices, and therefore also acts on S^n by permuting coordinates.

The idea of a symmetric or orthogonal spectrum is that the nth level of the spectrum has an action by Σ_n, respectively $O(n)$, which mimics this action on the n-sphere.

Definition 6.1.2. A **symmetric spectrum** is a spectrum X, with a basepoint-preserving left action of Σ_n on X_n for all $n \geq 0$, with the following condition. Define $\xi_{m,n}$ to be the composite of n bonding maps:

$$\xi_{m,n}\colon X_m \wedge S^n \xrightarrow{\xi_m \wedge \mathrm{id}} X_{m+1} \wedge S^{n-1} \xrightarrow{\xi_{m+1} \wedge \mathrm{id}} \dots \xrightarrow{\xi_{m+n-1}} X_{m+n}.$$

Then we require that $\xi_{m,n}$ is $\Sigma_m \times \Sigma_n$-equivariant. Here Σ_n acts on S^n by permuting coordinates, and $\Sigma_m \times \Sigma_n$ acts on X_{m+n} along the inclusion $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{m+n}$ given by block permutations.

Note that by Example 5.3.41, the based action of Σ_n on X_n can be described as a map out of the smash product

$$(\Sigma_n)_+ \wedge X_n \longrightarrow X_n.$$

A map of symmetric spectra is a map of spectra $X \to Y$ commuting with the Σ_n-action at each level n. We let $\mathbf{Sp}^{\Sigma}$ denote the category of symmetric spectra.

Definition 6.1.3. A **orthogonal spectrum** is a spectrum X, with a continuous basepoint-preserving left action

$$O(n)_+ \wedge X_n \longrightarrow X_n$$

for all $n \geq 0$, so that the composite maps $\xi_{m,n}$ are $O(m) \times O(n)$-equivariant.

A map of orthogonal spectra is a map of spectra $X \to Y$ commuting with the $O(n)$-action at each level n. We let $\mathbf{Sp}^O$ denote the category of orthogonal spectra.

Every orthogonal spectrum is a symmetric spectrum, and every symmetric spectrum is a (sequential) spectrum, by neglect of structure. So we have forgetful functors

$$\begin{tikzcd} \mathbf{Sp}^O \arrow[r, "U^O_\Sigma"] \arrow[rr, bend right, "U^O_{\mathbb{N}}"'] & \mathbf{Sp}^\Sigma \arrow[r, "U^\Sigma_{\mathbb{N}}"] & \mathbf{Sp}^{\mathbb{N}}. \end{tikzcd}$$

We will shortly define model structures on the categories $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$, so that these forgetful functors are right Quillen, and give equivalences of homotopy categories

$$\mathrm{Ho}\,\mathbf{Sp}^O \simeq \mathrm{Ho}\,\mathbf{Sp}^\Sigma \simeq \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}.$$

In other words, symmetric and orthogonal spectra are *different* point-set models for the *same* stable homotopy category.

Example 6.1.4. The sphere spectrum $\mathbb{S}$ is an orthogonal spectrum, and therefore also a symmetric spectrum. The group $O(n)$ acts on S^n as in Definition 6.1.1, and so Σ_n acts by permuting the coordinates. In fact, this is the only way to make $\mathbb{S}$ into a symmetric or orthogonal spectrum!

More generally, the suspension spectrum $\Sigma^\infty A$ is an orthogonal spectrum. At level n, $O(n)$ acts on $A \wedge S^n$ by acting just on the sphere S^n and leaving A alone.

Example 6.1.5. We can build a model for the Eilenberg-Maclane spectrum HG that is an orthogonal spectrum, and therefore a symmetric spectrum. The idea is to take level n to be the space of configurations of points in S^n, with labels in G. When points collide their labels add, and when a point goes to the basepoint of S^n, it disappears. The orthogonal group acts on the sphere as in Definition 6.1.1. See **??** for more details.

6.1.2 Recasting as diagrams

Recall from Lemma 5.3.32 that a spectrum is the same thing as a based diagram on the category $\mathbf{S}$ from Definition 5.3.31, with one object for each nonnegative integer $n \geq 0$, and morphism spaces the spheres

$$\mathbf{S}(m,n) = \begin{cases} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

The compositions use the canonical homeomorphisms of Definition 2.1.4 that concatenate coordinates.

To generalize this construction to symmetric spectra, we have to include more morphisms in $\mathbf{S}$, to account for the actions of the symmetric groups.

Definition 6.1.6. Let $\mathbf{S}^\Sigma$ be the based topological category with an object for each integer $n \geq 0$. The morphisms are

$$\mathbf{S}^\Sigma(m,n) = \begin{cases} (\Sigma_n)_+ \wedge_{\Sigma_{n-m}} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

In particular, we get a wedge of spheres S^{n-m}, one for each element of Σ_n, but then we glue these spheres together along the relation

$$(\sigma\tau, x) \sim (\sigma, \tau x) \qquad \text{for } \sigma \in \Sigma_n,\ \tau \in \Sigma_{n-m},\ x \in S^{n-m}.$$

To define the composition in $\mathbf{S}^\Sigma$, it is easiest to rewrite the morphism space in a different way. Recall from Definition 6.1.1 that $\underline{n} = \{1, 2, \ldots, n\}$. For each subset $A \subseteq \underline{n}$, we let the A-sphere $S^A \subseteq S^n$ be the one-point compactification of $\mathbb{R}^A \subseteq \mathbb{R}^n$. This is a sphere of dimension $|A|$. Note that for disjoint subsets A and B we have

$$S^A \wedge S^B \cong S^{A \amalg B}.$$

For each injective map $\alpha\colon \underline{m} \hookrightarrow \underline{n}$, we can therefore take the sphere on the complement of the image of α,

$$S^{\underline{n}-\alpha(\underline{m})} \cong S^{n-m}.$$

We can now rewrite the morphism space as the wedge sum of these spheres over all the injective maps $\alpha\colon \underline{m} \hookrightarrow \underline{n}$:

$$\mathbf{S}^\Sigma(m,n) = \bigvee_{\alpha\colon \underline{m} \hookrightarrow \underline{n}} S^{\underline{n}-\alpha(\underline{m})} \qquad \text{when } n \geq m.$$

With this rewriting, we can finally compose the maps. For two composable injective maps $\alpha\colon \underline{m} \hookrightarrow \underline{n}$ and $\beta\colon \underline{n} \hookrightarrow \underline{p}$, the complement of the image of $\beta\alpha$ splits into the disjoint union of two sets, the complement of β and the complement of α:

$$\begin{aligned} \underline{p} - \beta\alpha(\underline{m}) &= (\underline{p} - \beta(\underline{n})) \amalg \beta(\underline{n} - \alpha(\underline{m})) \\ &\cong (\underline{p} - \beta(\underline{n})) \amalg (\underline{n} - \alpha(\underline{m})) \end{aligned}$$

Therefore we get a homeomorphism of spheres

$$S^{\underline{p}-\beta\alpha(\underline{m})} \cong S^{\underline{p}-\beta(\underline{n})} \wedge S^{\underline{n}-\alpha(\underline{m})}. \tag{6.1.7}$$

The composition in $\mathbf{S}^\Sigma$ is now defined by composing the injective maps α and β, and using the homeomorphism (6.1.7).

Lemma 6.1.8. *There is an isomorphism of categories* $\mathbf{Top}_*^{\mathbf{S}^\Sigma} \cong \mathbf{Sp}^\Sigma$.

This and the next lemma are left to exercise 3.

Definition 6.1.9. Let $\mathbf{S}^O = \mathscr{J}$ be the based topological category with an object for each integer $n \geq 0$. The morphisms are

$$\mathbf{S}^O(m,n) = \begin{cases} O(n)_+ \wedge_{O(n-m)} S^{n-m} & \text{when } n \geq m, \\ * & \text{when } n < m. \end{cases}$$

Again we rewrite this composition in a different way. For each inclusion of inner product spaces $V \subseteq W$, let $W - V$ refer to the orthogonal complement of V in W, and S^{W-V} its one-point compactification. For each pair of integers $m, n \geq 0$, let $O(m,n)$ be the space of all linear isometries $\mathbb{R}^m \to \mathbb{R}^n$. That is, all linear maps that preserve the standard inner product, and are therefore injective. Note that $O(n,n) = O(n)$.

There is a vector bundle $E(m,n) \to O(m,n)$ whose fiber over each embedding $i\colon \mathbb{R}^m \to \mathbb{R}^n$ is the orthogonal complement $\mathbb{R}^n - i(\mathbb{R}^m)$. Taking the one-point compactification of $E(m,n)$, in other words its Thom space, gives the space $\mathbf{S}^O(m,n)$.

Intuitively, this is a "wedge sum" of spheres S^{n-m}, but the spheres are indexed by the space $O(m,n)$, so it is not really a wedge sum. Instead, the spheres are continuously glued together.

For any three integers $m \leq n \leq p$, there is a product map

$$E(m,n) \times E(n,p) \longrightarrow E(m,p)$$

that composes the embeddings $i\colon \mathbb{R}^m \to \mathbb{R}^n$ and $j\colon \mathbb{R}^n \to \mathbb{R}^p$, and identifies each pair of points $y \in \mathbb{R}^p - j(\mathbb{R}^n)$ and $x \in \mathbb{R}^n - i(\mathbb{R}^m)$ with the point $y + j(x) \in \mathbb{R}^p - ji(\mathbb{R}^m)$. Passing to one-point compactifications gives the composition map

$$\mathbf{S}^O(m,n) \wedge \mathbf{S}^O(n,p) \longrightarrow \mathbf{S}^O(m,p).$$

Lemma 6.1.10. *There is an isomorphism of categories* $\mathbf{Top}_*^{\mathbf{S}^O} \cong \mathbf{Sp}^O$.

6.1.3 Cellular spectra and the level model structure

By Theorem 5.3.37, there is a **level model structure** on the category of symmetric spectra $\mathbf{Sp}^\Sigma$, and also on the category of orthogonal spectra $\mathbf{Sp}^O$.

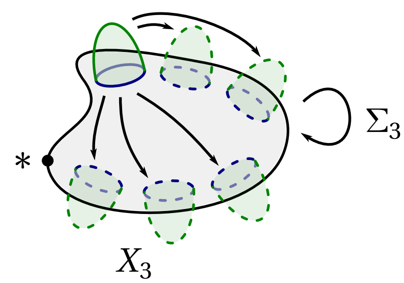
The weak equivalences in this model structure are the level equivalences, the maps of spectra $X \to Y$ that are weak equivalences on each level, $X_n \xrightarrow{\sim} Y_n$. Similarly, the fibrations are the maps of spectra $X \to Y$ that are fibrations at each level.

To describe the cofibrations, recall from Definition 5.3.33 that for each $m \geq 0$ and based space A, we can make a free symmetric spectrum on A at level m. We call this spectrum $F_m^\Sigma A$. At level n, it is the smash product

$$(F_m^\Sigma A)(n) = A \wedge \mathbf{S}^\Sigma(m,n) \simeq \bigvee_{\alpha\colon \underline{m} \hookrightarrow \underline{n}} (A \wedge S^{\underline{n}-\alpha(\underline{m})}).$$

A **cellular symmetric spectrum** is a spectrum built from $*$ by successively attaching "cells" of the form $F_m^\Sigma S_+^{n-1} \to F_m^\Sigma D_+^n$. These objects, and their retracts, are the cofibrant objects in the level model structure. The cofibrations are the retracts of the relative cell complexes.

Intuitively, this is like the notion of a cellular spectrum from Definition 2.6.2 and Proposition 2.6.11. The difference is that, when we attach a cell at level m, we not only have to attach its suspensions to the higher levels, but we *also* have to attach extra copies of the same cell so that the symmetric group actions Σ_n have somewhere to send them.



So if we attach a cell D_+^n at level 3, we have to attach 6 copies of that cell, so that Σ_3 can act by permuting them. If we only attached one, there would be no Σ_3 action on the result! Then we have to attach 24 copies of its suspension ΣD_+^n at level 4, with Σ_4 permuting them. Then 60 copies of $\Sigma^2 D_+^n$ at level 5, with Σ_5 acting by permuting them, and also flipping the two suspension coordinates on each of them. At level p, we attach as many copies of $\Sigma^{p-3} D_+^n$ as there are injective maps $\underline{3} \to \underline{p}$. The action of Σ_p permutes these cells and also their $(p-3)$ suspensions.

For orthogonal spectra $\mathbf{Sp}^O$, we get the same situation, except that the free spectra $F_m^O A$ now look like

$$(F_m^O A)(n) = A \wedge \mathbf{S}^O(m,n).$$

So when we attach a cell at level m, we also have to attach its image under all of the *orthogonal* group actions at each level. This is quite complicated to think about geometrically, so we encourage the reader to think about it more formally.

Proposition 6.1.11. *Using the level model structure, the forgetful functors*

$$\begin{tikzcd} \mathbf{Sp}^{\mathbb{N}} & \mathbf{Sp}^\Sigma \arrow[l, "U_{\mathbb{N}}^\Sigma"'] & \mathbf{Sp}^O \arrow[l, "U_\Sigma^O"'] \arrow[ll, bend left, "U_{\mathbb{N}}^O"] \end{tikzcd}$$

are right Quillen.

Proof. This is immediate – they clearly preserve both weak equivalences and fibrations. □

We call the left adjoint of each of these functors **prologation**. Prolongation is a special case of the left Kan extension from Section 5.4.3.

$$\begin{tikzcd} \mathbf{Sp}^{\mathbb{N}} \arrow[r, "P_{\Sigma}^{O}"] \arrow[rr, bend right, "P_{\mathbb{N}}^{O}"'] & \mathbf{Sp}^{\Sigma} \arrow[r, "P_{\mathbb{N}}^{\Sigma}"] & \mathbf{Sp}^{O} \end{tikzcd}$$

These can all be written very explicitly using coequalizers. For instance, the prolongation of a sequential spectrum X to a symmetric spectrum is written as the coequalizer

$$\left(\bigvee_{m_0,m_1\geq 0} X_{m_0}\wedge \mathbf{S}^{\mathbb{N}}(m_0,m_1)\wedge \mathbf{S}^{\Sigma}(m_1,n)\right)\rightrightarrows\left(\bigvee_{m\geq 0} X_m\wedge \mathbf{S}^{\Sigma}(m,n)\right)\longrightarrow (P_{\mathbb{N}}^{\Sigma}X)_n. \qquad (6.1.12)$$

The prolongation $P_{\mathbb{N}}^{O}$ is written the same way, only using $\mathbf{S}^{O}$ in the place of $\mathbf{S}^{\Sigma}$, and the prolongation P_{Σ}^{O} is written this way but swapping $\mathbf{S}^{O}$ for $\mathbf{S}^{\Sigma}$ and $\mathbf{S}^{\Sigma}$ for $\mathbf{S}^{\mathbb{N}}$.

The formula (6.1.12) looks pretty complicated. Here is a simpler way to think about it. By composition of adjoints, the prolongation of a free spectrum is always a free spectrum:

$$P_{\mathbb{N}}^{\Sigma}(F_n^{\mathbb{N}}A)\cong F_n^{\Sigma}A, \qquad P_{\mathbb{N}}^{O}(F_n^{\mathbb{N}}A)\cong F_n^{O}A, \qquad P_{\Sigma}^{O}(F_n^{\Sigma}A)\cong F_n^{O}A.$$

Prolongation also commutes with the formation of cell complexes, and preserves weak equivalences between cell complexes, because it is left Quillen. So, if we think of “cell complexes” abstractly, then when we use prolongation to move between these different categories, the cell complexes don’t change.

Therefore, if we replace all of our spectra by cellular spectra, then prolongation essentially does nothing, it just changes the exact meaning of the word “cell” as we pass from one category to another.

Example 6.1.13. The prolongation of a suspension spectrum is always a suspension spectrum on the same space. As a special case, the prolongation of $\mathbb{S}$ is $\mathbb{S}$ again.

Example 6.1.14. The prolongation of the spectrum $F_1^{\mathbb{N}}A$ to symmetric spectra is the symmetric spectrum $F_1^{\Sigma}A$. At level n, it is a wedge sum of n copies of the suspension $\Sigma^{n-1}A$, whereas the original spectrum F_1A had a single copy of $\Sigma^{n-1}A$ at each spectrum level.

For any space A and symmetric or orthogonal spectrum X, we define the tensor $A\wedge X$ just as in Definition 2.3.6, by smashing each level of X with A. This produces a symmetric or orthogonal spectrum in which Σ_n or $O(n)$ is acting on the X_n inside $A\wedge X_n$ and leaving the A alone.

