Homework 2

due on Friday, February 6

Read carefully sections 1.5, 2.1, 2.2 in the book. Solve the following problems.

Problem 1. Suppose that $a_1 = 2$ and $a_{n+1} = 3a_n + 2$. Prove that $a_n = 3^n - 1$ for every natural number n.

Problem 2. Let $F_n = 2^{2^n} + 1$, for $n = 0, 1, 2, ...$

a) Prove that $F_0 \cdot F_1 \cdot F_2 \cdot ... \cdot F_n = F_{n+1} - 2$ for every n.

b) Prove that $\gcd(F_n, F_m) = 1$ for $n \neq m$.

c) Use b) to give yet another proof that the set of primes is infinite.

Problem 3. Read the proof of Proposition 1.22 (page 32) in the book (we went over it in clas). Using simialr method prove that there are infinitely many prime numbers of the form $3n + 2$.

Problem 4. Recall that when p is a prime number and $n \neq 0$ an integer then $e_p(n)$ is the largest integer a such that $p^a | n$.

a) Prove that if $n > 1$ and $p > n$ is a prime then $e_p(n!) = 0$

b) Recall thal $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Prove that if n, k are positive integers then

$$\left\lfloor \frac{n+1}{k} \right\rfloor = \begin{cases} \lfloor \frac{n}{k} \rfloor & \text{if } k \nmid (n+1) \\ 1 + \lfloor \frac{n}{k} \rfloor & \text{if } k|(n+1) \end{cases}.$$

c) Prove that if $n > 1$ and $p \leq n$ is a prime then

$$e_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots$$

(note that the sum is actually finite since $\lfloor n/p^k \rfloor = 0$ when $p^k > n$.

Hint. There are several ways to prove this, but I suggest a proof by induction on n. Note that $e_p((n+1)!) = e_p(n!) + e_p(n+1)$ and use part b) (this is why b) is part of this problem).

d) Use c) to write the prime factorization of 20!.

e) Find the number of zeros with which the decimal representation of 99! terminates.

Problem 5. a) Prove that if a, b, c are integers and $\gcd(a, c) = 1 = \gcd(b, c)$ then $\gcd(ab, c) = 1$.

b) Prove that if $\gcd(a, b) = 1$ then $\gcd(a^n, b^m) = 1$ for all positive integers m, n.

c) Prove that if $\gcd(a^n, b^m) = 1$ for some positive integers m, n then $\gcd(a, b) = 1$.

d) Prove that if n is a positive integer and $a^n | b^n$ then $a | b$.