Decimal expansion of fractions.

$1 \leq a < b$, $\gcd(a,b)=1$. We write $b = 2^u \cdot 5^w \cdot B$, where $\gcd(B,10)=1$. Last time we proved:

$\frac{a}{b}$ has finite decimal expansion if and only if $B=1$.

Suppose $B>1$. We expect a periodic decimal expansion:

$$\frac{a}{b} = 0.a_1 \ldots a_k \overline{b_1 \ldots b_s}, \quad \underline{a_k \neq b_s}, \ \underline{s \geq 1}$$

Let $\alpha = a_1 10^{k-1} + a_2 10^{k-2} + \ldots + a_k$, $\beta = b_1 10^{s-1} + b_2 10^{s-2} + \ldots + b_s$.

Then: $10 \nmid \alpha - \beta$ (since $a_k \neq b_s$), $1 \leq \beta < 10^s - 1$ and

$$\frac{a}{b} = \frac{\alpha}{10^k} + \frac{1}{10^k}\left(\frac{\beta}{10^s} + \frac{\beta}{10^{2s}} + \frac{\beta}{10^{3s}} + \ldots\right) =$$

$$= \frac{\alpha}{10^k} + \frac{\beta}{10^k}\left(\frac{1}{10^s} + \frac{1}{10^{2s}} + \ldots\right) = \frac{\alpha}{10^k} + \frac{\beta}{10^k(10^s-1)}$$

Thus $\quad a \cdot 10^k(10^s-1) = b\left[\alpha(10^s-1)+\beta\right] = 2^u 5^w B\left(\alpha(10^s-1)+\beta\right)$

If $u>0$ then $2 \nmid a$ and $2 \nmid 10^s-1$, so $2^u \mid 10^k$ so $k \geq u$.

If $w>0$ then $5 \nmid a$ and $5 \nmid 10^s-1$, so $5^w \mid 10^k$ so $k \geq w$.

This means that $k \geq \max(u,w)$ and

$$(*) \quad a \cdot 2^{k-u} \cdot 5^{k-w}(10^s-1) = B(\alpha(10^s-1)+\beta) = B(\alpha \cdot 10^s + \beta - \alpha)$$

Note that $10 \nmid B(\alpha \cdot 10^s + \beta - \alpha)$ (since $10 \nmid \beta - \alpha$) so $10 \nmid 2^{k-u} \cdot 5^{k-w}$. This means that one of $k-u$, $k-w$ must be 0. Thus $\underline{k = \max(u,w)}$.

Now $B \mid a \cdot 2^{k-u} 5^{k-w}(10^s-1)$ and B is relatively prime to $a \cdot 2^{k-u} \cdot 5^{k-w}$. Thus $B \mid 10^s - 1$. This means that $\operatorname{ord}_B 10 \mid s$. This means that $s \geq \operatorname{ord}_B 10$. We predict that we may take $\boxed{s = \operatorname{ord}_B 10}$.

Now we return to $*$:

$$a \cdot 2^{k-u} \cdot 5^{k-w} \cdot B \cdot \frac{10^s-1}{B} = B(\alpha(10^s-1)+\beta). \text{ so}$$

$$a \cdot 2^{k-u} 5^{k-w} \cdot \frac{10^s-1}{B} = \alpha(10^s-1)+\beta.$$

This implies that $\frac{10^s-1}{B} \mid \beta$. Write $\beta = \frac{10^s-1}{B} \cdot \gamma$.

Then
$$a \cdot 2^{k-u} \cdot 5^{k-w} = \alpha \cdot B + \gamma \quad (**)$$

Since $1 \leq \beta = \frac{10^s-1}{B} \cdot \gamma < 10^s-1$, we see that $\underline{1 \leq \gamma < B}$

The division algorithm tells us that there exist unique α, γ such that $(**)$ holds and $1 \leq \gamma < B$.

Thus we get the following algorithm to find the decimal expansion of $\frac{a}{b}$:

(1) Write $b = 2^u 5^w \cdot B$, $\gcd(B, 10) = 1$

If $B = 1$, $\frac{a}{b}$ has finite decimal expansion.

If $B > 1$ set $k = \max(u, w)$. Find $s = \operatorname{ord}_B 10$.

Then $\frac{a}{b}$ has a periodic decimal expansion:

$$\frac{a}{b} = \frac{\alpha}{10^k} + \frac{1}{10^k}\left(\frac{\beta}{10^s} + \frac{\beta}{10^{2s}} + \dots\right)$$

where $\beta = \frac{10^s-1}{B}\gamma$ and α, γ are obtained from division algorithm: $a \cdot 2^{k-u} \cdot 5^{k-w} = \alpha B + \gamma$, $1 \leq \gamma < B$.

In particular: • $\operatorname{ord}_B 10$ is the shortest period

• the decimal expansion of $\frac{a}{b}$ is purely periodic iff $u = w = k = 0$.

Example: Find the decimal expansion of $\frac{24}{2275}$.

We have $2275 = 5^2 \cdot 91 = 5^2 \cdot 7 \cdot 13$.

Thus: $a = 24$, $b = 2275$, $u = 0$, $w = 2$, $k = 2$, $B = 91$.

We need to find $s = \text{ord}_B 10$. From homework we know that if $\gcd(m,n) = 1$ then $\text{ord}_{mn} a = \text{lcm}(\text{ord}_m a, \text{ord}_n a)$

Since $91 = 7 \cdot 13$, we get $\text{ord}_{91} 10 = \text{lcm}(\text{ord}_7 10, \text{ord}_{13} 10)$.

Note that $10^3 + 1 = 7 \cdot 11 \cdot 13$. This means that

$10^3 \equiv -1 \pmod 7$ and $10^3 \equiv -1 \pmod{13}$ so

$10^6 \equiv 1 \pmod 7$ and $10^6 \equiv 1 \pmod{13}$.

Thus $\text{ord}_7 10 \mid 6$ and $\text{ord}_{13} 10 \mid 6$. We easily check that $\text{ord}_7 10 \neq 1, 2, 3$ so $\text{ord}_7 10 = 6$. Similarly $\text{ord}_{13} 10 = 6$.

Thus $s = \text{ord}_{91} 10 = \text{lcm}(6,6) = 6$.

Now we find α, γ s.t. $a \cdot 2^{k-u} 5^{k-w} = \alpha B + \gamma$, $1 \leq \gamma < B$, i.e

$$24 \cdot 2^2 \cdot 5^0 = \alpha \cdot 91 + \gamma, \quad 1 \leq \gamma < B.$$

$$96 = \alpha \cdot 91 + \gamma, \quad 1 \leq \gamma < B$$

Thus $\alpha = 1$, $\gamma = 5$. Then $\beta = \frac{10^6 - 1}{B} \cdot \gamma = \frac{10^6 - 1}{91} \cdot 5 = 999 \cdot 11 \cdot 5 = 54945$

Thus: $\frac{24}{2275} = \frac{1}{10^2} + \frac{1}{10^2}\left(\frac{54945}{10^6} + \frac{54945}{10^{12}} + \dots\right) =$

$= 0.01\overline{054945}$.