Solutions to Homework 12

Problem 1. Let $d > 1$ be an integer which is not a square (so $\sqrt{d}$ is irrational). Suppose that positive integers m, n satisfy $m^2 - n^2 d = \pm 1$. Prove that m/n is a convergent of $\sqrt{d}$.
Hint Show that $|\sqrt{d} - m/n| < 1/2n^2$.

Solution. Note first that $n^2 + 1 \leq n^2 + n^2 = 2n^2 \leq dn^2 = m^2 \pm 1 \leq m^2 + 1$. Thus $n^2 \leq m^2$ and therefore $m \geq n$ (as both m, n are positive). It follows that

$$m + n\sqrt{d} \geq n + \sqrt{d}n = (1 + \sqrt{d})n > 2n.$$

Note that

$$1 = |m^2 - dn^2| = |(m - n\sqrt{d})(m + n\sqrt{d})| = n\left|\frac{m}{n} - \sqrt{d}\right|(m + n\sqrt{d})$$

so

$$\left|\frac{m}{n} - \sqrt{d}\right| = \frac{1}{n(m + n\sqrt{d})} < \frac{1}{n \cdot 2n} = \frac{1}{2n^2}.$$

A theorem we proved in class says that the last inequality implies that $m = p_k$, $n = q_k$ for some k, where p_k/q_k is the k-th convergent of $\sqrt{d}$.

Remark. For a given $d > 0$ which is not a square, the equation $x^2 - y^2 d = \pm 1$ is called **Pell's equation**. The problem asserts that if x, y are solutions to the Pell's equation in positive integers then $x = p_k$, $y = q_k$ for some k, where p_k/q_k is the k-th convergent for $\sqrt{d}$. Recall now that the continued fraction for $\sqrt{d}$ is of the form $[m_0, \overline{m_1, \ldots, m_n, 2m_0}]$ for some $n \geq 0$ ($n = 0$ means that $\sqrt{d} = [m_0, \overline{2m_0}]$). It can be proved that $p_k^2 - dq_k^2 = \pm 1$ if and only if $k = s(n+1) - 1$ for some integer $s \geq 0$. Moreover, for $k = s(n+1) - 1$ we have $p_k^2 - dq_k^2 = (-1)^{s(n+1)}$. Setting $x_s = p_k, y_s = q_k$ for $k = s(n+1) - 1$, we have the following identity:

$$(x_s + y_s\sqrt{d}) = (x_1 + y_1\sqrt{d})^s.$$

Problem 2. Let x be an irrational number, let the simple continued fraction expression for x be $x = [k_0, k_1, \ldots]$, and let p_i/q_i, $i = 0, 1\ldots$ be the convergents of x.

a) Prove that if $n > 0$ then

$$\frac{1}{(k_{n+1} + 2)q_n^2} < \left|x - \frac{p_n}{q_n}\right| < \frac{1}{k_{n+1}q_n^2}.$$

b) Prove that there is $c > 0$ such that $|x - \frac{a}{b}| > \frac{1}{cb^2}$ for every fraction a/b if and only if the sequence $k_0, k_1, \ldots$ is bounded. Numbers x with this property are called **poorly approximable**.

Solution. Recall that we have

$$\frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} > \ldots > x > \ldots > \frac{p_4}{q_4} > \frac{p_2}{q_2} > \frac{p_0}{q_0}$$

and

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n+1}}, \quad \frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n} = \frac{(-1)^n k_{n+2}}{q_n q_{n+2}}.$$

Now $q_{n+1} = k_{n+1}q_n + q_{n-1} \geq k_{n+1}q_n$. Since x is between p_n/q_n and p_{n+1}/q_{n+1}, we have

$$\left|x - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_{n+1}q_n} \leq \frac{1}{k_{n+1}q_n^2}.$$

Similarly,

$$q_{n+2} = k_{n+2}q_{n+1} + q_n = k_{n+2}k_{n+1}q_n + k_{n+2}q_{n-1} + q_n < k_{n+2}k_{n+1}q_n + k_{n+2}q_n + q_n$$
$$\leq k_{n+2}k_{n+1}q_n + 2k_{n+2}q_n = k_{n+2}(k_{n+1}+2)q_n$$