Lemma 6.1.15. *This defines Quillen tensors*

$$\wedge\colon \mathbf{Top}_* \times \mathbf{Sp}^\Sigma \to \mathbf{Sp}^\Sigma$$
$$\wedge\colon \mathbf{Top}_* \times \mathbf{Sp}^O \to \mathbf{Sp}^O$$

using the level model structure on spectra.

Proof. This is just a special case of Example 5.5.15. □

The right adjoints give us the space of maps $\mathrm{Map}_*(X, Y)$ between two diagram spectra, and the cotensor $F(A, X)$. The cotensor just applies $F(A, -)$ at each spectrum level, as in Definition 2.3.8. The space of maps is the subspace of the product $\prod_{n\geq 0} \mathrm{Map}_*(X_n, Y_n)$ of those maps that commute with both the bonding maps and the symmetric or orthogonal group actions.

We can similarly define the homotopy cofiber Cf and homotopy fiber Ff of any map of symmetric or orthogonal spectra, by doing the same construction we did for sequential spectra in Example 2.3.19 and Example 2.3.22. The result has an obvious Σ_n or $O(n)$-action, so that it is a symmetric or orthogonal spectrum.

Remark 6.1.16. More generally, the colimit of any diagram of symmetric or orthogonal spectra can be defined by taking the colimit of the underlying spectra. The Σ_n and $O(n)$ actions act in an obvious way on the colimit. In particular, we can take wedge sums $X \vee Y$ and pushouts $X \cup_A Y$ of symmetric or orthogonal spectra.

Similarly, we can take any limit, or homotopy colimit or limit, by doing the same construction to the underlying spectrum. For homotopy limits, we do want to restrict attention to Ω-spectra when doing this. We'll see a little later how to replace each symmetric or orthogonal spectrum by an equivalent Ω-spectrum.

6.1.4 π_*-isomorphisms and stable equivalences

Recall that a "diagram spectrum" is either a sequential spectrum in $\mathbf{Sp}^{\mathbb{N}}$, a symmetric spectrum in $\mathbf{Sp}^\Sigma$, or an orthogonal spectrum in $\mathbf{Sp}^O$. Every diagram spectrum X has an underlying sequential spectrum, and therefore has stable homotopy groups $\pi_*(X)$, defined as in Definition 2.1.2.

Definition 6.1.17. A **π_*-isomorphism** of diagram spectra is a map $X \to Y$ inducing an isomorphism $\pi_*(X) \xrightarrow{\cong} \pi_*(Y)$.

We might be tempted to define a stable equivalence to be a π_*-isomorphism, as we did in $\mathbf{Sp}^{\mathbb{N}}$. However, if we did that in symmetric spectra $\mathbf{Sp}^\Sigma$, then the forgetful functor on homotopy categories

$$U_{\mathbb{N}}^{\Sigma}\colon \mathrm{Ho}\,\mathbf{Sp}^\Sigma \longrightarrow \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}$$

would *not* be an equivalence of categories. In other words, symmetric spectra would not model the stable homotopy category Ho **Sp**.

To give a heuristic for why this fails, suppose that $U_{\mathbb{N}}^{\Sigma}$ is an equivalence of homotopy categories Ho $\mathbf{Sp}^{\Sigma} \simeq$ Ho $\mathbf{Sp}^{\mathbb{N}}$, and that the left adjoint $P_{\mathbb{N}}^{\Sigma}$ is left-deformable. Then the derived left adjoint $\mathbb{L}P_{\mathbb{N}}^{\Sigma}$ must be an equivalence of categories as well. If we take the free sequential spectrum on the circle, $F_1^{\mathbb{N}}S^1$, and move it back and forth through this equivalence, we get the free symmetric spectrum on the circle, $F_1^{\Sigma}S^1$, regarded as a sequential spectrum.

If this were an equivalence of categories, the two sequential spectra $F_1^{\mathbb{N}}S^1$ and $F_1^{\Sigma}S^1$ would have the same homotopy groups. However, we can see directly that

$$\pi_0(F_1^{\Sigma}S^1) \cong \bigoplus_{n\geq 0} \mathbb{Z},$$
$$\pi_0(F_1^{\mathbb{N}}S^1) \cong \mathbb{Z},$$

and we have our contradiction. Of course, we had to assume here that $P_{\mathbb{N}}^{\Sigma}$ is left-deformable. See exercise 6 for a different proof that doesn't use this assumption.

This is a pickle! To get around this problem, we take motivation from the idea of left Bousfield localization. We *want* $F_1^{\Sigma}S^1$ to have the correct homotopy type, the same as $F_0^{\Sigma}S^0 = \mathbb{S}$. There is a map of symmetric spectra $F_1^{\Sigma}S^1 \to F_0^{\Sigma}S^0$ that should be the equivalence. So, we just *formally* declare that this map is an equivalence, and see what class of stable equivalences this gives us in the category $\mathbf{Sp}^{\Sigma}$.

How will we characterize such maps? Well, maps of the form $F_1^{\Sigma}S^1 \to Z$ correspond to maps of based spaces $S^1 \to Z_1$. Maps $F_0^{\Sigma}S^0 \to Z$ correspond to based maps $S^0 \to Z_0$. If Z is an Ω-spectrum, then these are equivalent, because $Z_0 \simeq \Omega Z_1$. This gives us an idea: define the stable equivalences using maps to Ω-spectra.

Definition 6.1.18. A **stable equivalence** of diagram spectra is a map $f : X \to Y$ such that, for any diagram spectrum Z that is also an Ω-spectrum, the induced map

$$(-)\circ f\colon\ [Y,Z]_\ell \longrightarrow [X,Z]_\ell$$

is a bijection. Here $[-,-]_\ell$ means morphisms in the homotopy category of diagram spectra, with the level equivalences inverted.

In other words, if we replace X and Y by level-equivalent cellular spectra QX and QY, a map $f : X \to Y$ is a stable equivalence if it induces a bijection on homotopy classes of maps

$$(-)\circ Qf\colon\ [QY,Z]_h \xrightarrow{\cong} [QX,Z]_h.$$

Example 6.1.19. In sequential spectra $\mathbf{Sp}^{\mathbb{N}}$, a map is a stable equivalence iff it is a π_*-isomorphism. To see this, notice that $X \to Y$ is a π_*-isomorphism iff it induces a bijection on $[-,Z]_s$ for all Z. (See Section 3.5, exercises 4 and 7.) Without loss of generality

Z is an Ω-spectrum. Applying Theorem 5.2.26 twice, we see that stable maps $X \to Z$ are the same as the homotopy classes of maps $QX \to Z$, which are the same as maps in the level homotopy category $X \to Z$.

We will also see shortly that in orthogonal spectra $\mathbf{Sp}^O$, a map is a stable equivalence iff it is a π_*-isomorphism. So symmetric spectra are to blame for the extra complexity in Definition 6.1.18. However, symmetric spectra are very useful, because for instance orthogonal spectra can't be defined in the motivic setting. So we will bite the bullet and develop the theory in a way that works for symmetric spectra too.

Example 6.1.20. As discussed before, the canonical map $F_1^\Sigma S^1 \to F_0^\Sigma S^0$ is a stable equivalence of symmetric spectra, even though it is not a π_*-isomorphism. Similarly, for any well-based space A the canonical maps

$$F_{i+j}\Sigma^i A \longrightarrow F_j A$$

are all stable equivalences, in any of the three categories of diagram spectra $\mathbf{Sp}^{\mathbb{N}}$, $\mathbf{Sp}^{\Sigma}$, or $\mathbf{Sp}^O$. Therefore the maps $k_{i,j}$ in Definition 5.6.3 are all stable equivalences.

Our goal in the remainder of the section is to prove that every π_*-isomorphism is a stable equivalence. We restrict attention to symmetric spectra since the argument for orthogonal spectra is identical.

Definition 6.1.21. Let X be any symmetric spectrum. We define symmetric spectra ΣX and $\operatorname{sh} X$ as

$$(\Sigma X)_n = \Sigma X_n = S^1 \wedge X_n$$
$$(\operatorname{sh} X)_n = X_{1+n}$$

with the same bonding maps as X. We write $(1+n)$ instead of $(n+1)$ in the second subscript to indicate that we take the Σ_n-action on X_{1+n} that arises from the inclusion $\Sigma_n \leq \Sigma_{1+n}$ as the permutations on the *last* n letters, not the first n letters. This is necessary to actually make $\operatorname{sh} X$ a symmetric spectrum, since the bonding maps will add more elements to the right-hand side of the set $\{1, 2, \dots, n+1\}$, not the left.

We define a map of symmetric spectra

$$\lambda_*\colon \Sigma X \to \operatorname{sh} X$$

by sending $S^1 \wedge X_n \to X_{1+n}$ by applying the bonding map $\xi_n\colon X_n \wedge S^1 \to X_{n+1}$, then applying the action of the block permutation $\tau_{n,1} \in \Sigma_{n+1}$ that moves the first n elements past the last element. Intuitively, the permutation $\tau_{n,1}$ is "correcting" for the fact that we had to swap the S^1 past the X_n to define the bonding map.

It is an exercise that λ_* is a map of symmetric spectra, and that it would fail to be a map of symmetric spectra if we didn't include $\tau_{n,1}$ (exercise 8). You need to use the equivariance condition in Definition 6.1.2 when checking this!

Proposition 6.1.22. *If X is an Ω-spectrum, then the adjoint map $\lambda^*\colon X \to \Omega\,\mathrm{sh}\,X$ is a level equivalence.*

Proof. At each level it is the composite

$$X_n \xrightarrow[\sim]{\tilde{\xi}_n} \Omega X_{n+1} \xrightarrow[\cong]{\Omega\tau_{n,1}} \Omega X_{1+n}.$$

The first map is an equivalence because X is an Ω-spectrum. The second map is a homeomorphism because it has an inverse. Therefore their composite is a weak equivalence. □

Let LX be the based homotopy colimit (reduced mapping telescope) of iterations of the map λ^*.

Proposition 6.1.23. *There is a natural isomorphism $\pi_{k+n}((LX)_n) \cong \pi_k(X)$.*

Proof. If λ^* consisted only of the adjoint maps $\tilde{\xi}_n$, the colimit would be the spectrum RX from Proposition 2.2.9, and so this would be true because RX is an Ω-spectrum:

$$\pi_k(X) \cong \pi_k(RX) \cong \pi_{n+k}((RX)_n).$$

The full truth is that we added some isomorphisms $\tau_{n,1}$ into the colimit system. However, this does not change the colimit up to isomorphism, so the colimit group $\pi_{n+k}((LX)_n)$ is still isomorphic to $\pi_k(X)$. Interestingly, LX may fail to be an Ω-spectrum, even though $\pi_{k+n}((LX)_n)$ is abstractly isomorphic to $\pi_{k+n+1}((LX)_{n+1})$. □

Proposition 6.1.24. *Every π_*-isomorphism $X \to Y$ is a stable equivalence.*

Proof. Since homotopy colimits preserve equivalences of based spaces, L preserves level equivalences, so it induces a map on the level homotopy category $[X,Z]_\ell \to [LX,LZ]_\ell$. If we let $\iota\colon X \to LX$ denote the natural map, then for every map of spectra $f\colon X \to Z$ where Z is an Ω-spectrum, we have a commuting square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota"'] & Z \arrow[d, "\iota"] \\ LX \arrow[r, "Lf"'] & LZ, \end{tikzcd}$$

and therefore a commuting diagram

$$\begin{tikzcd}
{[LX,Z]_\ell} \arrow[d, "\iota\circ-", "\cong"'] & & {[X,Z]_\ell} \arrow[d, "\iota\circ-", "\cong"'] \arrow[dll, "L"'] \\
{[LX,LZ]_\ell} \arrow[rr, "-\circ\iota"'] & & {[X,LZ]_\ell}.
\end{tikzcd}$$

The isomorphisms are because Z is an Ω-spectrum, and so by applying Proposition 6.1.22 and taking the colimit, the map $\iota\colon Z \to LZ$ is a level equivalence.

We conclude that the set $[X,Z]_\ell$ is naturally a retract of the set $[LX,Z]_\ell$. Now suppose $f\colon X \to Y$ is a π_*-isomorphism. By Proposition 6.1.23, Lf is a level equivalence, so it induces an isomorphism $(-\circ Lf)\colon [LY,Z]_\ell \xrightarrow{\cong} [LX,Z]_\ell$. Since $(-\circ f)\colon [Y,Z]_\ell \to [X,Z]_\ell$ is a retract of this isomorphism, it is also an isomorphism. Therefore f is a stable equivalence. □

We round out this section with the observation:

Lemma 6.1.25. *Between Ω-spectra, every stable equivalence is a level equivalence.*

Proof. This is a simple application of the Yoneda lemma, to the category of Ω-spectra with level equivalences inverted. □

To summarize, we have implications

$$\text{level equivalence} \Rightarrow \pi_*\text{-isomorphism} \Rightarrow \text{stable equivalence}$$

and for Ω-spectra, all three are equivalent.

6.1.5 Stability theorems

Since our definition of stable equivalence has changed, we need to re-prove some of the theorems from the first few chapters of the book, for this new notion of stable equivalence. In the spirit of [MMSS01], we list these theorems in a huge block, then proceed to prove them.

Theorem 6.1.26. 1. *The map f is a stable equivalence iff Σf is a stable equivalence.*

2. *For stable equivalences $X_\alpha \to Y_\alpha$, the wedge sum $\bigvee_\alpha X_\alpha \to \bigvee_\alpha Y_\alpha$ is a stable equivalence.*

3. *A map $f\colon X \to Y$ is a stable equivalence iff the homotopy cofiber Cf is stably equivalent to $*$.*

4. *If $A \to X$ is a stable equivalence and either $A \to X$ or $A \to Y$ is a free cofibration, the pushout map $Y \to X \cup_A Y$ is also a stable equivalence.*[1]

5. *If we have a sequence of maps $X^{(n)} \to X^{(n+1)}$ that are stable equivalences and free cofibrations, their sequential composition is also a stable equivalence (and free cofibration).*

6. *A retract of a stable equivalence is a stable equivalence.*

We also get several more corollaries, since we already know how π_* behaves on the underlying sequential spectra. For example:

Corollary 6.1.27. 1. *Finite wedges are stably equivalent to finite products (because they are π_*-isomorphic).*

2. *The map f is a stable equivalence iff Ωf is a stable equivalence (because $\Sigma\Omega X \to X$ is a π_*-isomorphism).*

3. *There is a stable equivalence $\Sigma F f \xrightarrow{\sim} C f$ (because it is a π_*-isomorphism).*

4. *The map f is a stable equivalence iff $F f$ is stably equivalent to $*$.*

Proof of Theorem 6.1.26. The key idea is to think about the properties of the functor $[-,Z]_\ell$ for any Ω-spectrum Z. Since Z is an Ω-spectrum, the construction of Example 3.2.9 makes $[X,Z]_\ell$ into an abelian group in a natural way. This will be important for the steps below that form exact sequences out of these abelian groups.

1. There are natural isomorphisms

$$\begin{aligned}[]
[\Sigma X, Z]_\ell &\cong [X, \Omega Z]_\ell, \\
[X, Z]_\ell &\cong [X, \Omega\, \mathrm{sh}\, Z]_\ell \cong [\Sigma X, \mathrm{sh}\, Z]_\ell
\end{aligned}$$

   using Proposition 6.1.22. Also, the operations Ω and sh preserve Ω-spectra. It follows that f induces an isomorphism on $[-,Z]_\ell$ iff it induces an isomorphism on $[\Sigma(-),Z]_\ell$ for all Ω-spectra Z.

2. The proof of Proposition 3.2.2 gives a natural isomorphism

$$\left[\bigvee_\alpha X_\alpha, Z\right]_\ell \cong \prod_\alpha [X,Z]_\ell,$$

   so a wedge sum of stable equivalences is a stable equivalence.

[1] In points 4 and 5 of Theorem 6.1.26, it is not necessary to assume we have free cofibrations. It is enough if the maps of spectra have the homotopy extension property at each spectrum level.

3. We define a long exact sequence of abelian groups

$$[\Sigma Cf, Z]_\ell \longrightarrow [\Sigma Y, Z]_\ell \xrightarrow{-\circ\Sigma f} [\Sigma X, Z]_\ell \longrightarrow [Cf, Z]_\ell \longrightarrow [Y, Z]_\ell \xrightarrow{-\circ f} [X, Z]_\ell$$

just as in Proposition 3.2.19. If f is a stable equivalence then f and Σf induce bijections, so $[Cf, Z]_\ell = 0$, so Cf is stably equivalent to zero. Conversely, if Cf is stably equivalent to zero then so is ΣCf. By the exact sequence, Σf is a stable equivalence, and therefore f is a stable equivalence.

