Lecture, Friday May 11

Last time we proved the following result.

<u>Theorem 1</u>: For every rational number $\frac{a}{b}$ $(b>0)$ there are unique integers $m_0, m_1, m_2, \dots, m_k$ such that either $k=0$ or $m_k \geq 2$ and $\frac{a}{b} = [m_0, m_1, \dots, m_k]$. The only other expansion of $\frac{a}{b}$ as continued fraction is $[m_0, m_1, \dots, m_k - 1, 1]$.

The numbers $m_0, \dots, m_k$ are computed by the Euclid's algorithm for $\gcd(a,b)$. Alternatively, define $\alpha_0 = \frac{a}{b}$, $\alpha_{i+1} = \frac{1}{\alpha_i - \lfloor \alpha_i \rfloor}$ for $i<k$.

Then $m_i = \lfloor \alpha_i \rfloor$, $\alpha_i = [m_i, \dots, m_k]$.

<u>Definition</u>: Given real number $r_0, r_1, \dots, r_n$ with $r_i > 0$ for $i \geq 1$ define $[r_0, r_1, \dots, r_n] = r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{\ddots + \cfrac{1}{r_k}}}}$. We call this expression a <u>finite continued fraction</u>. It is called <u>simple</u> if all r_i's are integers.

Given any sequence $r_0, r_1, \dots$ (finite or infinite) define two sequences (p_n) and (q_n) as follows:

$$p_{-1} = 1,\ p_0 = r_0,\ p_k = r_k p_{k-1} + p_{k-2}, \text{ for } k \geq 1$$

$$q_{-1} = 0,\ q_0 = 1,\ q_k = r_k q_{k-1} + q_{k-2}, \text{ for } k \geq 1.$$

<u>Then</u>: ① $\begin{bmatrix} r_0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} r_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} r_k & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}$ for every $k \geq 0$.

② $[r_0, r_1, \dots, r_k] = \frac{p_k}{q_k}$ for every k

③ $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$ for every k.

(4) $q_0 = 1 < q_2 < q_4 < \dots$ and $q_1 < q_3 < q_5 < \dots$

If $r_i \geq 1$ for all $i \geq 1$ then $q_0 < q_1 < q_2 < \dots$ and $q_n > n$.

(5) From (3) we get $\boxed{\dfrac{p_{k+1}}{q_{k+1}} - \dfrac{p_k}{q_k} = \dfrac{(-1)^{k+2}}{q_{k+1}q_k}}$ $\boxed{A}$ and

$$\frac{p_{k+2}}{q_{k+2}} - \frac{p_k}{q_k} = \left(\frac{p_{k+2}}{q_{k+2}} - \frac{p_{k+1}}{q_{k+1}}\right) + \left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) = \frac{(-1)^{k+3}}{q_{k+2}q_{k+1}} + \frac{(-1)^{k+2}}{q_{k+1}q_k} =$$

$$= \frac{(-1)^{k+2}}{q_{k+1}}\left(\frac{1}{q_k} - \frac{1}{q_{k+2}}\right) = \frac{(-1)^{k+2}(q_{k+2} - q_k)}{q_k q_{k+1} q_{k+2}} = \frac{(-1)^{k+2} r_{k+2} q_{k+1}}{q_k q_{k+1} q_{k+2}}$$

Thus $\boxed{\dfrac{p_{k+2}}{q_{k+2}} - \dfrac{p_k}{q_k} = \dfrac{(-1)^{k+2} r_{k+2}}{q_k q_{k+2}}}$ $\boxed{B}$ (note: $(-1)^{k+2} = (-1)^k$)

(6) From $\boxed{B}$:

$$\begin{array}{ccccccc} \dfrac{p_0}{q_0} & < & \dfrac{p_2}{q_2} & < & \dfrac{p_4}{q_4} & < & \dots \\ \wedge| & & \wedge| & & \wedge| & & \\ \dfrac{p_1}{q_1} & > & \dfrac{p_3}{q_3} & > & \dfrac{p_5}{q_5} & > & \dots \end{array}$$