Since p_n/q_n and p_{n+2}/q_{n+2} are on the same side of x, we have

$$\left|x - \frac{p_n}{q_n}\right| > \left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| = \frac{k_{n+2}}{q_{n+2}q_n} > \frac{k_{n+2}}{k_{n+2}(k_{n+1}+2)q_n^2} = \frac{1}{(k_{n+1}+2)q_n^2}.$$

b) Suppose that there is $c > 0$ such that $|x - \frac{a}{b}| > \frac{1}{cb^2}$ for every fraction a/b. Then, for every n we have

$$\frac{1}{k_{n+1}q_n^2} > \left|x - \frac{p_n}{q_n}\right| > \frac{1}{cq_n^2}$$

so $k_{n+1} < c$. Thus the sequence $k_0, k_1, \ldots$ is bounded. Conversely, assume that $k_n < L$ for some $L > 0$ and all n. Let a/b be any fraction, $b > 0$. There is n such that $q_n \leq b < q_{n+1}$. We proved in class that $|bx - a| \geq |q_n x - p_n|$. Thus

$$\left|x - \frac{a}{b}\right| \geq \frac{q_n}{b}\left|x - \frac{p_n}{q_n}\right| > \frac{q_n}{b}\frac{1}{(k_{n+1}+2)q_n^2} > \frac{1}{b(L+2)q_n} \geq \frac{1}{b^2(L+2)}.$$

Thus $c = L + 2$ works, so x is poorly approximable.

Solution to Problem 18. This problem was already solved in homework 11. Here we give a different solution.

Recall than for any real numbers $r_0, r_1, \ldots$ with $r_i > 0$ for $i > 0$, we define sequences (p_i), (q_i) by

$$p_{-1} = 1, p_0 = r_0, p_k = r_k p_{k-1} + p_{k-2}; \quad q_{-1} = 0, q_0 = 1, q_k = r_k q_{k-1} + q_{k-2}.$$

Then for any $k \geq$ we have

$$\begin{bmatrix} r_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} r_k & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} \tag{1}$$

and

$$[r_0, r_1, \ldots, r_k] = \frac{p_k}{q_k} \text{ and } [r_0, r_1, \ldots, r_{k-1}] = \frac{p_{k-1}}{q_{k-1}}.$$

Suppose now that $r_0 > 0$. Then taking the transpose of (1) we get

$$\begin{bmatrix} r_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_{k-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} r_0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix}. \tag{2}$$

Thus

$$[r_k, r_{k-1}, \ldots, r_0] = \frac{p_k}{p_{k-1}} \text{ and } [r_k, r_{k-1}, \ldots, r_1] = \frac{q_k}{q_{k-1}}.$$

Solution to Problem 20e). Recall that if $\alpha_0 = \sqrt{19}$, $\alpha_{n+1} = \dfrac{1}{\alpha_n - \lfloor \alpha_n \rfloor}$ and $k_n = \lfloor \alpha_n \rfloor$ then $\sqrt{19} = [k_0, k_1, \ldots]$. We have $k_0 = \lfloor \alpha_0 \rfloor = 4$,

$$\alpha_1 = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3}, \quad k_1 = \lfloor \alpha_1 \rfloor = 2, \quad \alpha_2 = \frac{1}{\frac{\sqrt{19}+4}{3} - 2} = \frac{3}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{5},$$

$$k_2 = \lfloor \alpha_2 \rfloor = 1,\ \alpha_3 = \frac{1}{\frac{\sqrt{19}+2}{5} - 1} = \frac{5}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{2},\ k_3 = \lfloor \alpha_3 \rfloor = 3,$$

$$\alpha_4 = \frac{1}{\frac{3+\sqrt{19}}{2} - 3} = \frac{2}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{5},\ k_4 = \lfloor \alpha_4 \rfloor = 1,\ \alpha_5 = \frac{1}{\frac{3+\sqrt{19}}{5} - 1} = \frac{5}{\sqrt{19}-2} =$$

$$\frac{\sqrt{19}+2}{3},\ k_5 = \lfloor \alpha_5 \rfloor = 2,\ \alpha_6 = \frac{1}{\frac{2+\sqrt{19}}{3} - 2} = \frac{3}{\sqrt{19}-4} = \sqrt{19}+4,\ k_6 = \lfloor \alpha_5 \rfloor = 8,$$

$$\alpha_7 = \frac{1}{\sqrt{19}-4} = \alpha_1.$$

We see that $\alpha_7 = \alpha_1$. It follows that $\sqrt{19} = [4, \overline{2, 1, 3, 1, 2, 8}]$.

Solution to problem 29. Let α be an irrational number, $\alpha = [m_0, m_1, \dots,] = \lim_{k\to\infty}[m_0, m_1, \dots, m_k]$. Note that

$$[0, m_0, m_1, \dots, m_k] = \frac{1}{[m_0, m_1, \dots, m_k]}.$$

Letting k go to infinity and taking the limits we see that

$$[0, m_0, m_1, \dots,] = \lim_{k\to\infty}[0, m_0, m_1, \dots, m_k] = \lim_{k\to\infty}\frac{1}{[m_0, m_1, \dots, m_k]} = \frac{1}{\alpha}.$$

Thus the $(i+1)$-st convergent of $1/\alpha$ is $[0, m_0, m_1, \dots, m_i] = 1/[m_0, m_1, \dots, m_k]$ which is the reciprocal of the i-th convergent $[m_0, m_1, \dots, m_k]$ of α.