4. The cofibration assumption implies that the pushout $X \cup_A Y$ is equivalent to the homotopy pushout $X \cup_A^h Y$. As in Lemma 2.4.14, the homotopy cofiber of $f : A \to X$ is equivalent to the homotopy cofiber of $\bar{f} : Y \to X \cup_A Y$. So if f is a stable equivalence, Cf is stably equivalent to zero, hence $C\bar{f}$ is stably equivalent to zero, so $\bar{f}$ is a stable equivalence.

5. As in Proposition 3.2.20, we get a $\lim^1$ exact sequence of abelian groups

$$0 \longrightarrow \lim{}^1 [\Sigma X^{(n)}, Z]_\ell \longrightarrow [X^{(\infty)}, Z]_\ell \longrightarrow \lim [X^{(n)}, Z]_\ell \longrightarrow 0.$$

If all of the maps $[X^{(n)}, Z]_\ell \to [X^{(0)}, Z]_\ell$ are bijections, the $\lim^1$ term vanishes and the map $[X^{(\infty)}, Z]_\ell \to [X^{(0)}, Z]_\ell$ is therefore a bijection.

6. A retract of an isomorphism on $[-, Z]_\ell$ is an isomorphism, so a retract of a stable equivalence is a stable equivalence.

□

6.1.6 The stable model structure

Now that we have proven the stability theorems, we have all the ingredients we need to give symmetric and orthogonal spectra a model structure with the stable equivalences.

Theorem 6.1.28 (Hovey-Shipley-Smith)**.** *The category of symmetric spectra* $\mathbf{Sp}^\Sigma$ *has a **stable model structure** in which*

- *the cofibrations are the retracts of the relative cellular spectra,*
- *the weak equivalences are the stable equivalences, and*
- *the fibrations are the stable fibrations, i.e. the maps that are fibrations at each spectrum level and such that the squares*

$$\begin{array}{ccc} X_i & \longrightarrow & \Omega^j X_{i+j} \\ \big\downarrow{\scriptstyle p_i} & & \big\downarrow{\scriptstyle \Omega^j p_{i+j}} \\ Y_i & \longrightarrow & \Omega^j Y_{i+j} \end{array}$$

are homotopy pullbacks.

Theorem 6.1.29 (Mandell-May-Schwede-Shipley)**.** *The category of orthogonal spectra* $\mathbf{Sp}^O$ *has a stable model structure with the same description.*

Proof. All of the arguments in Section 5.6 now apply, using Lemma 6.1.15, Theorem 6.1.26, and Corollary 6.1.27 instead of the earlier results on sequential spectra. □

Remark 6.1.30. This model structure is proper in the sense of Definition 5.2.29. It is left proper by part (4) of Theorem 6.1.26. It is right proper by the dual of the proof of (4), taking homotopy fibers in a pullback square and using part (4) of Corollary 6.1.27.

We have seen that in symmetric spectra, not every stable equivalence is a π_*-isomorphism. For orthogonal spectra, we are a bit luckier, and the two classes of maps coincide. We first check this on the free spectra.

Lemma 6.1.31. *If A is a well-based space, the canonical map from the free sequential spectrum on A to the free orthogonal spectrum on A,*

$$F_i^{\mathbb{N}} A \longrightarrow F_i^O A,$$

is a π_*-isomorphism.

Proof. At spectrum level n, this is the map

$$A \wedge S^{n-i} \longrightarrow A \wedge \mathbf{S}^O(i,n).$$

It suffices to check that the map $S^{n-i} \to \mathbf{S}^O(i,n)$, that includes the compactification of a single fiber into the entire Thom space, becomes more highly connected as $n \to \infty$. This follows from the geometric fact that the space of embeddings of $\mathbb{R}^i$ into $\mathbb{R}^n$ becomes more highly connected as $n \to \infty$. □

Proposition 6.1.32. *For orthogonal spectra, every stable equivalence is a π_*-isomorphism.*

Proof. Since level equivalences are π_*-isomorphisms, without loss of generality X and Y are cofibrant. We want to show that every stable equivalence $X \to Y$ is a π_*-isomorphism.

Let J be the generating acyclic cofibrations for orthogonal spectra. The maps of J are built from free orthogonal spectra, each of which has the same homotopy groups as the corresponding free sequential spectrum by Lemma 6.1.31. It now straightforward to argue that each J-cofibration is a π_*-isomorphism. By Ken Brown's Lemma, therefore every stable equivalence of cofibrant orthogonal spectra $X \to Y$ is a π_*-isomorphism. □

Remark 6.1.33. For symmetric spectra, we have shown that stable equivalences and π_*-isomorphisms coincide when we restrict attention to Ω-spectra. There is a somewhat larger class of **semistable** symmetric spectra, on which the stable equivalences also agree with the π_*-isomorphisms.

A spectrum is semistable if the symmetric group actions Σ_n induce no interesting action on the stable homotopy groups $\pi_*(X)$, only the action that multplies by the sign of the permutation. Equivalently, if the map λ^* in Proposition 6.1.22 is a π_*-isomorphism. Every Ω-spectrum is semistable, as is the underlying symmetric spectrum of any orthogonal spectrum. See [Sch07] for a much more extensive discussion.

Proposition 6.1.34. *The prolongation and forgetful functors form Quillen equivalences* $\mathbf{Sp}^{\mathbb{N}} \simeq \mathbf{Sp}^{\Sigma} \simeq \mathbf{Sp}^{O}$.

Proof. It is easy to see that the forgetful functors preserve fibrations. The acyclic fibrations are the same thing as the I-injective maps, which in every case are the level acyclic fibrations, so they are also preserved. Therefore all of the forgetful functors are right Quillen.

We next check that $(P_{\mathbb{N}}^{O}, U_{\mathbb{N}}^{O})$ is a Quillen equivalence. We abbreviate the forgetful and prolongation functors to P and U. The unit of the adjunction $(\mathbb{L}P \dashv \mathbb{R}U)$ can be described on any cofibrant sequential spectrum X as the composite

$$X \longrightarrow UPX \longrightarrow URPX$$

that applies the unit of $(P \dashv U)$ and then a fibrant replacement in the middle. But this is equivalent to the plain old unit map $X \to UPX$, in other words the canonical map from each sequential spectrum to the free orthogonal spectrum on it.

If X is a free spectrum $F_i^{\mathbb{N}}A$ on a cell complex A, this canonical map is π_*-isomorphism by Lemma 6.1.31. We observe using (6.1.12) that prolongation P commutes with homotopy cofibers. So by Lemma 2.4.9 and the 5-lemma, if the unit map is an equivalence for any two spectra in a cofiber sequence $X \to Y \to Cf$, it is also an equivalence on the third. Using this, we can show by induction that the unit map is an equivalence for each of the skeleta $X^{(n)}$ of a cellular spectrum X. Since P preserves sequential compositions of cofibrations, we conclude also that the unit map is an equivalence on X itself.

The forgetful functor $\mathbb{R}U \simeq U$ reflects weak equivalences, in the sense that $UX \to UY$ is a weak equivalence iff $X \to Y$ is a weak equivalence. It follows formally that the derived counit map $PQUY \to PUY \to Y$ must also be an equivalence. (See exercise 10.) Therefore we have a Quillen equivalence between sequential spectra and orthogonal spectra.

Next, we apply the same argument to the pair $(P_{\Sigma}^{O}, U_{\Sigma}^{O})$. We need to check that:

- U_{Σ}^{O} preserves weak equivalences. This is true because weak equivalences of orthogonal spectra are π_*-isomorphisms by Proposition 6.1.32, which are weak equivalences of symmetric spectra by Proposition 6.1.24.

- U^O_Σ reflects weak equivalences. This is true because up to equivalence, we may as well assume the orthogonal spectra are Ω-spectra, and if a map of orthogonal Ω-spectra $X \to Y$ is a stable equivalence as a map of symmetric spectra, then by Lemma 6.1.25 it's a level equivalence, and therefore it's a stable equivalence as a map of orthogonal spectra too.
- The unit map $X \to U^O_\Sigma P^O_\Sigma X$ is a stable equivalence (not a π_*-isomorphism!) when X is a free spectrum $F^\Sigma_n A$. Using the five-lemma and Theorem 6.1.26, we reduce this to the case where $A = S^n$. Then we compare to $F^\Sigma_0 S^0$:

$$\begin{tikzcd}
F^\Sigma_n S^n \arrow[r, "\sim"] \arrow[d, "\eta"'] & F^\Sigma_0 S^0 \arrow[d, "\eta", "\cong"'] \\
U^O_\Sigma F^O_n S^n \arrow[r, "\sim"] & U^O_\Sigma F^O_0 S^0
\end{tikzcd}$$

  The horizontal maps are stable equivalences of symmetric spectra by Example 6.1.20, and the unit map on the right-hand side is an isomorphism $\mathbb{S} \cong \mathbb{S}$, so the unit map on the left-hand side must be a stable equivalence of symmetric spectra as well.

Finally, we give a gasp of dismay: this argument does not apply to $(P^\Sigma_\mathbb{N}, U^\Sigma_\mathbb{N})$, because of the mismatch between π_*-isomorphisms and stable equivalences in $\mathbf{Sp}^\Sigma$. But it doesn't matter! We know that $\mathbb{L}P^O_\Sigma$ is an equivalence of categories, and the composite

$$\mathbb{L}P^O_\mathbb{N} \simeq (\mathbb{L}P^O_\Sigma) \circ (\mathbb{L}P^\Sigma_\mathbb{N})$$

is an equivalence of categories too. It follows that $\mathbb{L}P^\Sigma_\mathbb{N}$ must also be an equivalence of categories. This proves that $(P^\Sigma_\mathbb{N}, U^\Sigma_\mathbb{N})$ is a Quillen equivalence and finishes the proof. □

Since the stable equivalences between Ω-spectra are π_*-isomorphisms, we can use the fibrant replacement functor R to right-derive the homotopy groups of a symmetric spectrum.

Definition 6.1.35. In symmetric spectra, the right-derived functor of the homotopy groups

$$\mathbb{R}\pi_*(X) := \pi_*(RX)$$

is called the **true homotopy groups**.

A map is a stable equivalence iff it induces isomorphisms on the true homotopy groups. Along the Quillen equivalences of Proposition 6.1.34, the true homotopy groups of a symmetric spectrum X correspond to the homotopy groups of its associated orthogonal spectrum $\mathbb{L}P_{\Sigma}^{O}X$ or sequential spectrum $\mathbb{R}U_{\mathbb{N}}^{\Sigma}X$. We therefore get all of the same theorems and calculations for the true homotopy groups that we did back in sequential spectra – for instance, for a symmetric spectrum X we have

$$[\mathbb{S}^k, X]_s \cong \mathbb{R}\pi_*(X)$$

where $\mathbb{S}^k = F_0^{\Sigma}S^k$ when $k \geq 0$ and $F_{|k|}^{\Sigma}S^0$ when $k < 0$. For this reason, we often drop the $\mathbb{R}$ and think of these as the real homotopy groups, and the ones we defined before as the "naïve homotopy groups" of symmetric spectra.

6.2 The smash product

In this section we define the smash product $X \wedge Y$ of symmetric or orthogonal spectra. We show it is equivalent to the handicrafted smash product of Definition 2.3.23, and that it is associative, commutative, and unital up to isomorphism:

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z), \qquad X \wedge Y \cong Y \wedge X, \qquad \mathbb{S} \wedge X \cong X.$$

We also show that it has right adjoints in each variable, and plays well with the stable equivalences, so that it defines a Quillen tensor, and passes to a smash product on the homotopy category Ho **Sp**. The construction of this smash product was a major development in the subject in the 1990s.

6.2.1 Explicit definition

In the handicrafted smash product, $(X \wedge Y)_n$ is defined to be $X_p \wedge Y_q$ for *some* values of p and q that add to n. However, it's much better to take *all* values of p and q that add to n, and use the structure of a symmetric spectrum to glue them together.

Definition 6.2.1. Let X and Y be symmetric spectra. The **smash product** $X \wedge Y$ is the symmetric spectrum that at level n is the quotient

$$(X \wedge Y)_n = \bigvee_{p+q=n} (\Sigma_{p+q})_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)/\sim$$

that identifies the images of the following maps for all p, q, and r adding to n:

$$\begin{tikzcd}
(\Sigma_{p+q+r})_+ \wedge X_p \wedge Y_q \wedge S^r \arrow[r, "\mathrm{id}\wedge\mathrm{id}\wedge\upsilon_{q,r}"] \arrow[d, "(-\circ(1\times\tau_{r,q}))\wedge\mathrm{id}\wedge\gamma"] & (\Sigma_{p+q+r})_+ \wedge X_p \wedge Y_{q+r} \\
(\Sigma_{p+q+r})_+ \wedge X_p \wedge S^r \wedge Y_q \arrow[r, "\mathrm{id}\wedge\xi_{p,r}\wedge\mathrm{id}"] & (\Sigma_{p+q+r})_+ \wedge X_{p+r} \wedge Y_q
\end{tikzcd}$$

Here $G \wedge_H X$ denotes the quotient of $G \wedge X$ by the relation $(gh, x) \sim (g, hx)$, $\xi_{p,r}$ and $\upsilon_{q,r}$ are the iterated bonding maps from Definition 6.1.2, γ is the symmetry isomorphism $Y_q \wedge S^r \cong S^r \wedge Y_q$, and $- \circ (1 \times \tau_{r,q})$ is the operation that pre-composes the permutation $\sigma \in \Sigma_{p+q+r}$ with the block permutation $1 \times \tau_{r,q}$ that leaves the first p letters alone, and switches the next r letters past the last q letters.

The symmetric group Σ_n acts on $(X \wedge Y)_n$ by multiplying into Σ_{p+q} on the left, and the bonding map $(X \wedge Y)_n \wedge S^1 \to (X \wedge Y)_{n+1}$ is obtained from the bonding map of Y. (However, because of the quotient we applied, we get the same result if we use the bonding map of X, along with some extra permutations.)

Intuitively, the smash product is the disjoint union of all of the terms along one diagonal in the grid in Definition 2.3.23, modulo relations that glue these terms together along bonding maps coming from the terms below that diagonal. We also add in the symmetric group $(\Sigma_{p+q})_+$ so that the full symmetric group will act on the resulting space. For instance, at spectrum level 2, we get a single space built out of the following six spaces:

$$\begin{tikzcd}
X_0 \wedge Y_2 \arrow[d, dash] & & \\
X_0 \wedge Y_1 \arrow[r, dash] \arrow[d, dash] & X_1 \wedge Y_1 \arrow[d, dash] & \\
X_0 \wedge Y_0 \arrow[r, dash] & X_1 \wedge Y_0 \arrow[r, dash] & X_2 \wedge Y_0
\end{tikzcd}$$

This identification comes with an extra factor of $\tau_{r,q}$, which you should think of as the Koszul sign rule for spectra.

You can also think of $\tau_{r,q}$ as a correcting factor that undoes the permutation we do when we switch the S^r past Y_q. That switch feels like a block permutation $\tau_{q,r}$ that wasn't there before, so we add in the inverse $\tau_{r,q}$ to "undo" it. See also Remark 6.2.5.

If X and Y are orthogonal spectra, the smash product is defined the same way, but with orthogonal groups in the place of the symmetric groups:

$$(X \wedge Y)_n = \bigvee_{p+q=n} O(p+q)_+ \wedge_{O(p)\times O(q)} (X_p \wedge Y_q)/\sim$$

Example 6.2.2. The smash product $\mathbb{S} \wedge \mathbb{S}$ is isomorphic to $\mathbb{S}$ again. To see this, we observe that all of the terms $X_p \wedge Y_q$ are n-spheres S^n, and all of the relations give the identity map of S^n. The permutations $\tau_{r,q}$ are absolutely necessary for this! If we didn't work them in, we would get the orbit space $(S^n)_{\Sigma_n}$ at level n, which is contractible!

Example 6.2.3. The argument of the previous example also shows that $\Sigma^\infty A \wedge \Sigma^\infty B \cong \Sigma^\infty(A \wedge B)$. So a smash product of suspension spectra is a suspension spectrum again.

