# Solutions to Homework 9

Solutions to problems 16b is coming soon.

**Solution to Problem 4.** This is similar to problem 2 from homework 3. Let $x$ be the number of people attending the banquet who ordered a steak dinner for \$ 8 and let $y$ be the number of people attending the banquet who ordered a lobster dinner for \$ 13. The problem is about solutions to the equation $8x + 13y = 1571$ in non-negative integers. Among such solutions, problems askes what are the smallest and largest possible values of $x + y$ (which is the number of people attending).

Note that $\gcd(8, 13) = 1$. From Euclid's algorithm we see that $1 = 5 \cdot 8 + (-3) \cdot 13$. Multiplying by 1571 we get

$$1571 = 8 \cdot (5 \cdot 1571) + 13 \cdot (-3 \cdot 1571) = 8 \cdot 7855 + 13 \cdot (-4713).$$

All integer solutions to $8x + 13y = 1571$ are given by $x = 7855 + 13k$, $y = -4713 - 8k$, $k \in \mathbb{Z}$. To get non-negative solutions we need $7855 + 13k \geq 0$ and $-4713 - 8k \geq 0$, i.e.

$$\frac{-4713}{8} \geq k \geq \frac{-7855}{13}$$

which for integers $k$ is the same as

$$-604 \leq k \leq -590.$$

Note that $x + y = 7855 + 13k + (-4713 - 8k) = 3142 + 5k$. Thus $x + y$ is largest when $k$ is largest possible, i.e. $k = -590$ and then $x + y = 192$. Similarly, $x + y$ is smallest when $k$ is smallest possible, i.e. $k = -604$ and then $x + y = 122$. Thus the largest possible number of people attending is 192 and the smallest possible number is 122.

**Solution to problem 11d).** Suppose that $x^3 - 5 = 7y^3$ for some integers $x, y$. Looking at it modulo 7, we get $x^3 - 5 \equiv 0 \pmod 7$, i.e. 5 is a cubic residue modulo 7. But the only cubic residues modulo 7 are $-1, 1$ (if $x^3 \equiv a \pmod 7$ and $7 \nmid a$ then $1 \equiv x^6 \equiv a^2 \pmod 7$, so $a \equiv \pm 1 \pmod 7$), so 5 is not a cubic residue modulo 7 and therefore the equation has no solutions in integers

**Solution to problem 12d).** Suppose that $x^2 + 2y^2 + 3 = 8z$ for some integers $x, y, z$. This implies that $x^2 + 3$ is even and therefore $x$ is odd. Recall that $x^2 \equiv 1 \pmod 8$ for any odd integer $x$. Going modulo 8 we see that $1 + 2y^2 + 3 \equiv 0 \pmod 8$, i.e. $2y^2 = 4 \pmod 8$. This implies that $y^2 \equiv 2 \pmod 4$. However this congruence has no solutions (any solution $y$ would have to be even and then $y^2$ would be divisible by 4). Thus our original equation has no solutions.

**Solution to problem 14.** Let $x, y, x$ be a primitive Pythagorean triple with $y$ even. Thus $x^2 + y^2 = z^2$ and $\gcd(x, y) = \gcd(x, z) = \gcd(y, z) = 1$. We know that there are relatively prime positive integers $m > n$ of different parity such that $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$.

a) Note that if $3 \nmid a$ then $a^2 \equiv 1 \pmod 3$. Thus, if neither $x$ nor $y$ is divisible by 3, then $x^2 \equiv 1 \equiv y^2 \pmod 3$. This however implies that $z^2 = x^2 + y^2 \equiv 2 \pmod 3$, which is not possible. Thus at least one of $x, y$ must be divisible by 3. Since $\gcd(x, y) = 1$, exactly one of $x, y$ is divisible by 3.

b) We know that $x = m^2 - n^2$ and $y = 2mn$, where $m, n$ have different parity. This means that $x$ is odd and $4 | y$ (as one of $m, n$ is even).