Thus $\lim_{n\to\infty} \dfrac{p_{2n}}{q_{2n}} = x$ exists and $\lim_{n\to\infty} \dfrac{p_{2n+1}}{q_{2n+1}} = y$ exists

and $y \geq x$. By $\boxed{A}$ $y - x = \lim_{n\to\infty}\left(\dfrac{p_{2n+1}}{q_{2n+1}} - \dfrac{p_{2n}}{q_{2n}}\right) = \lim_{n\to\infty} \dfrac{1}{q_{2n+1}q_{2n}}$

Thus, if q_n tends to infinity then $y = x = \lim_{n\to\infty} \dfrac{p_n}{q_n}$

<u>Exercise</u>: Show that if $\sum_{n=1}^{\infty} r_n$ diverges then $y = x$.

Since $q_n \geq n$ when all $r_n \geq 1$ we get

Theorem 2. For any sequence $m_0, m_1, m_2, \dots$ of integers with $m_i \geq 1$ for $i > 0$ the limit

$$\lim_{n\to\infty} [m_0, m_1, m_2, \dots, m_n] \text{ exists.}$$

Definition: $[m_0, m_1, m_2, \dots\ ] = \lim_{n\to\infty} [m_0, m_1, \dots, m_n]$ is called an <u>infinite simple continued fraction</u>, or just <u>simple continued fraction</u>.

For every $k \geq 0$ the rational number $\frac{p_k}{q_k} = [m_0, \dots, m_k]$ is called the <u>k-th reduct</u> or <u>k-th convergent</u> of the continued fraction.

Last time we proved

Theorem 3: For any infinite continued fraction $x = [m_0, m_1, \dots\ ]$ the number x is irrational. If (α_k) is defined by $\alpha_0 = x$, $\alpha_{k+1} = \frac{1}{\alpha_k - \lfloor \alpha_k \rfloor}$ then $\alpha_k = [m_k, m_{k+1}, \dots\ ]$ and $m_k = \lfloor \alpha_k \rfloor$.

Theorem 4: Let x be any irrational real number. Define $\alpha_0 = x$, $\alpha_{k+1} = \frac{1}{\alpha_k - \lfloor \alpha_k \rfloor}$, $m_k = \lfloor \alpha_k \rfloor$. Then $x = [m_0, m_1, \dots\ ]$

Proof: Since x is irrational, all α_k are irrational so $\alpha_k \neq \lfloor \alpha_k \rfloor$, i.e. the sequence α_k is well defined. Note that for any k:

$$[m_0, m_1, \dots, m_k, \alpha_{k+1}] = [m_0, m_1, \dots, m_{k-1}, [m_k, \alpha_{k+1}]] = [m_0, m_1, \dots, m_{k-1}, \alpha_k]$$

since $[m_k, \alpha_{k+1}] = m_k + \frac{1}{\alpha_{k+1}} = m_k + \frac{1}{\frac{1}{\alpha_k - \lfloor \alpha_k \rfloor}} = m_k + \alpha_k - \lfloor \alpha_k \rfloor = \alpha_k$

(recall that $m_k = \lfloor \alpha_k \rfloor$)

Thus:

$$x = [\alpha_0] = [m_0, \alpha_1] = [m_0, m_1, \alpha_2] = [m_0, m_1, m_2, \alpha_3] = \dots$$

From $\boxed{A}$ we know that for k even we have

$$x = [m_0, m_1, \dots, m_k, \alpha_{k+1}] > [m_0, m_1, \dots, m_k]$$

and for k odd we have

$$x = [m_0, m_1, \dots, m_k, \alpha_{k+1}] < [m_0, m_1, \dots, m_k]$$

In other works

$$[m_0, m_1, \dots, m_{2n+1}] > x > [m_0, m_1, \dots, m_{2n}]$$

for every x. Since $\lim_{k\to\infty} [m_1, \dots, m_k]$ exists, it must be equal to x: $\quad x = [m_0, m_1, \dots] = \lim_{k\to\infty} [m_0, m_1, m_2, \dots m_k]$.