Example 6.2.4. Generalizing in a different direction, if X is any symmetric spectrum then $X \wedge \mathbb{S} \cong X$. This is also not hard to show: all the terms $X_p \wedge S^q$ get identified to $X_n \wedge S^0$, and we're left with just X_n at spectrum level n.

Remark 6.2.5. You should imagine a symmetric spectrum as a spectrum in which X_n has "secret sphere coordinates," and Σ_n permutes those coordinates around. Of course, there actually is a map $\Sigma^n X_0 \to X_n$ that commutes with the Σ_n-action, so on the image of that map, the points really do have sphere coordinates that Σ_n is permuting around.

With this intuition, the smash product identifies all copies of $X_p \wedge Y_q$ together along bonding maps, but when we use the bonding maps to move some spheres from X to Y, the sphere coordinates get shuffled out of order. The Koszul sign rule (in other words the permutation $\tau_{r,q}$) is there to shuffle everything back into the correct order.

Proposition 6.2.6. *There are natural isomorphisms*

$$X \wedge \mathbb{S} \cong X, \qquad X \wedge Y \cong Y \wedge X, \qquad (X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z).$$

Proof sketch. We won't do the proof in much detail since it is a tedious exercise to check everything carefully. We will just give the general idea.

For the first isomorphism, we use the equivalence relation to identify all the points in $X_p \wedge S^q$ to X_{p+q} using the bonding map of X.

For the middle isomorphism, we switch X with Y, but introduce a permutation to correct for the switch, another instance of the spectral Koszul sign rule.

For the last isomorphism, we define a three-fold variant of the smash product $X \wedge Y \wedge Z$, and then identify both $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ with this three-fold smash product. □

Remark 6.2.7. It's important to emphasize how strong this is: $\mathbb{S} \wedge X$ is *isomorphic* to X. Not just stably equivalent, or something wishy-washy like that. They are isomorphic.

One can proceed further and carefully check that the isomorphisms in Proposition 6.2.6 are coherent in the sense outlined above Theorem 4.1.4. We conclude:

Theorem 6.2.8. *Symmetric spectra* $\mathbf{Sp}^\Sigma$ *and orthogonal spectra* $\mathbf{Sp}^\Sigma$ *are both symmetric monoidal categories under the smash product* $\wedge$.

There is also a somewhat easier way to do this step, see Theorem 6.2.22 below.

6.2.2 Explicit function spectra

To define function spectra $F(X, Y)$, it's convenient to first capture the universal property of the smash product of symmetric spectra, in a way that's similar to the universal property of the tensor product $\otimes$ of abelian groups.

Definition 6.2.9. Let X, Y, and Z be symmetric spectra, with bonding maps labeled ξ, υ, and ζ, respectively. A **bimorphism** $(X,Y) \longrightarrow Z$ is a collection of $\Sigma_p \times \Sigma_q$-equivariant maps of pointed spaces

$$b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$$

such that the following diagram commutes.

$$\begin{tikzcd}
X_p \wedge Y_q \wedge S^1 \arrow[r, equal] \arrow[d, "b_{p,q}\wedge \mathrm{id}"] & X_p \wedge Y_q \wedge S^1 \arrow[r, "\mathrm{id}\wedge\tau"] \arrow[d, "\mathrm{id}\wedge \upsilon_q"] & X_p \wedge S^1 \wedge Y_q \arrow[d, "\xi_p \wedge \mathrm{id}"] \\
Z_{p+q} \wedge S^1 \arrow[d, "\zeta_{p+q}"] & X_p \wedge Y_{q+1} \arrow[d, "b_{p,q+1}"] & X_{p+1} \wedge Y_q \arrow[d, "b_{p+1,q}"] \\
Z_{p+q+1} \arrow[r, equal] & Z_{p+q+1} \arrow[r, "1\times\tau_{q,1}"'] & Z_{p+1+q}
\end{tikzcd} \tag{6.2.10}$$

The following is a straightforward check of the definitions.

Lemma 6.2.11. *The smash product $X \wedge Y$ is the universal symmetric spectrum that receives a bimorphism from (X,Y).*

Definition 6.2.12. We define the **function spectrum** $F(Y,Z)$ as follows. At level p, it is the subspace of the product

$$F(Y,Z)_p \subseteq \prod_{q\geq 0} \mathrm{Map}_*(Y_q, Z_{p+q})$$

consisting of all tuples of Σ_q-equviariant maps $f_q \colon Y_q \to Z_{p+q}$ that commute with the bonding maps of Y and Z:

$$\begin{tikzcd}
Y_q \wedge S^1 \arrow[r, "\upsilon_q"] \arrow[d, "f_q\wedge\mathrm{id}"'] & Y_{q+1} \arrow[d, "f_{q+1}"] \\
Z_{p+q} \wedge S^1 \arrow[r, "\zeta_{p+q}"] & Z_{p+q+1}
\end{tikzcd}$$

We define a Σ_p-action on this space by composing the maps f_q with the action of $\Sigma_p \times 1 \leq \Sigma_{p+q}$ on the space Z_{p+q}. We define the bonding map

$$F(Y,Z)_p \to \Omega F(Y,Z)_{p+1} \cong F(Y,\Omega Z)_{p+1}$$

by taking each collection of maps $f_q \colon Y_q \to Z_{p+q}$ to the collection of maps

$$Y_q \xrightarrow{f_q} Z_{p+q} \xrightarrow{\tilde{\zeta}_{p+q}} \Omega Z_{p+q+1} \xrightarrow{(1\times\tau_{q,1})} \Omega Z_{p+1+q}.$$

It is straightforward to check this satisfies the definition of a symmetric spectrum (Definition 6.1.2).

Now, if we have a map of symmetric spectra $X \to F(Y,Z)$, the individual maps $X_p \to F(Y,Z)_p$ rearrange to maps $b_{p,q}\colon X_p \wedge Y_q \to Z_{p+q}$, making the left-hand region in (6.2.10) commute. Furthermore, requiring that $X_p \to F(Y,Z)_p$ is Σ_p-equivariant and that each f_q is Σ_q-equivariant is equivalent to asking that each $b_{p,q}$ is $\Sigma_p \times \Sigma_q$-equivariant. Finally, the maps $X_p \to F(Y,Z)_p$ must commute with the bonding maps:

$$\begin{tikzcd}
X_p \arrow[r, "\tilde{\tilde{\xi}}_p"] \arrow[d] & \Omega X_{p+1} \arrow[d] \\
F(Y,Z)_p \arrow[r] & F(Y,\Omega Z)_{p+1}.
\end{tikzcd}$$

This rearranges to the commutativity of the outside rectangle in (6.2.10). All together, this proves:

Lemma 6.2.13. *Maps of symmetric spectra $X \to F(Y,Z)$ correspond to maps of symmetric spectra $X \wedge Y \to Z$. In other words, $F(Y,-)$ is the right adjoint of $(-)\wedge Y$.*

It is clear that we can form the same constructions using orthogonal spectra, and the same result holds, with the same proof. In conclusion:

Theorem 6.2.14. *The symmetric monoidal structures on $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$ from Theorem 6.2.14 are both closed.*

As mentioned in Example 4.1.34, this means that we also have isomorphisms of spectra

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)), \qquad F(\mathbb{S}, X) \cong X,$$

and that a function spectrum out of a suspension spectrum $F(\Sigma^\infty K, X)$ is isomorphic to the cotensor spectrum $F(K,X)$ where we just apply $\mathrm{Map}_*(K,-)$ to each level of X separately. See also exercise 11.

6.2.3 Abstract definition

In this section we give a second definition of the smash product. It is isomorphic to the first definition, but it is described in a more abstract and global way, which will make it easier to prove some of its properties.

Recall from Lemma 6.1.8 that symmetric spectra are the same thing as diagrams of based spaces on the category $\mathbf{S}^\Sigma$, whose mapping spaces are

$$\mathbf{S}^\Sigma(m,n) = \bigvee_{\alpha\colon \underline{m} \hookrightarrow \underline{n}} S^{\underline{n}-\alpha(\underline{m})} \qquad \text{when } n \geq m.$$

If X and Y are two symmetric spectra, we can consider all of the smash products $X_p \wedge Y_q$ for $p,q \geq 0$. These form a diagram on the smash product category $\mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma$ defined in Section 5.8, exercise 30. The objects of this category are pairs of integers, and the maps are the smash product of the mapping spaces:

$$\begin{aligned}(\mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma)((p_0,q_0),(p_1,q_1)) &= \mathbf{S}^\Sigma(p_0,p_1) \wedge \mathbf{S}^\Sigma(q_0,q_1) \\ &= \left(\bigvee_{\alpha\colon \underline{p_0} \hookrightarrow \underline{p_1}} S^{\underline{p_1} - \alpha(\underline{p_0})} \right) \wedge \left(\bigvee_{\beta\colon \underline{q_0} \hookrightarrow \underline{q_1}} S^{\underline{q_1} - \beta(\underline{q_0})} \right).\end{aligned}$$

Definition 6.2.15. We define a functor $\amalg\colon \mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma \to \mathbf{S}^\Sigma$ as follows. On objects, we take each pair (p,q) to the sum $p+q$. On morphisms, we take each pair of maps $\alpha\colon \underline{p_0} \hookrightarrow \underline{p_1}$ and $\beta\colon \underline{q_0} \hookrightarrow \underline{q_1}$ to the disjoint union

$$(\alpha \amalg \beta)\colon \underline{p_0+q_0} \hookrightarrow \underline{p_1+q_1}.$$

We have bijections of sets

$$\underline{p_1+q_1} - (\alpha \amalg \beta)(\underline{p_0+q_0}) \cong (\underline{p_1} - \alpha(\underline{p_0})) \amalg (\underline{q_1} - \beta(\underline{q_0}))$$

which give homeomorphisms of spheres

$$S^{\underline{p_1+q_1} - (\alpha\amalg\beta)(\underline{p_0+q_0})} \cong S^{\underline{p_1} - \alpha(\underline{p_0})} \wedge S^{\underline{q_1} - \beta(\underline{q_0})}, \tag{6.2.16}$$

and together these define the desired maps

$$\begin{array}{ccc} \left(\bigvee_{\alpha\colon \underline{p_0} \hookrightarrow \underline{p_1}} S^{\underline{p_1} - \alpha(\underline{p_0})}\right) \wedge \left(\bigvee_{\beta\colon \underline{q_0} \hookrightarrow \underline{q_1}} S^{\underline{q_1} - \beta(\underline{q_0})}\right) & \longrightarrow & \bigvee_{\gamma\colon \underline{n_0} \hookrightarrow \underline{n_1}} S^{\underline{n_1} - \gamma(\underline{n_0})} \\ \mathbf{S}^\Sigma(p_0,p_1) \wedge \mathbf{S}^\Sigma(q_0,q_1) & \longrightarrow & \mathbf{S}^\Sigma(p_0+q_0, p_1+q_1) \end{array}$$

finishing the definition of the functor $\mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma \to \mathbf{S}^\Sigma$. It is straightforward to check this respects composition (exercise 15). We call this functor $\amalg$, because in essence it is taking the disjoint union of finite sets, and concatenating sphere coordinates.

Definition 6.2.17. The **smash product** $X \wedge Y$ is the $\mathbf{S}^\Sigma$-diagram obtained from the smash products $(X_p \wedge Y_q)_{p,q\geq 0}$ by left Kan extension along $\amalg$ (Definition 5.4.13):

$$X \wedge Y = (\amalg)_!(X_- \wedge Y_-).$$

We can also define an analogous functor $\oplus\colon \mathbf{S}^O \wedge \mathbf{S}^O \to \mathbf{S}^O$ for orthogonal spectra, see exercise 16.

Proposition 6.2.18. *This abstract smash product is isomorphic to the explicit smash product from Definition 6.2.1.*

Proof. The expression for the left Kan extension from (5.4.15), adapted to based spaces (so that the Cartesian products $\times$ becomes smash products $\wedge$), gives in this case the coequalizer of the two maps

$$\left(\bigvee_{(p_0,q_0),(p_1,q_1)} X_{p_0}\wedge Y_{q_0}\wedge \mathbf{S}^{\Sigma}(p_0,p_1)\wedge \mathbf{S}^{\Sigma}(q_0,q_1)\wedge \mathbf{S}^{\Sigma}(q_0+q_1,n)\right)\rightrightarrows\left(\bigvee_{(p,q)} X_p\wedge Y_q\wedge \mathbf{S}^{\Sigma}(p+q,n)\right)$$

that either have $\mathbf{S}^{\Sigma}(p_0,p_1)\wedge\mathbf{S}^{\Sigma}(q_0,q_1)$ act on $X_{p_0}\wedge Y_{q_0}$, or apply $\amalg$ to $\mathbf{S}^{\Sigma}(p_0,p_1)\wedge\mathbf{S}^{\Sigma}(q_0,q_1)$ and then compose the result with $\mathbf{S}^{\Sigma}(q_0+q_1,n)$.

Along these relations, the terms in which $p+q<n$ get identified to terms in which $p+q=n$, so that we get a quotient of the sum

$$\bigvee_{p+q=n} X_p\wedge Y_q\wedge \mathbf{S}^{\Sigma}(p+q,n)\cong \bigvee_{p+q=n} X_p\wedge Y_q\wedge (\Sigma_{p+q})_+.$$

The remaining relations turn out to give the relations we imposed in Definition 6.2.1, but it is tedious to check this and we omit the details. $\square$

The advantage of this definition of the smash product is that we can prove the following by a formal argument:

Lemma 6.2.19. *The smash product of two free spectra is free:*

$$F_d A\wedge F_e B\cong F_{d+e}(A\wedge B).$$

This holds if the free spectra and smash product are both taken in the category of symmetric spectra, or if they are both taken in the category of orthogonal spectra.

Proof. The smash products of the levels of $F_d A$ and $F_e B$ become

$$(F_d A)_p\wedge (F_e B)_q\cong A\wedge \mathbf{S}^{\Sigma}(d,p)\wedge B\wedge \mathbf{S}^{\Sigma}(e,q).$$

This is the free diagram on the category $\mathbf{S}^{\Sigma}\wedge\mathbf{S}^{\Sigma}$, at the object (d,e), using the space $(A\wedge B)$. The left Kan extension preserves free diagrams (Section 5.8, exercise 16) and therefore we get $F_{d+e}(A\wedge B)$ as the result. $\square$

Recall that $\Sigma^\infty A\wedge\Sigma^\infty B\cong\Sigma^\infty(A\wedge B)$ is a special case of this result. As in Section 5.8, exercise 20, we can also conclude that the smash product preserves cell complex spectra:

Corollary 6.2.20. *$\wedge$ preserves cellular spectra, and $X\wedge Y$ has a stable $(m+n)$-cell for every stable m-cell of X and stable n-cell of Y.*

This is illustrated at the beginning of Chapter 4.

It is likewise easier to prove that the smash products for symmetric and orthogonal spectra agree strictly in the following sense.

Proposition 6.2.21. *The prolongation from symmetric to orthogonal spectra commutes with the smash product,*

$$(P_\Sigma^O X) \wedge^{(O)} (P_\Sigma^O Y) \cong P_\Sigma^O (X \wedge^{(\Sigma)} Y).$$

Proof. Let $i\colon \mathbf{S}^\Sigma \to \mathbf{S}^O$ be the obvious inclusion. Using the fact that the smash product preserves colimits in each slot, the smash product of the levels of $P_\Sigma^O X$ and $P_\Sigma^O Y$ can be rewritten as the left Kan extension of $X_- \wedge Y_-$ along $i \wedge i\colon \mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma \to \mathbf{S}^O \wedge \mathbf{S}^O$. To finish taking the smash product in orthogonal spectra, we left Kan extend to $\mathbf{S}^O$ along the sum functor $\oplus$.

On the other hand, if we smashed X and Y in symmetric spectra first, we'd take the left Kan extension of $X_- \wedge Y_-$ along the sum functor $\amalg$ to get a symmetric spectrum, then left Kan extend along i to get an orthogonal spectrum. So, both sides of the desired equation are left Kan extensions along the two routes in the following commuting square:

$$\begin{tikzcd}
\mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma \arrow[r, "\amalg"] \arrow[d, "i\wedge i"'] & \mathbf{S}^\Sigma \arrow[d, "i"] \\
\mathbf{S}^O \wedge \mathbf{S}^O \arrow[r, "\oplus"'] & \mathbf{S}^O.
\end{tikzcd}$$

The results agree up to isomorphism because left Kan extensions are closed under composition (Section 5.8, exercise 17). □

6.2.4 Point-set rigidity of the smash product

The smash product has a convenient property that keeps us from getting errors when we pass between different definitions, or when we check that the associativity and commutativity isomorphisms are coherent.