c) We know that at most one of $x, y, z$ can be divisible by 5. Recall that if $5 \nmid a$ then $a^2 \equiv \pm 1 \pmod 5$ (since $(a^2)^2 = a^4 \equiv 1 \pmod 5$ by Fermats Little Theorem). It suffices to show then that one of $x, y, z$ is divisible by 5. If $x$ or $y$ is divisible by 5 we are done. Suppose that neither $x$ nor $y$ are divisible by 5. Then $x^1 \equiv \pm 1 \pmod 5$ and $y^2 \equiv \pm 1 \pmod 5$ which means that $z^2 = x^2 + y^2 \mod 5$ is one of $(-1) + (-1) = -2, -1 + 1 = 0, 1 + (-1) = 0, 1 + 1 = 2$. Since a square can not be congruent to -2 or 2 modulo 5, we must have $z^2 \equiv 0 \pmod 5$, so $5|z$.

d) We have $x + y = m^2 - n^2 + 2mn = (m+n)^2 - 2n^2 = 2m^2 - (m-n)^2$. Note that $m + n$ and $m - n$ are odd so $(m+n)^2 \equiv 1 \equiv (m-n)^2 \pmod 8$. If $m$ is even then $2m^2 \equiv 0 \pmod 8$ and $x + y \equiv -1 \equiv 7 \pmod 8$. If $n$ is even then $2n^2 \equiv 0 \pmod 8$ and $x + y \equiv 1 \pmod 8$. Thus $x + y \equiv 1, 7 \pmod 8$. Now $(x+y) - (x-y) = 2y \equiv 0 \pmod 8$, since we showed in b) that $y$ is divisible by 4. Thus $x + y \equiv x - y \pmod 8$.

e) Note that $xyz$ is divisible by 3) by part a), it is divisible by 4 by part b), and it is divisible by 5 by part c). Since $3, 4, 5$ are pairwise relatively prime, $60 = 3 \cdot 4 \cdot 5$ divides $xyz$.

**Solution to problem 29 e,f.** Note that $1105 = 5 \cdot 13 \cdot 17$ and $16133 = 13 \cdot 17 \cdot 73$. We have $5 = 2^2 + 1^2$, $13 = 3^2 + 2^2$, $17 = 4^2 + 1^2$, $73 = 8^2 + 3^2$. We will use the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + (ad - bc)^2.$$

Thus

$$13 \cdot 17 = (3^2 + 2^2)(4^2 + 1^2) = (12 - 2)^2 + (3 + 8)^2 = 10^2 + 11^2.$$

**Remark.** Note that this is not the only possibility. We could also get $13 \cdot 17 = 14^2 + 5^2$, which would lead to different solutions to e) and f).

e) We have

$$1105 = 5 \cdot (13 \cdot 17) = (2^2 + 1^2)(10^2 + 11^2) = (20 - 11)^2 + (22 + 10)^2 = 9^2 + 32^2.$$

f) We have

$$16133 = (13 \cdot 17) \cdot 73 = (10^2 + 11^2)(8^2 + 3^2) = (80 - 33)^2 + (30 + 88)^2 = 47^2 + 118^2.$$

**Solution to problem 32d.** Note that $2926 = 2 \cdot 7 \cdot 11 \cdot 19$. Now $7 = 2^2 + 1^2 + 1^2 + 1^2$, $11 = 3^2 + 1^1 + 1^1 + 0^2$, $19 = 4^1 + 1^1 + 1^2 + 1^2$. We will use the identities:

$$(a^2 + b^2 + c^2 + d^2)(a_1^2 + b_1^2 + c_1^2 + d_1^2) =$$

$$(aa_1 + bb_1 + cc_1 + dd_1)^2 + (-ab_1 + ba_1 - cd_1 + dc_1)^2 + (-ac_1 + bd_1 + ca_1 - db_1)^2 + (-ad_1 - bc_1 + cb_1 + da_1)^2$$

and

$$2(a^2 + b^2 + c^2 + d^2) = (a+b)^2 + (a-b)^2 + (c+d)^2 + (c-d)^2.$$

Thus

$$11 \cdot 19 = (3^2 + 1^1 + 1^1 + 0^2)(4^1 + 1^1 + 1^2 + 1^2) =$$

$$(12 + 1 + 1 + 0)^2 + (-3 + 4 - 1 + 0)^2 + (-3 + 1 + 4 - 0)^2 + (-3 - 1 + 1 + 0)^2 = 14^2 + 0^2 + 2^2 + 3^2.$$

