

Solutions to Homework 10

Read carefully chapter 3 in the book. Solve problems 4, 14, 15, 39, 40, 47, 61, 64, 66 in chapter 3.

Solution to Problem 4. a) Let $p(n)$ be the number of distinct prime divisors of n . Clearly, if m, n are relatively prime, then they do not share any prime divisors. Thus $p(mn) = p(m) + p(n)$. The function ρ is defined by $\rho(n) = 2^{p(n)}$. Thus, for m, n relatively prime we have

$$\rho(mn) = 2^{p(mn)} = 2^{p(m)+p(n)} = 2^{p(n)}2^{p(m)} = \rho(m)\rho(n)$$

which shows that ρ is multiplicative. If p is a prime number, then $\rho(p) = 2 = \rho(p^2)$. This shows that ρ is not completely multiplicative, as $\rho(p^2)$ is not equal to $\rho(p)^2$.

b) We have $f = \rho * \mathbb{1}$. Thus f is multiplicative, being a convolution of two multiplicative functions. For any prime p and $k > 0$ we have $\rho(p^k) = 2$. Thus,

$$f(p^k) = \sum_{i=0}^k \rho(p^i) = 1 + 2k.$$

It follows that

$$f(p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}) = (1 + 2a_1)(1 + 2a_2) \dots (1 + 2a_m).$$

Solution to Problem 14. a) Take $n = 2 \cdot 3^k$ for $k \geq 1$. Then

$$\phi(n) = \phi(2)\phi(3^k) = 1 \cdot (3 - 1) \cdot 3^{k-1} = 2 \cdot 3^{k-1} = n/3.$$

b) Let p be the largest prime divisor of n , $n = p^k m$, $k \geq 1$. Then $\phi(n) = \phi(m)p^{k-1}(p - 1)$ and all prime divisors of $\phi(m)$ are smaller than p . If $n = 4\phi(n)$ then $p^k | n = 4\phi(m)(p - 1)p^{k-1}$, so $p | 4\phi(m)(p - 1)$. Since p does not divide $p - 1$ nor $\phi(m)$, we must have $p | 4$, i.e. $p = 2$. This implies that $n = 2^k$ and $\phi(n) = 2^{k-1} \neq n/4$. Thus $\phi(n) = n/4$ is not possible.

Solution to problem 15. Let n be such that $\phi(n) = k$. If $n = p_1^{s_1} \dots p_m^{s_m}$ with $s_i > 0$ for all i then $p_i^{s_i-1}(p_i - 1) | \phi(n) = k$. This implies that

$$k \geq p_i^{s_i-1}(p_i - 1) = p_i^{s_i-1}(1 - 1/p_i) \geq p_i^{s_i}/2.$$

Thus $p_i^{s_i} \leq 2k$. Since there are finitely many prime powers smaller or equal than $2k$, there is only a finite number of possibilities for n .

Solution to problem 39. Note that

$$2^k - 1 = 2^0 + 2^1 + 2^2 + \dots + 2^{k-1} \geq 1 + \dots + 1 = k$$

(we replaced each power of 2 by 1). If $n = p^k$, $k \geq 1$, is a prime power then $\nu(n) = k + 1 \leq 2^k \leq p^k = n$. Since ν is multiplicative, if $n = p_1^{s_1} \dots p_m^{s_m}$ then

$$\nu(n) = \nu(p_1^{s_1}) \dots \nu(p_m^{s_m}) \leq p_1^{s_1} \dots p_m^{s_m} = n$$

so the result follows.

Solution to Problem 40. The problem asks as to prove that $(\nu * \mathbb{1}(n))^2 = (\nu^3 * \mathbb{1})(n)$ for every n . As ν is multiplicative, so are ν^3 , $(\nu * \mathbb{1})^2$, and $\nu^3 * \mathbb{1}$. It suffices then to show that $(\nu * \mathbb{1}(p^k))^2 = (\nu^3 * \mathbb{1})(p^k)$ for every prime power p^k . Now

$$(\nu * \mathbb{1})(p^k) = \sum_{i=0}^k \nu(p^i) = \sum_{i=0}^k (i + 1) = \frac{(k + 1)(k + 2)}{2}$$

and

$$(\nu^3 * \mathbb{1})(p^k) = \sum_{i=0}^k \nu^3(p^i) = \sum_{i=0}^k (i+1)^3.$$

It suffices then to prove that for every positive integer n we have

$$1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}.$$

This can be done, for example, by induction: if

$$1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

then

$$\begin{aligned} 1^3 + 2^3 + \dots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2 \left(\frac{n^2}{4} + n + 1 \right) = \\ &= (n+1)^2 \frac{(n+2)^2}{4} = \frac{(n+1)^2(n+2)^2}{4}. \end{aligned}$$

Solution to Problem 47. We have

$$\sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} = n \sum_{d|n} \frac{1}{d}$$

so the result follows.