Think of the smash product as a functor $\wedge\colon \mathbf{Sp}^\Sigma \times \mathbf{Sp}^\Sigma \to \mathbf{Sp}^\Sigma$, and consider natural transformations from this functor to itself. These are maps

$$\eta_{X,Y}\colon X \wedge Y \to X \wedge Y$$

that commute with maps in each variable.

Theorem 6.2.22 (Point-set rigidity of $\wedge$)**.** *The only such natural transformations are the identity transformation, and the zero transformation sending everything to the basepoint. Moreover, the same is also true for:*

- the n-fold smash product

  $$(X_1, X_2 \dots, X_n) \mapsto (((X_1 \wedge X_2) \wedge \dots) \wedge X_n)$$

  for every $n \geq 0$,

- the n-fold smash product of orthogonal spectra for every $n \geq 0$,
- the suspension spectrum functor Σ^∞, to either symmetric or orthogonal spectra,
- the n-fold smash product of spaces, and also the suspension spectrum of the n-fold smash product of spaces.

This is a useful fact because it allows us to “lock down” isomorphisms between different models of the smash product.

Corollary 6.2.23. *For any two functors isomorphic to the n-fold smash product, there is a unique natural isomorphism between them.*

Proof. Any two isomorphisms α, β can be composed to give a natural transformation $\beta^{-1}\alpha$ from the n-fold smash product to itself, that is not zero. This transformation must be the identity by Theorem 6.2.22, and therefore $\beta = \alpha$. □

In particular, the two definitions of the smash product we gave in Definition 6.2.1 and Definition 6.2.17 are isomorphic by a unique isomorphism.

Here are a few more applications. The first is another proof that symmetric spectra and orthogonal spectra form symmetric monoidal categories.

Second proof of Theorem 6.2.8. We explained how to define the associativity, commutativity, and unit isomorphisms in Proposition 6.2.6. Now we have to check the coherences listed in Theorem 4.1.4. But in each of these diagrams, the maps are all natural transformations. Going around the diagram gives a natural transformation from the n-fold smash product to itself. This must be the identity by Theorem 6.2.22. Therefore the diagram commutes. □

Lemma 6.2.24. *Σ^∞ is a strong symmetric monoidal functor* $\mathbf{Top}_* \to \mathbf{Sp}^\Sigma$ *or* $\mathbf{Sp}^O$.

Proof. The isomorphisms $\Sigma^\infty A \wedge \Sigma^\infty B \cong \Sigma^\infty(A \wedge B)$ were constructed in Lemma 6.2.19, and $\Sigma^\infty S^0 \cong \mathbb{S}$ is obvious. As in the previous proof, all of the coherences for a symmetric monoidal functor are automatically satisfied, because the functor $\Sigma^\infty X_1 \wedge \ldots \wedge X_n$ has no nontrivial automorphisms. □

Proof of Theorem 6.2.22. We handle the n-fold smash product of symmetric spectra; the other cases are analogous. Let η be a natural transformation from the n-fold smash product functor to itself.

We first examine what η does to n-tuples of free spectra $(F^\Sigma_{d_1} A_1, \ldots, F^\Sigma_{d_n} A_n)$. By Lemma 6.2.19, the smash product of these n spectra is equivalent to the free spectrum $F^\Sigma_{d_1+\ldots+d_n}(A_1 \wedge \ldots \wedge A_n)$.

The simplest case is the n-tuple consisting entirely of sphere spectra, $(F_0^\Sigma S^0, \ldots, F_0^\Sigma S^0)$, whose smash product is $F_0^\Sigma S^0$. Here η gives a map of spectra

$$F_0^\Sigma S^0 \longrightarrow F_0^\Sigma S^0.$$

But $F_0^\Sigma S^0$ is a free spectrum, so this is determined a choice of point in S^0 at level 0. There are only two possibilities here: η is the identity map, or the zero map.

Let's examine the case where η is the identity on $(F_0^\Sigma S^0, \ldots, F_0^\Sigma S^0)$. Moving next to $(F_{d_1}^\Sigma S^0, \ldots, F_{d_n}^\Sigma S^0)$, η gives a map

$$F_{d_1+\ldots+d_n}^\Sigma S^0 \longrightarrow F_{d_1+\ldots+d_n}^\Sigma S^0.$$

Again this is determined by what it does at spectrum level $(d_1 + \ldots + d_n)$, where it gives a point in the space $(\Sigma_{d_1+\ldots+d_n})_+$. In other words, a permutation σ, or the basepoint.

We claim that this point has to be the identity permutation $\sigma = \mathrm{id}$. To see this, consider any point $(t_1, \ldots, t_n) \in S^{d_1} \wedge \ldots \wedge S^{d_n}$. We can choose maps of spectra $F_{d_i}^\Sigma S^0 \to F_0^\Sigma S^0$ which at level d_i send the non-basepoint of S^0 to the point $t_i \in S^{d_i}$. Since η is a natural transformation, this square commutes for all choices of $(t_1, \ldots, t_n)$:

$$\begin{tikzcd}
(\Sigma_{d_1+\ldots+d_n})_+ \arrow[r, "\cdot\sigma"] \arrow[d, "(-)(t_1,\ldots,t_n)"'] & (\Sigma_{d_1+\ldots+d_n})_+ \arrow[d, "(-)(t_1,\ldots,t_n)"] \\
S^{d_1+\ldots+d_n} \arrow[r, "\mathrm{id}"] & S^{d_1+\ldots+d_n}
\end{tikzcd}$$

Since $\Sigma_{d_1+\ldots+d_n}$ acts faithfully on the sphere $S^{d_1+\ldots+d_n}$, we must have $\sigma = \mathrm{id}$. In summary, η acts as the identity on the n-tuple $(F_{d_1}^\Sigma S^0, \ldots, F_{d_n}^\Sigma S^0)$.

Next we examine an arbitrary n-tuple of free spectra $(F_{d_1}^\Sigma A_1, \ldots, F_{d_n}^\Sigma A_n)$. Each collection of choices of point $a_i \in A_i$ gives a sequence of maps $F_{d_i}^\Sigma S^0 \to F_{d_i}^\Sigma A_i$, and applying η to this sequence of maps gives a commuting square

$$\begin{tikzcd}
F_{d_1+\ldots+d_n}^\Sigma S^0 \wedge \ldots \wedge S^0 \arrow[r, "\mathrm{id}"] \arrow[d] & F_{d_1+\ldots+d_n}^\Sigma S^0 \wedge \ldots \wedge S^0 \arrow[d] \\
F_{d_1+\ldots+d_n}^\Sigma A_1 \wedge \ldots \wedge A_n \arrow[r, "\eta"] & F_{d_1+\ldots+d_n}^\Sigma A_1 \wedge \ldots \wedge A_n
\end{tikzcd}$$

The commutativity of the square implies that the bottom map must be the identity on any point of the form

$$(a_1, \ldots, a_n, f) \in A_1 \wedge \ldots \wedge A_n \wedge \mathbf{S}^\Sigma(d_1 + \ldots + d_n, m).$$

But by varying $(a_1, \ldots, a_n)$ this gives every point, and so η is the identity on every n-tuple of free spectra.

It is left as an exercise to conclude that η is therefore the identity on all n-tuples of spectra (exercise 22). In the remaining case, we assume that $\eta = 0$ on the n-tuple $(F_0^\Sigma S^0, \ldots, F_0^\Sigma S^0)$, and follow the same steps to conclude that $\eta = 0$ on all n-tuples of spectra. $\square$

We can prove the same for prolongation, which also helps us check it is a symmetric monoidal functor. We combine the two results into one:

Theorem 6.2.25. *Prolongation P_Σ^O is a strong symmetric monoidal functor.*

Proof. As in the previous proofs, it suffices to show that the functor that takes an n-tuple of symmetric spectra, takes their smash product, and prolongs the result to orthogonal spectra, has no nontrivial automorphisms. The proof is the same as the proof of Theorem 6.2.22. For the part deferred to exercise 22, we observe that if $X \to Y$ is a levelwise surjection of symmetric spectra, the map of prolongations $P_\Sigma^O X \to P_\Sigma^O Y$ is also levelwise surjective by examination of the formula (5.4.21). □

6.2.5 Symmetric and orthogonal ring spectra

Recall from Definition 4.1.16 that a ring spectrum is a monoid for the smash product. Now that we have an honest definition of the smash product, we get an honest definition of ring spectra:

Definition 6.2.26. A **symmetric ring spectrum** is a monoid in $\mathbf{Sp}^\Sigma$. So it consists of a symmetric spectrum R with maps

$$\mu\colon R \wedge R \longrightarrow R, \qquad \eta\colon \mathbb{S} \longrightarrow R,$$

such that the diagrams in Definition 4.1.11 commute. A **commutative symmetric ring spectrum** is a commutative monoid object in $\mathbf{Sp}^\Sigma$, and an R**-module spectrum** is a module object over R, with action map

$$\alpha\colon R \wedge M \longrightarrow M,$$

such that the diagrams in Definition 4.1.11 commute. An **orthogonal ring spectrum** and modules over it are defined in the same way in $\mathbf{Sp}^O$.

Remark 6.2.27. Using the definition of a bimorphism from Definition 6.2.9, we can see that the ring structure is determined by a collection of maps

$$R_p \wedge R_q \longrightarrow R_{p+q}$$

that interact well with the bonding maps of R. Similarly an R-module is determined by maps

$$R_p \wedge M_q \longrightarrow M_{p+q}.$$

Lemma 6.2.28. *The prolongation P_Σ^O of a symmetric ring spectrum is an orthogonal ring spectrum, and the forgetful functor U_Σ^O of an orthogonal ring spectrum is a symmetric ring spectrum. These functors likewise preserve modules over ring spectra, and commutative ring spectra.*

Proof. Follows immediately from Proposition 4.1.23 and Section 4.3, exercise 12. □

Example 6.2.29. The sphere spectrum $\mathbb{S}$ is an orthogonal ring spectrum in a canonical way. It is therefore also a symmetric ring spectrum.

Example 6.2.30. If G is a topological group or monoid, the **spherical group ring**

$$\mathbb{S}[G] = \mathbb{S} \wedge G_+ = \Sigma^\infty_+ G$$

is the orthogonal ring spectrum given by the suspension spectrum of G, with multiplication coming from that of G. This is a ring spectrum by Proposition 4.1.23 and the fact that Σ^∞_+ is a strong symmetric monoidal functor (Lemma 6.2.24).

Remark 6.2.31. We will prove in **??** that for any ordinary ring R, the Eilenberg-Maclane spectrum HR can also be made into an orthogonal ring spectrum.

Definition 6.2.32. We define the category of symmetric ring spectra $\mathbf{Sp}^\Sigma_{\mathrm{Alg}}$ by saying that an object is a symmetric ring spectrum R, and a morphism is a map of spectra $f: R \to S$ that commutes with the multiplication and unit maps:

$$\begin{tikzcd} R \wedge R \arrow[r, "f \wedge f"] \arrow[d, "\mu_R"'] & S \wedge S \arrow[d, "\mu_S"] \\ R \arrow[r, "f"'] & S \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \arrow[r, equal] \arrow[d, "\eta_R"'] & \mathbb{S} \arrow[d, "\eta_S"] \\ R \arrow[r, "f"'] & S. \end{tikzcd}$$

We also sometimes call such a map a **ring homomorphism**. The category of commutative ring spectra $\mathbf{Sp}^\Sigma_{\mathrm{CAlg}}$ is the full subcategory on the rings that are commutative. (In other words, the morphisms are the same.)

For a fixed symmetric ring spectrum R, the category of module spectra R-$\mathbf{Mod}^\Sigma$ has objects the R-modules, and morphisms the maps of spectra $f: M \to N$ that commute with the R-action:

$$\begin{tikzcd} R \wedge M \arrow[r, "\mathrm{id} \wedge f"] \arrow[d, "\alpha_M"'] & R \wedge N \arrow[d, "\alpha_N"] \\ M \arrow[r, "f"'] & N. \end{tikzcd}$$

We also sometimes call this an R-**linear map**. We can also do the same definitions with orthogonal spectra.

Example 6.2.33. By Section 4.3, exercise 10, the category of $\mathbb{S}$-modules in symmetric spectra, $\mathbb{S}$-$\mathbf{Mod}^\Sigma$, is equivalent to the category of symmetric spectra $\mathbf{Sp}^\Sigma$. Likewise, the category of $\mathbb{S}$-modules in orthogonal spectra, $\mathbb{S}$-$\mathbf{Mod}^O$, is equivalent to $\mathbf{Sp}^O$. In other words, every spectrum is a module over the sphere spectrum in a canonical way.

Example 6.2.34. We will later see that $H\mathbb{Z}$-module spectra up to stable equivalence correspond to unbounded chain complexes over $\mathbb{Z}$.

Lemma 6.2.28 tells us that we get functors on ring spectra

$$P^O_\Sigma \colon \mathbf{Sp}^\Sigma_{\mathrm{Alg}} \longleftrightarrow \mathbf{Sp}^O_{\mathrm{Alg}} \colon U^O_\Sigma.$$

As with ordinary spectra, we really want to think about ring spectra up to stable equivalence.

Definition 6.2.35. A **stable equivalence of ring spectra** is a homomorphism of ring spectra $R \to S$ that is a stable equivalence on the underlying spectra. A stable equivalence of modules, or of commutative rings, is defined similarly.

We said after Definition 2.1.10 that stable homotopy theory is the study of spectra up to stable equivalence. By analogy with this:

Higher algebra is the study of ring spectra and their modules up to stable equivalence.

To do this properly, we will need to define a model category on ring spectra with these stable equivalences. This will be done in **??** and exercise 24. For now, we state the result:

Theorem 6.2.36. *There are model structures on $\mathbf{Sp}^\Sigma_{\mathrm{Alg}}$ and $\mathbf{Sp}^O_{\mathrm{Alg}}$ where the weak equivalences and fibrations are the ring homomorphisms that on the underlying spectra are equivalences or fibrations in the stable model structure from Theorem 5.2.11.*

The same is also true for modules over a fixed ring R-**Mod**. We also get such a model structure for commutative ring spectra $\mathbf{Sp}_{\mathrm{CAlg}}$ if we use the "positive" stable model structure (see Theorem 6.4.2). Every stable equivalence of rings $R \xrightarrow{\sim} S$ gives a Quillen equivalence R-**Mod** $\simeq$ S-**Mod**. We also have the following compatibility with our earlier model structure:

Lemma 6.2.37. *A cofibrant ring always has a cofibrant underlying spectrum. A cofibrant module over a cofibrant ring is also cofibrant as an ordinary spectrum.*

Remark 6.2.38. The theory of commutative ring spectra is more subtle – cofibrant objects in the model structure on *commutative* rings are actually not cofibrant as spectra. See **??**.

These results are enough to start working with rings and modules up to stable equivalence. For instance, it shows that we can left-derive the prolongation $P^O_\Sigma \colon \mathbf{Sp}^\Sigma_{\mathrm{Alg}} \to \mathbf{Sp}^O_{\mathrm{Alg}}$. In detail: if X is a symmetric ring spectrum, and we cofibrantly replace it as a symmetric spectrum, there is no reason to expect the result QX to also be a ring spectrum. However, if we cofibrantly replace X in the model structure on symmetric *ring spectra*, then

QX is a ring and $QX \to X$ is a ring homomorphism by definition. Then the prolongation $P_\Sigma^O QX$ is an orthogonal ring spectrum. By Lemma 6.2.37, QX is also cofibrant as a spectrum, so $P_\Sigma^O QX$ preserves weak equivalences, as desired.

This shows that we can effectively pass between symmetric and orthogonal ring spectra without losing information, so that the theory of *ring spectra up to equivalence* is the same in both cases.

6.3 Interaction with the model structure

Now that we understand how the smash product works *up to isomorphism*, let us consider how it works *up to weak equivalence*. The big thing to show is that the smash product preserves weak equivalences on cofibrant spectra. To do that, we prove that it plays nicely with the stable model structure, making spectra into something called a symmetric monoidal model category.

6.3.1 Symmetric monoidal model categories

Suppose that **C** is both a model category and a closed symmetric monoidal category. For instance, **C** might be:

- (CGWH) topological spaces $(\mathbf{Top}, \times, *)$,
- based spaces $(\mathbf{Top}_*, \wedge, S^0)$,
- chain complexes $(\mathbf{Ch}(k), \otimes_k, k[0])$,
- symmetric spectra $(\mathbf{Sp}^\Sigma, \wedge, \mathbb{S})$, or
- orthogonal spectra $(\mathbf{Sp}^O, \wedge, \mathbb{S})$.