Solution to problem 36. Let α be a quadratic irrational. Thus $\alpha = \dfrac{a+\sqrt{d}}{c}$ for some integers a, c, d such that $d > 0$ is not a perfect square. Let u, w, s, t be integers. Then

$$\frac{u\alpha + w}{s\alpha + t} = \frac{ua + u\sqrt{d} + cw}{sa + s\sqrt{d} + ct} = \frac{m + u\sqrt{d}}{n + s\sqrt{d}},$$

where $m = ua + cw$, $n = sa + ct$ are integers. Now

$$\frac{m + u\sqrt{d}}{n + s\sqrt{d}} = \frac{(m + u\sqrt{d})(n - s\sqrt{d})}{n^2 - s^2 d} = \frac{(mn - usd) + (nu - ms)\sqrt{d}}{n^2 - s^2 d} = \frac{A + B\sqrt{d}}{C},$$

where $A = mn - usd$, $B = nu - ms$, $C = n^2 - s^2 d$ are integers. If $B = 0$ then $\dfrac{m + u\sqrt{d}}{n + s\sqrt{d}} = \dfrac{A}{C}$ is a rational number, and if $B \neq 0$ then $\dfrac{A + B\sqrt{d}}{C}$ is an irrational number which is a root of the quadratic equation $C^2x^2 - 2ACx + (A^2 - B^2 d) = 0$, so it is a quadratic irrational.

Solution to problem 39. b) Let $x = [\overline{1, n-2, 1, 2n-2}]$. Then $x = [1, n-2, 1, 2n-2, x]$. We have

$$[1, n-2, 1, 2n-2, x] = 1 + \cfrac{1}{n - 2 + \cfrac{1}{1 + \cfrac{1}{2n - 2 + \cfrac{1}{x}}}} = 1 + \cfrac{1}{n - 2 + \cfrac{1}{1 + \cfrac{x}{(2n-2)x + 1}}} =$$

$$1+\cfrac{1}{n-2+\cfrac{(2n-2)x+1}{(2n-1)x+1}}=1+\frac{(2n-1)x+1}{(2n^2-3n)x+n-1}=\frac{(2n^2-n-1)x+n}{(2n^2-3n)x+n-1}.$$

Thus we get the equation

$$x=\frac{(2n^2-n-1)x+n}{(2n^2-3n)x+n-1}$$

from which we get

$$n(2n-3)x^2-2n(n-1)x-n=0,\ \text{i.e. } (2n-3)x^2-2(n-1)x-1=0.$$

The discriminant of the last equation is $4(n-1)^2+4(2n-3)=4(n^2-2)$. Thus

$$x=\frac{2(n-1)\pm\sqrt{4(n^2-2)}}{2(2n-3)}=\frac{n-1\pm\sqrt{n^2-2}}{2n-3}.$$

Since $x>0$, we have $x=\dfrac{n-1+\sqrt{n^2-2}}{2n-3}$. Now

$$[n-1,\overline{1,n-2,1,2n-2}]=(n-1)+\frac{1}{[\overline{1,n-2,1,2n-2}]}=n-1+\frac{1}{x}=$$

$$n-1+\frac{1}{\frac{n-1+\sqrt{n^2-2}}{2n-3}}=n-1+\frac{2n-3}{n-1+\sqrt{n^2-2}}=n-1+\frac{(2n-3)(\sqrt{n^2-2}-n+1)}{(n^2-2)-(n-1)^2}=$$

$$n-1+\frac{(2n-3)(\sqrt{n^2-2}-n+1)}{2n-3}=\sqrt{n^2-2}.$$

c) We have $531=23^2+2$ so $\sqrt{531}=[23,\overline{23,46}]$.

We have $674=26^2-2$ so $\sqrt{674}=[25,\overline{1,24,1,50}]$.

Solution to problem 42. We are given that

$$4+\sqrt{23}=[\overline{8,1,3,1}]=8+\frac{1}{[\overline{1,3,1,8}]}.$$

Thus

$$\sqrt{23}=-4+8+\frac{1}{[\overline{1,3,1,8}]}=4+\frac{1}{[\overline{1,3,1,8}]}=[4,\overline{1,3,1,8}].$$