Now

$$7 \cdot (11 \cdot 19) = (2^2 + 1^2 + 1^2 + 1^2)(14^2 + 3^2 + 2^2 + 0^2) =$$

$$(28+3+2+0)^2+(-6+14-0+2)^2+(-4+0+14-3)^2+(-0-2+3+14)^2 = 33^2+10^2+7^2+15^2.$$

Finally,

$$2926 = 2 \cdot (7 \cdot 11 \cdot 19) = 43^2 + 23^2 + 22^2 + 8^2.$$

**Solution to problem 37.** Note that

$$169 = 13^2 = 5^2 + 12^2 = 3^2 + 4^2 + 12^2 = 4^2 + 6^2 + 6^2 + 9^2$$

. Let $n > 169$. Then $n - 169$ is a sum of 1, 2, 3, or 4 squares of positive numbers by Lagrange's Theorem.

If $n - 169 = a^2$ is a square, then $n = a^2 + 4^2 + 6^2 + 6^2 + 9^2$.

If $n - 169 = a^2 + b^2$ is a sum of two positive squares, then $n = a^2 + b^2 + 3^2 + 4^2 + 12^2$.

If $n-169 = a^2+b^2+c^2$ is a sum of three positive squares, then $n = a^2+b^2+c^2+5^2+12^2$.

If $n - 169 = a^2 + b^2 + c^2 + d^2$ is a sum of four positive squares, then $n = a^2 + b^2 + c^2 + d^2 + 13^2$.

We see that in each case $n$ is a sum of five positive squares.

**Solution to problem 21.** Suppose there exist solutions to the equation

$$X^4 - Y^4 = Z^2 \tag{1}$$

in integers $X, Y, Z$ which are all non zero. Among all such solutions choose one $x, y, z$ for which $x > 0$ is smallest possible and $y > 0$, $z > 0$. We will arrive at a contradiction by showing that there is a solution with smaller $X > 0$.

Let $d = \gcd(x, y)$. Then $d^4 | x^4 - y^4 = z^2$. Thus $d^2 | z$ and $X = x/d, Y = y/d, Z = z^2/d$ is also a solution to (1). By our choice of $x, y, z$ we have $x/d \geq x$, so $d = 1$ . In the same way we show that $\gcd(x, z) = 1 = \gcd(y, z)$. It follows that $z, y^2, x^2$ is a primitive Pythagorean triple($z^2 + (y^2)^2 = (x^2)^2$). In particular, $x$ is odd and one of $z, y$ is even.

If $z$ is even, then $x^2 = m^2 + n^2$, $y^2 = m^2 - n^2$ and $z = 2mn$ for some relatively prime $m > n$ of different parity. Multiplying the first two equations we get

$$(xy)^2 = (m^2 + n^2)(m^2 - n^2) = m^4 - n^4.$$

Thus $X = m, Y = n, Z = xy$ is a solution to (1) and $x^2 = m^2 + n^2 > m^2$, so $x > m$. This is not possible by our choice of $x, y, z$.

If $z$ is odd then $x^2 = m^2 + n^2$, $z = m^2 - n^2$ and $y^2 = 2mn$ for some relatively prime $m > n$ of different parity. Assume $n$ is even (if $m$ is even the argument is exactly the same). Note that $m, n, x$ is a primitive Pythagorean triple. Thus $x = a^2 + b^2$, $n = 2ab$, $m = (a^2 - b^2)$ for some relatively prime $a, b$. Thus $mab = (y/2)^2$ and $m, a, b$ are pairwise relatively prime. This means that each of $m, a, b$ must be a square: $m = p^2$, $a = s^2$, $b = t^2$. Thus $p^2 = m = a^2 - b^2 = s^4 - t^4$. Clearly none of $p, s, t$ is 0 and $X = s, Y = t, Z = p$ is a solution to (1) with $s \leq a < a^2 + b^2 = x$. Again this is not possible.