At the moment, we haven't asked for the model structure and the symmetric monoidal structure on **C** to be compatible at all. But it is reasonable to ask for something.

Recall that the **pushout-product** of two maps $f\colon A \to X$ and $g\colon B \to Y$ in a symmetric monoidal category **C** is the map

$$f \square g\colon (X \otimes B) \cup_{A\otimes B} (A \otimes Y) \longrightarrow X \otimes Y.$$

We say that $\otimes$ is a **Quillen tensor** if $X \otimes (-)$ always has a right adjoint $\mathrm{Hom}(X,-)$, and if the pushout-product preserves cofibrations and acyclic cofibrations:

$$C \square C \subseteq C, \qquad C \square (W \cap C) \subseteq (W \cap C).$$

(We first saw pushout-products in Definition 5.5.6, and Quillen tensors in Definition 5.5.13. One of the three conditions in Definition 5.5.13 has become redundant because of the symmetry of $\otimes$.)

Definition 6.3.1. A **symmetric monoidal model category** is a closed symmetric monoidal category $(\mathbf{C}, \otimes, I)$ that is also a model category, such that

- $\otimes$ is a Quillen tensor, in other words if f and g are cofibrations then $f \square g$ is a cofibration, and if additionally one of them is a weak equivalence then $f \square g$ is a weak equivalence, and
- either the unit I is cofibrant, or the map $QI \otimes QX \to I \otimes QX$ is a weak equivalence for all $X \in \mathbf{C}$.

Remark 6.3.2. By Proposition 5.5.14, for a cofibrantly generated model category $\mathbf{C}$, it suffices to check the pushout-product condition on the generating cofibrations I and generating acyclic cofibrations J. We also recalled in Remark 5.5.17 a common rearrangement of the pushout-product axiom called **Quillen's SM7 axiom**.

Example 6.3.3. All of the examples in the above list are closed symmetric monoidal model categories. For the last two, we use the following proposition.

Proposition 6.3.4. *The smash product $\wedge$ is a Quillen tensor, both in symmetric spectra $\mathbf{Sp}^\Sigma$ and orthogonal spectra $\mathbf{Sp}^O$.*

Proof. We use the generic notation for a free spectrum $F_d A$ since our proof will apply equally well to symmetric or orthogonal spectra.

We need to calculate the pushout-product of two maps using the smash product. We use the fact that smashing with a suspension spectrum $\Sigma^\infty K$ is the same as smashing with the space K, and that the free spectrum $F_d A$ is the smash product of the spectrum $F_d S^0$ with the space A.

Using the simplifications of the pushout-product from Example 5.5.7 and Example 5.5.9, the pushout-product of two maps in I from Definition 5.6.1 is rewritten as

$$\begin{aligned} F_d(S^{k-1} \to D^k)_+ \square F_e(S^{\ell-1} \to D^\ell)_+ &\cong (\mathrm{id}_{F_d S^0}) \square (S^{k-1} \to D^k)_+ \square (\mathrm{id}_{F_e S^0}) \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong (\mathrm{id}_{F_d S^0}) \square (\mathrm{id}_{F_e S^0}) \square (S^{k-1} \to D^k)_+ \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong (\mathrm{id}_{F_{d+e} S^0}) \square (S^{k+\ell-1} \to D^{k+\ell})_+ \\ &\cong F_{d+e}(S^{k+\ell-1} \to D^{k+\ell})_+. \end{aligned}$$

This is a map in I, so we've shown that $I \square I \subseteq C$. (Recall from Section 5.8, exercise 1 that any map isomorphic to a cofibration is again a cofibration.)

Next we take pushout-product of maps in I with maps in J. For the first set of maps in J from Definition 5.6.3, the proof is largely the same:

$$\begin{aligned} F_d(S^{k-1} \to D^k)_+ \square F_e(D^\ell \to D^\ell \times I)_+ &\cong (\mathrm{id}_{F_d S^0}) \square (S^{k-1} \to D^k)_+ \square (\mathrm{id}_{F_e S^0}) \square (D^\ell \to D^\ell \times I)_+ \\ &\cong (\mathrm{id}_{F_d S^0}) \square (\mathrm{id}_{F_e S^0}) \square (S^{k-1} \to D^k)_+ \square (D^\ell \to D^\ell \times I)_+ \\ &\cong (\mathrm{id}_{F_{d+e} S^0}) \square (D^{k+\ell} \to D^{k+\ell} \times I)_+ \\ &\cong F_{d+e}(S^{k+\ell-1} \to D^{k+\ell})_+, \end{aligned}$$

which is another map in J. For the second set of maps from Definition 5.6.3,

$$\begin{aligned} F_d(S^{k-1} \to D^k)_+ \square k_{i,j} \square (S^{\ell-1} \to D^\ell)_+ &\cong (\mathrm{id}_{F_d S^0}) \square (S^{k-1} \to D^k)_+ \square k_{i,j} \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong (\mathrm{id}_{F_d S^0}) \square k_{i,j} \square (S^{k-1} \to D^k)_+ \square (S^{\ell-1} \to D^\ell)_+ \\ &\cong k_{d+i,j} \square (S^{k+\ell-1} \to D^{k+\ell})_+, \end{aligned}$$

which is another map in J. The last manipulation uses the fact that smashing a free spectrum $F_d S^0$ into the definition of $k_{i,j}$ increases the indices on all of the free spectra in the definition by d, giving the map $k_{d+i,j}$. In conclusion, we have shown that $I \square J \subseteq (W \cap C)$, so the smash product is a Quillen tensor. $\square$

By far, the biggest reason why we care about this is that

Corollary 6.3.5. *The smash product $X \wedge Y$ preserves stable equivalences when X and Y are cofibrant. The function spectrum $F(X, Y)$ preserves stable equivalences when X is cofibrant and Y is fibrant.*

Proof. This follows from Lemma 5.5.19 and Lemma 5.5.22, and the just-established fact that $\wedge$ is a Quillen tensor. $\square$

Remark 6.3.6. In fact, more is true: the smash product $X \wedge Y$ preserves stable equivalences when X is cofibrant and Y is *arbitrary*. See exercise 13. Also, the function spectrum $F(X, Y)$ preserves stable equivalences if X is a finite cell spectrum and Y is arbitrary. See exercise 14.

We therefore have a smash product of spectra that is well-behaved on the point-set level, and preserves stable equivlences when we need it to!

More generally, in any symmetric monoidal model category $\mathbf{C}$, the tensor $X \otimes Y$ preserves equivalences when X and Y are cofibrant, and the $\mathrm{Hom}(X, Y)$ preserves equivalences when X is cofibrant and Y is fibrant. We can therefore left-derive the tensor and right-derive the hom:

$$X \otimes^{\mathbb{L}} Y := QX \otimes QY, \qquad \mathbb{R}\mathrm{Hom}(X, Y) := \mathrm{Hom}(QX, RY).$$

These functors preserve all equivalences, so they define functors on the homotopy category as well. We proved in Corollary 5.5.27 that these make the homotopy category $\operatorname{Ho}\mathbf{C}$ into a closed symmetric monoidal category.

Since we have equivalences of homotopy categories

$$\operatorname{Ho}\mathbf{Sp}^{O} \simeq \operatorname{Ho}\mathbf{Sp}^{\Sigma} \simeq \operatorname{Ho}\mathbf{Sp}^{\mathbb{N}},$$

by Section 4.3, exercise 14, a symmetric monoidal structure on any one of these categories becomes a symmetric monoidal structure on all of them. We have therefore, finally, achieved the goal of placing a closed symmetric monoidal structure on the stable homotopy category $\operatorname{Ho}\mathbf{Sp}$.[2]

Corollary 6.3.7. *The stable homotopy category* $\operatorname{Ho}\mathbf{Sp}$ *has a closed symmetric monoidal structure using* $\wedge^{\mathbb{L}}$ *and* $\mathbb{R}F(-,-)$*, and satisfying all of the properties from Example 4.1.9 and Example 4.1.34.*

There were more promised properties in Example 4.1.27, but those have to do with functors between this an other categories, and Example 4.2.11, but that had to do with duality. See exercise **??**.

6.3.2 Symmetric monoidal Quillen adjunctions

We now turn to functors between symmetric monoidal model categories.

Definition 6.3.8. Suppose $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal model categories. A **strong symmetric monoidal Quillen adjunction** is

- a Quillen adjunction $(F \dashv G)$ in which
- the left Quillen functor $F : \mathbf{C} \to \mathbf{D}$ is strong symmetric monoidal, and
- the map of units $F(QI_{\mathbf{C}}) \to F(I_{\mathbf{C}}) \cong I_{\mathbf{D}}$ is a weak equivalence.

A **strong symmetric monoidal Quillen equivalence** is the same thing, except that $(\mathbb{L}F \dashv \mathbb{R}G)$ is a Quillen equivalence, i.e. the derived functors also give an equivalence on the homotopy category. See also exercise 26 for a weaker notion.

To check the first two conditions, we need to check that F preserves cofibrations and acyclic cofibrations, and preserves tensors and the units up to isomorphism:

$$F(C) \subseteq C, \quad F(W \cap C) \subseteq W \cap C, \quad F(X) \otimes_{\mathbf{D}} F(Y) \cong F(X \otimes_{\mathbf{C}} Y), \quad F(I_{\mathbf{C}}) \cong I_{\mathbf{D}}.$$

[2] The following section checks the important fact that $\mathbf{Sp}^{\Sigma}$ and $\mathbf{Sp}^{O}$ give the *same* symmetric monoidal structure, up to isomorphism.

Also, the third condition on the map $F(QI_{\mathbf{C}}) \to F(I_{\mathbf{C}}) \cong I_{\mathbf{D}}$ is automatically true if $I_{\mathbf{C}}$ is cofibrant.

Example 6.3.9. The following adjoint pairs are strong symmetric monoidal Quillen adjunctions:

- The disjoint basepoint $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$ and its right adjoint the forgetful functor $U : \mathbf{Top} \to \mathbf{Top}_*$.
- Suspension spectrum $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}^\Sigma$ and its right adjoint the evaluation functor $\mathrm{ev}_0 : \mathbf{Sp}^\Sigma \to \mathbf{Top}_*$.
- Prolongation $P^O_\Sigma : \mathbf{Sp}^\Sigma \to \mathbf{Sp}^O$ and its right adjoint the forgetful functor $U^O_\Sigma : \mathbf{Sp}^O \to \mathbf{Sp}^\Sigma$.

These are closed under composition. So for instance, suspension with a disjoint basepoint Σ^∞_+ is also a symmetric monoidal left Quillen functor. We summarize with the following diagram of strong symmetric monoidal Quillen left adjoints:

$$\mathbf{Top} \xrightarrow{(-)_+} \mathbf{Top}_* \xrightarrow{\Sigma^\infty} \mathbf{Sp}^\Sigma \xrightarrow[\sim]{P^O_\Sigma} \mathbf{Sp}^O. \tag{6.3.10}$$

Remark 6.3.11. As in Remark 4.1.28, any time $F : \mathbf{C} \to \mathbf{D}$ is a strong symmetric monoidal functor, we can think of $\mathbf{D}$ as a "module" or "algebra" over $\mathbf{C}$. So unbased spaces act on the category of based spaces by

$$X \cdot Y := X_+ \wedge Y, \qquad X \in \mathbf{Top},\ Y \in \mathbf{Top}_*,$$

based spaces act on symmetric or orthogonal spectra by

$$X \cdot Y := X \wedge Y \cong (\Sigma^\infty X) \wedge Y, \qquad X \in \mathbf{Top}_*,\ Y \in \mathbf{Sp},$$

and so on. In the diagram (6.3.10), every category "acts" in this way on every category to its right.

If $(F \dashv G)$ is a strong symmetric monoidal Quillen adjunction, it follows from Section 4.3, exercise 12 that the right Quillen functor $G : \mathbf{D} \to \mathbf{C}$ is lax symmetric monoidal. We also get:

Lemma 6.3.12. *The left-derived functor $\mathbb{L}F : \mathrm{Ho}\,\mathbf{C} \to \mathrm{Ho}\,\mathbf{D}$ is strong monoidal, and the right-derived functor $\mathbb{R}G : \mathrm{Ho}\,\mathbf{D} \to \mathrm{Ho}\,\mathbf{C}$ is lax monoidal.*

The proof is left to exercise 25. As a consequence of Proposition 4.1.23, we also get

Corollary 6.3.13. *F and G both preserve monoid objects, and $\mathbb{L}F$ and $\mathbb{R}G$ preserve monoid objects in the homotopy category.*

We already observed this in Lemma 6.2.28, where we concluded that the forgetful and prolongation functors between $\mathbf{Sp}^\Sigma$ and $\mathbf{Sp}^O$ preserve ring spectra.

6.4 Other symmetric monoidal models of spectra

We spent this chapter focusing on symmetric and orthogonal spectra, but there are other important models of spectra that also have a smash product. We develop a few of them in this section.

6.4.1 The positive stable model structure

Our first example is not really a new example, because it's just symmetric spectra again. However, the *model structure* is different.

Definition 6.4.1. A **positive cofibration** in $\mathbf{Sp}^\Sigma$ is a retract of a relative cell complex, in which all of the cells $F_i^\Sigma(S^{n-1} \to D^n)_+$ that appear have $i > 0$. There is no restriction on n.

In other words, we can attach cells of any stable dimension, but we can only attach them at spectrum level 1 and above. A positive cofibration in $\mathbf{Sp}^O$ is defined the same way.

Theorem 6.4.2 (Mandell-May-Schwede-Shipley)**.** *The category of symmetric spectra* $\mathbf{Sp}^\Sigma$ *has a* ***positive stable model structure*** *in which*

- *the cofibrations are the positive cofibrations,*
- *the weak equivalences are the stable equivalences, and*
- *the fibrations are the maps* $X \to Y$ *such that* $X_i \to Y_i$ *is a Serre fibration, and the squares*

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "p_i"] & \Omega^j X_{i+j} \arrow[d, "\Omega^j p_{i+j}"] \\ Y_i \arrow[r] & \Omega^j Y_{i+j} \end{tikzcd}$$

  are homotopy pullbacks, for all positive values of i.

The category of orthogonal spectra $\mathbf{Sp}^O$ *has a positive stable model structure with the same description.*

The proof is the same as the proof of Theorem 6.1.28 and its earlier variant Theorem 5.2.11, only we change the sets of maps I and J from Definition 5.6.1 and Definition 5.6.3 to only use positive values of i:

$$\begin{aligned} I^+ &= \{\, F_i S^{n-1}_+ \longrightarrow F_i D^n_+ \;:\; n \geq 0,\; i > 0 \,\}, \\ J^+ &= \{\, F_i(D^n \times \{0\})_+ \longrightarrow F_i(D^n \times I)_+ \;:\; n \geq 0,\; i > 0 \,\} \\ &\quad \cup \{\, k_{i,j} \,\square\, \left(S^{n-1}_+ \longrightarrow D^n_+\right) \;:\; j, n \geq 0,\; i > 0 \,\}. \end{aligned}$$

A fibrant object in this model structure is a **positive Ω-spectrum**: it is a spectrum X such that the maps $X_n \to \Omega X_{n+1}$ are all weak equivalences for $n \geq 1$.

Lemma 6.4.3. *The identity functor of* $\mathbf{Sp}^\Sigma$ *gives Quillen equivalence between the positive stable model structure and the stable model structure.*

Proof. This can be checked directly from the definitions of the two model structures in Theorems 6.1.28 and 6.4.2. The identity is left Quillen when going from the positive one to the ordinary one, and right Quillen when going from the ordinary one to the positive one. □

We let $\mathbf{Sp}^\Sigma_+$ denote the category of symmetric spectra with the positive stable model structure. So it is the same category as $\mathbf{Sp}^\Sigma$, but the model structure is different.