Solution to Problem 61. Let n be an odd perfect number. Factor $n = p_1^{a_1} \dots p_m^{a_m}$. Then

$$2n = \sigma(n) = \sigma(p_1^{a_1}) \dots \sigma(p_m^{a_m}).$$

Since n is odd, each p_i is odd. Since 4 does not divide $2n$, exactly one of the numbers $\sigma(p_1^{a_1}), \dots, \sigma(p_m^{a_m})$ is even and none is divisible by 4. Recall that for p odd prime

$$\sigma(p^a) = 1 + p + p^2 + \dots + p^a$$

is the sum of $a+1$ odd numbers, so it is even if and only if $a+1$ is even, i.e. a is odd. It follows that exactly one of the exponents a_1, \dots, a_s is odd. We may assume that $a_1 = a$ is odd and the other exponents are all even. Thus $p_2^{a_2} \dots p_m^{a_m} = m$ is a square and $n = p^a m$, where $p = p_1$ and $a = a_1$ is odd. Since 4 does not divide $\sigma(p^a)$ we have

$$\sigma(p^a) = 1 + p + p^2 + \dots + p^a \equiv 2 \pmod{4}.$$

We have $p \equiv \pm 1 \pmod{4}$. If $p \equiv 1 \pmod{4}$ then

$$\sigma(p^a) = 1 + p + p^2 + \dots + p^a \equiv a + 1 \pmod{4}$$

so $a + 1 \equiv 2 \pmod{4}$, i.e. $a \equiv 1 \equiv p \pmod{4}$. If $p \equiv -1 \pmod{4}$ then

$$\sigma(p^a) = 1 + p + p^2 + \dots + p^a \equiv 1 - 1 + 1 - 1 + \dots - 1 \equiv 0 \pmod{4}$$

so $a + 1 \equiv 0 \pmod{4}$, i.e. $a \equiv -1 \equiv p \pmod{4}$. This completes our solution.

Solution to Problem 64. In problem 4 we showed that the function $\rho(n) = 2^{\omega(n)}$ is multiplicative (we denoted $\omega(n)$ by $p(n)$ in the solution). The function $h(n) = |\mu(n)|$ is

also multiplicative, since μ is. The problem asks us to prove that $h * \mathbb{1} = \rho$. Since both sides are multiplicative, it suffices to prove that $h * \mathbb{1}(p^k) = \rho(p^k)$ for every prime power p^k . Now, $\rho(p^k) = 2^1 = 2$ and

$$h * \mathbb{1}(p^k) = \sum_{i=0}^k |\mu(p^i)| = 2 = \rho(p^k)$$

(since $|\mu(1)| = |\mu(p)| = 1$ and $|\mu(p^i)| = 0$ for $i > 1$). This completes the solution.

Solution to Problem 66. For $n = p_1^{a_1} \dots p_m^{a_m}$ define $h(n) = \prod_{i=1}^m (1 - f(p_i))$. Clearly h is multiplicative. Note that $g(n) = \mu(n)f(n)$ is also multiplicative. The problem asks us to prove that $g * \mathbb{1} = h$. Since both sides are multiplicative, it suffices to check that $g * \mathbb{1}(p^k) = h(p^k)$ for every prime power p^k . Clearly $h(p^k) = 1 - f(p)$. Now

$$g * \mathbb{1}(p^k) = \sum_{i=0}^k \mu(p^i) f(p^i) = 1 - f(p) = h(p^k)$$

since $\mu(1) = 1$, $\mu(p) = -1$, $\mu(p^i) = 0$ for $i > 1$, $f(1) = 1$ (this holds for any multiplicative function f which is not identically 0; if $f = 0$ the result is clearly false).

Problem 1. For two arithmetic functions f, g we can define their usual product $f \cdot g$ by $(f \cdot g)(n) = f(n)g(n)$ for every positive integer n . Prove that f has the property that $f \cdot (g * h) = (f \cdot g) * (f \cdot h)$ for any arithmetic functions g, h if and only if f is completely multiplicative.

Solution. Suppose first that f is completely multiplicative. Then for any $d|n$ we have $f(n) = f(d)f(n/d)$ so

$$\begin{aligned} f \cdot (g * h)(n) &= f(n) \sum_{d|n} g(d)h(n/d) = \sum_{d|n} f(n)g(d)h(n/d) = \\ &= \sum_{d|n} f(d)g(d)f(n/d)h(n/d) = (f \cdot g) * (f \cdot h)(n). \end{aligned}$$

Conversely, suppose that f has the stated property. For any positive integer a define J_a as follows:

$$J_a(n) = \begin{cases} 0 & \text{if } n < a \\ 1 & \text{if } n \geq a. \end{cases}$$

Note that for any arithmetic functions g, h we have

$$(g \cdot J_a) * (h \cdot J_b)(ab) = \sum_{d|ab} g(d)J_a(d)h(ab/d)J_b(ab/d).$$

If $d|ab$ and $d < a$ then $J_a(d) = 0$. If $d|ab$ and $d > a$ then $ab/d < b$ and $J_b(ab/d) = 0$. Thus

$$(g \cdot J_a) * (h \cdot J_b)(ab) = g(a)J_a(a)h(b)J_b(b) = g(a)h(a).$$

In particular, $J_a * J_b(ab) = 1$ and $(f \cdot J_a) * (f \cdot J_b)(ab) = f(a)f(b)$. It follows that $(f \cdot (J_a * J_b))(ab) = f(ab)$. Since $f \cdot (J_a * J_b) = (f \cdot J_a) * (f \cdot J_b)$, we see that $f(ab) = f(a)f(b)$. This proves that f is completely multiplicative.