Lemma 6.4.4. $\mathbf{Sp}^\Sigma_+$ *is a symmetric monoidal model category. The identity functor* $\mathbf{Sp}^\Sigma_+ \to \mathbf{Sp}^\Sigma$ *is a strong symmetric monoidal left Quillen adjoint.*

Proof. This is proven just as in Proposition 6.3.4, only the variables d, e, and i in that proof all have to be positive. □

To summarize, the diagram (6.3.10) can be expanded to the following diagram of strong symmetric monoidal Quillen left adjoints:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "(-)_+"] & \mathbf{Top}_* \arrow[r, "\Sigma^\infty"] & \mathbf{Sp}^\Sigma \arrow[r, "P_\Sigma^O", "\sim"'] & \mathbf{Sp}^O \\ & & \mathbf{Sp}^\Sigma_+ \arrow[u, "\mathrm{id}", "\sim"'] \arrow[r, "P_\Sigma^O", "\sim"'] & \mathbf{Sp}^O_+ . \arrow[u, "\mathrm{id}", "\sim"'] \end{tikzcd} \tag{6.4.5}$$

Remark 6.4.6. The suspension spectrum functor to the *positive* stable model structure, $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{Sp}^\Sigma_+$, is not left Quillen!

6.4.2 EKMM spectra

In the early 1990s, Hovey, Shipley, and Smith developed the theory of symmetric spectra, providing a solution to the problem of giving the category of spectra a well-behaved smash product. At about the same time, a second group of authors developed in parallel an *entirely different* category of spectra that has a smash product. We call it the category of **EKMM spectra** after its creators.[3]

[3]Michael Cole also provided significant simplifications to the theory, see [EKMM97] for more details.

Theorem 6.4.7 (Elmendorff, Kriz, Mandell, and May)**.** *There is a category $\mathscr{M}$, whose objects we call EKMM spectra, with the following properties.*

- $\mathscr{M}$ is a closed symmetric monoidal category, whose product we call the smash product $\wedge$ and whose unit is the sphere spectrum $\mathbb{S}$.
- $\mathscr{M}$ is also a model category, whose weak equivalences we call the stable equivalences.
- Every object in $\mathscr{M}$ is fibrant. The sphere spectrum $\mathbb{S}$ fails to be cofibrant.
- The smash product $\wedge$ is a Quillen tensor, and the map $Q\mathbb{S} \wedge QX \to \mathbb{S} \wedge QX$ is always a stable equivalence. Therefore, $\mathscr{M}$ is a symmetric monoidal model category.
- There is a strong symmetric monoidal Quillen equivalence $(\mathbb{N}, \mathbb{N}^{\#})$ between positive orthogonal spectra $\mathbf{Sp}^O_+$ and $\mathscr{M}$.
- The left adjoint $\mathbb{N}\colon \mathbf{Sp}^O \to \mathscr{M}$ preserves all stable equivalences between cofibrant orthogonal spectra, not just the ones that are positive cofibrant. However, if there are any cells at spectrum level 0, they don't go to cofibrant objects of $\mathscr{M}$.

We don't give the definition of EKMM spectra in this book. We simply remark that the definition is more complicated than that of symmetric or orthogonal spectra, so if you want to get your hands dirty and work with a definition directly, diagram spectra are usually the way to go. The main advantage of EKMM spectra is that every object is fibrant. This is very useful sometimes.

You can work effectively with EKMM spectra by using Theorem 6.4.7 as a black box. For instance, since $(\mathbb{N}, \mathbb{N}^{\#})$ is a Quillen equivalence, the homotopy category $\operatorname{Ho}\mathscr{M}$ is equivalent to the stable homotopy category $\operatorname{Ho}\mathbf{Sp}$. Since $\mathscr{M}$ is a symmetric monoidal category, we can define EKMM ring spectra just as before. And since the Quillen equivalence $(\mathbb{N}, \mathbb{N}^{\#})$ is strong symmetric monoidal, EKMM ring spectra up to equivalence are the same thing as symmetric or orthogonal ring spectra up to equivalence.

We define the EKMM suspension spectrum of a based space A to be $\mathbb{N}\Sigma^\infty A$. This is usually not cofibrant, even if A is a cell complex. One might find it odd that the sphere is not cofibrant, but this is actually impossible:

Theorem 6.4.8. *There does not exist a symmetric monoidal model category of spectra* $\mathbf{Sp}$ *with the following properties:*

- The homotopy category $\operatorname{Ho}\mathbf{Sp}$ is equivalent to the classical stable homotopy category, as a symmetric monoidal category.
- The unit object of $\mathbf{Sp}$ is both cofibrant and fibrant.

Proof sketch. If there were such a category, the space of maps from the unit object to itself would be equivalent to $QS^0 = \Omega^\infty \mathbb{S}$, and would also be a commutative monoid. Any commutative monoid in $(\mathbf{Top}_*, \times, *)$ is equivalent to a product of Eilenberg-Maclane spaces, but QS^0 is not equivalent to such a product, so we have a contradiction. □

Remark 6.4.9. This is a weaker form of a theorem of Gaunce Lewis. Lewis's result does not assume that Ho **Sp** is equivalent to the classical stable homotopy category: it is enough to ask for a strong symmetric monoidal adjunction $(\Sigma^\infty, \Omega^\infty)$ between $\mathbf{Top}_*$ and **Sp**, such that $\Omega^\infty \Sigma^\infty X$ has the homotopy type we expect. See [Lew91].

Remark 6.4.10. The category of symmetric spectra gets around this theorem by making the sphere spectrum $\mathbb{S}$ cofibrant but not fibrant. The category of EKMM spectra gets around this theorem by making the sphere spectrum $\mathbb{S}$ fibrant but not cofibrant. It is impossible to do both.

To summarize, the diagram (6.4.5) can be further expanded to the following diagram of strong symmetric monoidal Quillen left adjoints:

$$
\begin{tikzcd}
\mathbf{Top} \arrow[r, "(-)_+"] & \mathbf{Top}_* \arrow[r, "\Sigma^\infty"] & \mathbf{Sp}^\Sigma \arrow[r, "P^O_\Sigma", "\sim"'] & \mathbf{Sp}^O & \\
 & & \mathbf{Sp}^\Sigma_+ \arrow[u, "\mathrm{id}", "\sim"'] \arrow[r, "P^O_\Sigma", "\sim"'] & \mathbf{Sp}^O_+ \arrow[u, "\mathrm{id}", "\sim"'] \arrow[r, "\mathbb{N}", "\sim"'] & \mathscr{M}.
\end{tikzcd}
\tag{6.4.11}
$$

6.4.3 Spectra as functors

Another important perspective on spectra is that they are equivalent to functors from spaces to spaces with certain properties.

Definition 6.4.12. Let $F : \mathbf{Top}_* \to \mathbf{Top}_*$ be a functor. We say F is

- a **homotopy functor** if every weak equivalence $A \xrightarrow{\sim} B$ is sent to a weak equivalence $F(A) \xrightarrow{\sim} F(B)$,
- **excisive** if for each homotopy pushout square

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "h"] \\
C \arrow[r, "k"'] & D,
\end{tikzcd}
$$

applying F gives a homotopy pullback square:

$$\begin{tikzcd} F(A) \arrow[r, "F(g)"] \arrow[d, "F(f)"'] & F(B) \arrow[d, "F(h)"] \\ F(C) \arrow[r, "F(k)"'] & F(D), \end{tikzcd}$$

- **reduced** if $F(*)$ is contractible, and
- **finitary** if for any CW complex X and its finite subcomplexes $X_\alpha \subseteq X$ the canonical map
$$\operatorname*{hocolim}_{\alpha} F(X_\alpha) \longrightarrow F(X)$$
is an equivalence.

A finitary homotopy functor is determined up to equivalence by its behavior on finite CW complexes, so for simplicity, we restrict the domain of F to the category of finite based CW complexes $\mathbf{CW}_*^{\mathrm{fin}}$, and drop the finitary assumption.

We let $\mathscr{F}$ denote the category whose objects are homotopy functors $F : \mathbf{CW}_*^{\mathrm{fin}} \to \mathbf{Top}_*$, and whose morphisms are natural transformations. Let $\mathscr{F}^{\mathrm{re}} \subseteq \mathscr{F}$ be the full subcategory of reduced excisive homotopy functors.

Example 6.4.13. If $E \in \mathbf{Sp}^{\mathbb{N}}$ is any sequential spectrum, the functor of finite based CW complexes $A \in \mathbf{CW}_*^{\mathrm{fin}}$ given by
$$F(A) = \Omega^\infty(A \wedge E)$$
is excisive, and therefore $F \in \mathscr{F}^{\mathrm{re}}$. See exercise 27.

In fact, we can show that every reduced excisive functor is of this form. To go backwards, given $F \in \mathscr{F}^{\mathrm{re}}$, we define an Ω-spectrum E by

$$E_n = F(S^n).$$

We have the homotopy pullback square

$$\begin{tikzcd} F(S^n) \arrow[r] \arrow[d] & F(D^{n+1}) \arrow[d] \\ F(D^{n+1}) \arrow[r] & F(S^{n+1}), \end{tikzcd}$$

where $F(D^{n+1})$ is contractible. Therefore the homotopy pullback is equivalent to the based loopspace ΩE_{n+1}. Since this is a homotopy pullback square, we therefore get an equivalence $E_n \xrightarrow{\sim} \Omega E_{n+1}$. This makes E into an Ω-spectrum.

Theorem 6.4.14 (Goodwillie)**.** *These two operations give an equivalence of homotopy categories*

$$\mathrm{Ho}\,\mathscr{F}^{\mathrm{re}} \simeq \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}.$$

In other words, reduced excisive functors are equivalent to spectra.

It is easier to describe this equivalence if we give the functors a little more structure.

Definition 6.4.15. A functor $F\colon \mathbf{CW}_*^{\mathrm{fin}} \to \mathbf{Top}_*$ is **enriched in spaces**, or **topological**, if for every pair of spaces A, B the map

$$\mathrm{Map}_*(A,B) \xrightarrow{F} \mathrm{Map}_*(F(A),F(B))$$

induced by F is continuous.[4] A map of enriched functors is the same as a map of underlying functors. See exercise 28.

We let $\mathscr{E}$ denote the category of enriched functors on CW complexes, and $\mathscr{E}^{\mathrm{re}} \subseteq \mathscr{E}$ the full subcategory of reduced excisive enriched functors.

Every enriched functor has **assembly maps**

$$A \wedge F(B) \longrightarrow F(A \wedge B) \tag{6.4.16}$$

for all based CW complexes A and B, defined by sending each $a \in A$ to the map $F(B) \to F(A \wedge B)$ that is F applied to the inclusion $\{a\} \times B \to A \wedge B$. If F is enriched then these assembly maps are continuous – this is left to exercise 29.

Lemma 6.4.17. *Every enriched functor is a homotopy functor.*

Proof. The key observation is a homotopy of maps $A \to B$ is the same thing as a map $I_+ \to \mathrm{Map}_*(A,B)$. Since F is enriched, this becomes a homotopy of maps $F(A) \to F(B)$. It follows from this observation that F sends homotopy equivalences to homotopy equivalences. Since the domain is restricted to CW complexes, this means that F is a homotopy functor. □

Lemma 6.4.18. *Every homotopy functor is equivalent to an enriched functor.*

The proof uses simplicial sets and is deferred to **??**, exercise **??**.

Corollary 6.4.19. *The inclusions $\mathscr{E} \subseteq \mathscr{F}$ and $\mathscr{E}^{\mathrm{re}} \subseteq \mathscr{F}^{\mathrm{re}}$ give equivalences of homotopy categories.*

[4]Notice that this is a condition on F, not extra data, though in general giving a functor an enrichment requires extra data. We get away with murder here because the forgetful functor **Top** → **Set** is faithful.

We therefore have

$$\mathrm{Ho}\,\mathscr{E}^{\mathrm{re}} \simeq \mathrm{Ho}\,\mathscr{F}^{\mathrm{re}} \simeq \mathrm{Ho}\,\mathbf{Sp}.$$

This model of spectra can also be made symmetric monoidal. We don't give the details, but the rough idea is to define the smash product of two enriched functors $F \wedge G$ by imitating Definition 6.2.17, using the functor $\wedge\colon \mathbf{CW}_*^{\mathrm{fin}} \times \mathbf{CW}_*^{\mathrm{fin}} \to \mathbf{CW}_*^{\mathrm{fin}}$ in the place of $\amalg$. The resulting notion of a ring spectrum is called a **functor with smash product (FSP)**:

Definition 6.4.20. A functor with smash product (FSP) is an enriched functor $F\colon \mathbf{CW}_*^{\mathrm{fin}} \to \mathbf{Top}_*$ with maps

$$S^n \longrightarrow F(S^n), \qquad F(A) \wedge F(B) \longrightarrow F(A \wedge B)$$

that are associative and unital in the sense given in [MMSS01, 22.5].

Goodwillie's theorem generalizes to say that these functors with smash product are equivalent to orthogonal ring spectra, and therefore to all other notions of ring spectra. The equivalence is simple: just evaluate F on spheres, and notice that the resulting spaces are an orthogonal ring spectrum in a natural way.

6.5 Exercises

1. Given a symmetric spectrum E, define the E-homology and E-cohomology of another symmetric spectrum X by:

$$E_n(X) = [\Sigma^n \mathbb{S}, X \wedge^{\mathbb{L}} E]_s = \pi_n(X \wedge^{\mathbb{L}} E)$$

$$E^n(X) = [\Sigma^{-n} X, E]_s = \pi_{-n}(\mathbb{R}F(X, E)).$$

   Explain what the Eilenberg-Steenrod axioms should be in this setting, and verify that they hold for these theories.

   Note that the $\mathbb{L}$ decoration on the smash product can be removed if E and X are cofibrant, and the $\mathbb{R}$ decoration on F can be removed if X is cofibrant and E is fibrant.

   By exercise 20, these definitions of homology and cohomology agree with the earlier ones from Definition 2.6.36, Example 3.2.16, and Example 3.2.21.

2. Suppose that E is a symmetric ring spectrum. Show that the cohomology of any *space* X inherits a "cup product" by taking any two maps in the stable homotopy category

$$\Sigma^{-m} X \xrightarrow{f} E, \Sigma^{-n} X \xrightarrow{g} E$$

   to the composite

$$\Sigma^{-(m+n)} X \xrightarrow{\Delta} \Sigma^{-(m+n)}(X \wedge X) \xrightarrow{f \wedge g} E \wedge E \xrightarrow{\mu} E.$$

3. Prove Lemma 6.1.8 and Lemma 6.1.10.

4. In this exercise we give the inverse to the shift functor $\mathrm{sh}^1 \colon \mathbf{Sp}^\Sigma \to \mathbf{Sp}^\Sigma$ from Definition 6.1.21. Recall that $(\mathrm{sh}^1 X)_n = X_{1+n}$, with Σ_n acting as the subgroup $1 \times \Sigma_n \le \Sigma_{n+1}$ that permutes only the last n letters.

   (a) Construct a symmetric spectrum by the rule

$$(\mathrm{sh}^{-1} X)_n = (\Sigma_n)_+ \wedge_{\Sigma_{n-1}} X_{n-1},$$

   with bonding maps coming from those of X.

   (b) Prove that this defines the left adjoint to sh^1.

   (c) Argue that both of these functors preserve level equivalences, and that we get an adjunction in the level homotopy category

$$[\mathrm{sh}^{-1} X, Z]_\ell \cong [X, \mathrm{sh}^1 Z]_\ell.$$

(d) Conclude that sh^{-1} preserves stable equivalences.

5. Generalize exercise 4 to orthogonal spectra. To prove that

$$O(n)_+ \wedge_{O(n-1)} A$$

preserves weak equivalences of spaces, you might find it helpful to observe that $O(n)$ is built out of cells of the form $O(n-1)\times D^m$ for varying m.

6. This exercise gives a rigorous proof that the π_*-isomorphisms in symmetric spectra give the wrong homotopy category. Suppose we form the homotopy category $\mathrm{Ho}\,\mathbf{Sp}^\Sigma$ using the π_*-isomorphisms (Definition 6.1.17). The forgetful functor to sequential spectra $U_{\mathbb{N}}^{\Sigma}$ clearly preserves these weak equivalences, so we get a map of homotopy categories

$$U_{\mathbb{N}}^{\Sigma}\colon \mathrm{Ho}\,\mathbf{Sp}^\Sigma \longrightarrow \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}.$$

(a) Recall from exercise 4 the shift functor sh^1, and that the inverse of sh^1 on sequential spectra is $(\mathrm{sh}_{\mathbb{N}}^{-1}X)_n = X_{n-1}$. Explain why this does not define a symmetric spectrum.

(b) Show that the left adjoint $\mathrm{sh}_\Sigma^{-1}\colon \mathbf{Sp}^\Sigma \to \mathbf{Sp}^\Sigma$ to sh^1 from exercise 4 is *not* π_*-isomorphic to $\mathrm{sh}_{\mathbb{N}}^{-1}$.

(c) Observe that sh^1 commutes with the forgetful functor, and conclude using Lemma 3.4.13 that we have a square of functors of homotopy categories

$$\begin{tikzcd}
\mathrm{Ho}\,\mathbf{Sp}^\Sigma \arrow[r, "\mathrm{sh}_\Sigma^1"] \arrow[d, "U_{\mathbb{N}}^{\Sigma}"'] & \mathrm{Ho}\,\mathbf{Sp}^\Sigma \arrow[d, "U_{\mathbb{N}}^{\Sigma}"] \\
\mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}} \arrow[r, "\mathrm{sh}_{\mathbb{N}}^1"'] & \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}
\end{tikzcd}$$

that commutes up to isomorphism.

Now, *assume* that $U_{\mathbb{N}}^{\Sigma}$ gives an equivalence of homotopy categories. This implies that the left adjoints agree up to isomorphism as well:

$$\begin{tikzcd}
\mathrm{Ho}\,\mathbf{Sp}^\Sigma \arrow[d, "U_{\mathbb{N}}^{\Sigma}"', "\sim"] & \mathrm{Ho}\,\mathbf{Sp}^\Sigma \arrow[l, "\mathrm{sh}_\Sigma^{-1}"'] \arrow[d, "\sim"', "U_{\mathbb{N}}^{\Sigma}"] \\
\mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}} & \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}} \arrow[l, "\mathrm{sh}_{\mathbb{N}}^{-1}"]
\end{tikzcd}$$

But this contradicts part (c), so $U_{\mathbb{N}}^{\Sigma}$ cannot be an equivalence of homotopy categories $\mathrm{Ho}\,\mathbf{Sp}^\Sigma \simeq \mathrm{Ho}\,\mathbf{Sp}^{\mathbb{N}}$. This proves that the π_*-isomorphisms are the "wrong" class of weak equivalences to take in $\mathbf{Sp}^\Sigma$.

7. What happens to the argument in exercise 6 in the setting of orthogonal spectra? Which parts change, and how do we not get a contradiction?

8. Prove that the map λ_* in Definition 6.1.21 would fail to be a map of symmetric spectra if we did not include the permutation $\tau_{n,1}$ in its definition.

9. Define "unitary spectra." Show that every orthogonal spectrum gives a unitary spectrum using the inclusion of groups $U(n) \to O(2n)$.

10. Suppose $(F \dashv G)$ is a Quillen adjunction, the derived unit map

$$\eta \colon X \to (\mathbb{R}G)(\mathbb{L}F)X$$

is a weak equivalence, and $\mathbb{R}G$ reflects weak equivalences, in the sense that $X \to Y$ is a weak equivalence iff $\mathbb{R}GX \to \mathbb{R}GY$ is a weak equivalence. Prove that the derived counit map

$$\epsilon \colon (\mathbb{L}F)(\mathbb{R}G)Y \to Y$$

is also a weak equivalence, and therefore $(F \dashv G)$ is a Quillen equivalence.

(Hint: one of the triangle identities tells you that the composite

$$(\mathbb{R}G)Y \xrightarrow{\eta\circ\mathrm{id}} (\mathbb{R}G)(\mathbb{L}F)(\mathbb{R}G)Y \xrightarrow{\mathrm{id}\circ\epsilon} (\mathbb{R}G)Y$$

is equal to the identity.)

11. Check directly that $F(\mathbb{S}, X) \cong X$, where F is the function spectrum from Definition 6.2.12.

12. (a) Generalizing exercise 11, check that $F(F_1 S^0, X) \cong \mathrm{sh}^1 X$, where the shift functor sh^1 is defined in exercise 4.

    (b) Use this to conclude that $F(F_p S^0, X) \cong \mathrm{sh}^p X$, where sh^n is defined by iterating sh^1. More concretely, $(\mathrm{sh}^p X)_n = X_{p+n}$, with Σ_n acting through the permutations on $(p+n)$ letters that only permute the last n letters.

    (c) Deduce formally from this that for any based space K,

    $$F(F_p K, X) \cong F(K, \mathrm{sh}^p X) \cong \mathrm{sh}^p F(K, X)$$

    where $F(K, -)$ denotes cotensor spectrum, and also

    $$F_p K \wedge X \cong K \wedge \mathrm{sh}^{-p} X$$

    where $\mathrm{sh}^{-p} = (\mathrm{sh}^{-1})^{\circ p}$ is the p-fold iterate of the functor sh^{-1} from exercise 4.

    (d) Conclude using exercise 4 that if K is a cell complex, smashing with the free spectrum $F_p K \wedge (-)$ preserves all stable equivalences of spectra.

13. Prove that the smash product $X \wedge Y$ of symmetric or orthogonal spectra preserves stable equivalences when X is cofibrant and Y is *arbitrary*, generalizing Corollary 6.3.5.

    (You'll need to do exercises 4 and 12 first, which show that $X \wedge (-)$ preserves equivalences if X is a free spectrum on a based cell complex K. Then you can use Theorem 6.1.26 to work your way up from this to any cofibrant spectrum X. Finally, you can use Corollary 6.3.5 to show that equivalences in the X variable are preserved as well.)

14. Use the argument of exercise 13 to show that the function spectrum $F(X,Y)$ preserves equivalences if X is a *finite* cell spectrum and Y is arbitrary. (That is, if X is cofibrant but only has finitely many cells.)

    What goes wrong if X has infinitely many cells? (You might want to recall Section 2.7, exercise 25.)

15. Check that the functor $\amalg\colon \mathbf{S}^\Sigma \wedge \mathbf{S}^\Sigma \to \mathbf{S}^\Sigma$ defined in Definition 6.2.15 respects composition and identity maps, so that it is in fact a functor.

16. Generalize $\amalg$ to a functor $\oplus\colon \mathbf{S}^O \wedge \mathbf{S}^O \to \mathbf{S}^O$, where $\mathbf{S}^O = \mathscr{J}$ is the category from Definition 6.1.9. You should be taking direct sum of vector spaces and the direct sum of their orthogonal complements. The isomorphisms in Section 2.7, exercise 5 will probably be helpful.

17. Show that the sum functor $\amalg$ from Definition 6.2.15 makes $\mathbf{S}^\Sigma$ into a symmetric monoidal category. Similarly the functor $\oplus$ from exercise 16 makes $\mathbf{S}^O$ into a symmetric monoidal category.

18. Suppose $(\mathbf{I}, +, 0)$ and $(\mathbf{C}, \otimes, I)$ are symmetric monoidal categories, $\mathbf{I}$ is small, $\mathbf{C}$ has all colimits, and $\otimes$ preserves colimits in each slot. The **Day convolution** is a tensor product defined on the category of diagrams

    $$\boxtimes\colon \mathbf{C}^\mathbf{I} \times \mathbf{C}^\mathbf{I} \to \mathbf{C}^\mathbf{I}$$

    in the following way. Given two diagrams $X, Y\colon \mathbf{I} \to \mathbf{C}$, the tensor products $X(i) \otimes Y(j)$ form a diagram

    $$X \otimes Y\colon \mathbf{I} \times \mathbf{I} \to \mathbf{C}.$$

    We define the Day convolution to be the left Kan extension of this diagram along the sum functor $+$:

    $$X \boxtimes Y = (+)_!(X \otimes Y)\colon \mathbf{I} \to \mathbf{C}$$

    The smash product of diagram spectra from Definition 6.2.17 is a variant of this construction where $\mathbf{I}$ is enriched in based spaces and $\mathbf{C}$ has a tensor product with based spaces, but for the purposes of this exercise, let $\mathbf{I}$ and $\mathbf{C}$ be ordinary categories, enriched only in sets.

- Prove that there is an isomorphism $(X \boxtimes Y) \boxtimes Z \cong X \boxtimes (Y \boxtimes Z)$. You might find Section 5.8, exercise 17 helpful.
- Prove that there is an isomorphism $X \boxtimes Y \cong Y \boxtimes X$.
- Identify the unit diagram U, and prove that there is an isomorphism $X \boxtimes U \cong X$.

Together with the coherences between these isomorphisms, this makes $\mathbf{C}^{\mathbf{I}}$ into a symmetric monoidal category. This gives a more formal way of proving Theorem 6.2.8.

19. Recall from Section 5.8, exercise 31 that a bispectrum is a spectrum that has two directions. Formally, it is a diagram over the smash product category $\mathbf{S} \wedge \mathbf{S}$, where $\mathbf{S}$ is the sphere category $\mathbf{S}(m, n) = S^{n-m}$ from Definition 5.3.31.

    (a) Define a symmetric bispectrum to be a diagram on $\mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma}$. Explain how these arise in the definition of the smash product in Definition 6.2.17.

    (b) Taking on faith that bispectra have a stable model structure in which the cofibrations are the relative cellular bispectra, and that suspension bispectrum $\mathbf{Top}_*^{\mathbf{S}^{\Sigma}} \to \mathbf{Top}_*^{\mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma}}$ is a Quillen equivalence, prove that the left Kan extension $(\amalg)_! \colon \mathbf{Top}_*^{\mathbf{S}^{\Sigma} \wedge \mathbf{S}^{\Sigma}} \to \mathbf{Top}_*^{\mathbf{S}^{\Sigma}}$ is also a Quillen equivalence.

20. Using exercise 19, show that the homotopy groups of the smash product of orthogonal spectra $X \wedge Y$ agrees with the homotopy groups of the bispectrum $X \overline{\wedge} Y$, which is the colimit of the grid of groups from Definition 2.3.23:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \text{colim} = \pi_k(X \wedge Y) \\
\pi_{k+3}(X_0 \wedge Y_3) \arrow[r, "-"] \arrow[u, "+"'] & \pi_{k+4}(X_1 \wedge Y_3) \arrow[r, "-"] \arrow[u, "+"'] & \pi_{k+5}(X_2 \wedge Y_3) \arrow[r, "-"] \arrow[u, "+"'] & \cdots \\
\pi_{k+2}(X_0 \wedge Y_2) \arrow[r, "+"] \arrow[u, "+"'] & \pi_{k+3}(X_1 \wedge Y_2) \arrow[r, "+"] \arrow[u, "+"'] & \pi_{k+4}(X_2 \wedge Y_2) \arrow[r, "+"] \arrow[u, "+"'] & \cdots \\
\pi_{k+1}(X_0 \wedge Y_1) \arrow[r, "-"] \arrow[u, "+"'] & \pi_{k+2}(X_1 \wedge Y_1) \arrow[r, "-"] \arrow[u, "+"'] & \pi_{k+3}(X_2 \wedge Y_1) \arrow[r, "-"] \arrow[u, "+"'] & \cdots \\
\pi_{k}(X_0 \wedge Y_0) \arrow[r, "+"] \arrow[u, "+"'] & \pi_{k+1}(X_1 \wedge Y_0) \arrow[r, "+"] \arrow[u, "+"'] & \pi_{k+2}(X_2 \wedge Y_0) \arrow[r, "+"] \arrow[u, "+"'] & \cdots
\end{tikzcd}$$

Of course, the same conclusion follows in symmetric spectra as well if we use the true homotopy groups from Definition 6.1.35.

21. Prove that the most "obvious" definition of a functor $\mathbf{S} \wedge \mathbf{S} \to \mathbf{S}$, does not in fact define a functor. This is one way of explaining why we couldn't define a good smash product on sequential spectra.

22. Finish the proof of Theorem 6.2.22 using the following ideas. We say that a map of spectra $X \to Y$ is levelwise surjective if each map $X_n \to Y_n$ is surjective.

    (a) Show that if $X \to Y$ and $W \to Z$ are levelwise surjective then so is their smash product $X \wedge W \to Y \wedge Z$.

    (b) If $f : X \to Y$ is levelwise surjective and we have a commuting square of the form
$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\eta"'] & Y, \end{tikzcd}$$
    explain why we must have $\eta = \mathrm{id}$.

    (c) Prove that for any spectrum X, the canonical map
$$\bigvee_{n \geq 0} F_n X_n \longrightarrow X$$
    is levelwise surjective.

    (d) Use these ideas to prove that the natural transformation η in the proof of Theorem 6.2.22 must be the identity on any n-tuple of spectra.

23. Fix a symmetric or orthogonal ring spectrum R and let R-**Mod** be the category of R-module spectra from Definition 6.2.32.

    (a) Show that $\mathbb{S}$-**Mod** $\cong$ **Sp**, in other words $\mathbb{S}$-modules are the same thing as spectra.

    (b) Show that the forgetful functor R-**Mod** $\to$ **Sp** has as its left adjoint the functor $R \wedge -$ that takes the smash product with R, and has R act by
$$R \wedge R \wedge X \xrightarrow{\mu \wedge \mathrm{id}} R \wedge X.$$

24. Fix a symmetric or orthogonal ring spectrum R. Prove that the category R-**Mod** of R-module spectra from Definition 6.2.32 has a model stucture in which the weak equivalences and fibrations are determined by forgetting the R-action. The cofibrations and acyclic cofibrations are generated by smashing R with the maps from Definition 5.6.1 and Definition 5.6.3:
$$\begin{aligned} R \wedge I &= \{\, R \wedge f \,:\, f \in I \,\} = \{\, R \wedge F_i S^{n-1}_+ \longrightarrow R \wedge F_i D^n_+ \,:\, n, i \geq 0 \,\}, \\ R \wedge J &= \{\, R \wedge f \,:\, f \in J \,\} = \{\, R \wedge F_i (D^n \times \{0\})_+ \longrightarrow R \wedge F_i (D^n \times I)_+ \,:\, n, i \geq 0 \,\} \\ &\qquad \cup \{\, R \wedge k_{i,j} \,\square\, \left(S^{n-1}_+ \longrightarrow D^n_+\right) \,:\, i, j, n \geq 0 \,\}. \end{aligned}$$

(Essentially, the proofs in Section 5.6 carry through here as well. You do need exercise 13 though, because R might not be cofibrant.)

25. Prove Lemma 6.3.12. You might want to use ideas from the proof of Lemma 4.1.7.

26. Define a **symmetric monoidal Quillen adjunction** to be

    - a Quillen adjunction $(F \dashv G)$ in which
    - the left Quillen functor $F: \mathbf{C} \to \mathbf{D}$ is *oplax* symmetric monoidal, meaning the maps go in the direction
    $$F(X \otimes_{\mathbf{C}} Y) \longrightarrow F(X) \otimes_{\mathbf{D}} F(Y), \quad I_{\mathbf{D}} \longrightarrow F(I_{\mathbf{C}}),$$
    - the map of units $F(QI_{\mathbf{C}}) \to F(I_{\mathbf{C}}) \to I_{\mathbf{D}}$ is a weak equivalence, and
    - for *cofibrant* X and Y, the map
    $$F(X \otimes_{\mathbf{C}} Y) \longrightarrow F(X) \otimes_{\mathbf{D}} F(Y)$$
    is a weak equivalence.

    Prove Lemma 6.3.12 under these weaker assumptions. In other words, the left-derived functor $\mathbb{L}F: \operatorname{Ho}\mathbf{C} \to \operatorname{Ho}\mathbf{D}$ is strong monoidal, and the right-derived functor $\mathbb{R}G: \operatorname{Ho}\mathbf{D} \to \operatorname{Ho}\mathbf{C}$ is lax monoidal.

27. Prove that the functor
    $$F(A) = \Omega^\infty(A \wedge E)$$
    of Example 6.4.13 is excisive. Here E is a sequential spectrum and A is a based finite CW complex.

28. Suppose that F and G are enriched functors and $\eta: F \to G$ is a natural transformation on the underlying functors. Explain why this gives a commuting square of topological spaces, not just sets,
    $$\begin{tikzcd} \operatorname{Map}_*(A,B) \arrow[r, "F"] \arrow[d, "G"'] & \operatorname{Map}_*(F(A),F(B)) \arrow[d, "\eta\circ-"] \\ \operatorname{Map}_*(G(A),G(B)) \arrow[r, "-\circ\eta"] & \operatorname{Map}_*(F(A),G(B)). \end{tikzcd}$$
    This is called the **enriched naturality condition**. If this weren't automatically true, we would have to assume it as a condition on η in order to get a good theory of enriched functors.

29. If F is an enriched functor, prove that the assembly map $A \wedge F(B) \to F(A \wedge B)$ described in (6.4.16) is continuous.

Chapter 7

More to come!

This is a draft of what is planned to be an open-access textbook. The list at the end of Section 0.2 indicates the plan for the remaining chapters. One of them is about the bar construction. Here is a picture!

Feedback on this draft is warmly welcomed.

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