<u>Summarizing</u> our results:

$$\boxed{\begin{array}{l}\text{Every irrational number } x \text{ has unique expansion as} \\ \text{infinite continued fractions } x = [m_0, m_1, m_2, \dots] \\ \text{where } \alpha_0 = x,\ \alpha_{k+1} = \dfrac{1}{\alpha_k - \lfloor \alpha_k \rfloor} \text{ and } m_k = \lfloor \alpha_k \rfloor.\end{array}}$$

<u>Examples</u>: We have seen that $\frac{1+\sqrt{5}}{2} = [1, 1, 1, \dots]$

$$\sqrt{2} = [1, 2, 2, 2, \dots]$$

Euler proved that $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \dots]$

$\boxed{\text{Open problem}}$: When $x = \sqrt[3]{2}$, or $x = \sqrt{2} + \sqrt{3}$ let $x = [m_0, m_1, m_2, \dots]$. Is the sequence (m_i) bounded?

What is $[1,1,2,1,2,1,2,\dots]$?

Let $x=[1,2,1,2\cdots]$. Then $[1,1,2,1,2\cdots]=1+\frac{1}{x}$

Also $x=[1,2,1,2,\dots]=[1,2,[1,2,1,2,\dots]]=[1,2,x]$ so

$$x=1+\frac{1}{2+\frac{1}{x}}=1+\frac{x}{2x+1}=\frac{3x+1}{2x+1}$$

so $2x^2+x=3x+1$, i.e. $2x^2-2x-1=0$. We have $\Delta=4+8=12$

and $x=\frac{2\pm2\sqrt{3}}{4}$. Since $x>0$, $x=\frac{1+\sqrt{3}}{2}$. Thus

$$[1,1,2,1,2,\dots]=1+\frac{1}{\frac{1+\sqrt{3}}{2}}=1+\frac{2}{1+\sqrt{3}}=1+\frac{2(\sqrt{3}-1)}{2}=\sqrt{3}.$$

Let us compute some reducts of $\sqrt{3}$:

	-1	0	1	2	3	4	5	6	7	8	9	10	11	
		1	1	2	1	2	1	2	1	2	1	2	1	2
p	1	1	2	5	7	19	26	71	97	265	362	989	1351	
q	0	1	1	3	4	11	15	41	56	153	209	571	780	

so $\frac{p_{11}}{q_{11}}=\frac{1351}{780}>\sqrt{3}>\frac{p_{10}}{q_{10}}>\frac{p_8}{q_8}=\frac{265}{153}$

These two approximations were known to Archimedes (200 BC).

<u>Thm</u>: If x has purely periodic continued fraction then $x=\frac{A+\sqrt{D}}{C}$ for some integers A, D, C s.t. D is not a square.

<u>Pf</u>: Suppose $x=[\overline{a_0, a_1, \ldots, a_k}]=[a_0, a_1, \ldots, a_k, a_0, a_1, \ldots, a_k, a_0, a_1, \ldots a_k, \ldots]$

Then $x=[a_0, a_1, \ldots, a_k, x]$. Let

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} p_k x+p_{k-1} & p_k \\ q_k x+q_{k-1} & q_k \end{bmatrix}.$$

This means that $[a_0, a_1, \ldots, a_k, x]=\frac{p_k x+p_{k-1}}{q_k x+q_{k-1}}$, so

$$x=\frac{p_k x+p_{k-1}}{q_k x+q_{k-1}} \Rightarrow q_k x^2+(q_{k-1}-p_k)x-p_{k-1}=0$$

$$\Delta=(q_{k-1}-p_k)^2+4q_k p_{k-1}=q_{k-1}^2+p_k^2-2p_k q_{k-1}+4q_k p_{k-1}.$$

Recall: $q_k p_{k-1}=p_k q_{k-1}-(-1)^{k+1}=p_k q_{k-1}+(-1)^k$. Thus

$$\Delta=q_{k-1}^2+p_k^2-2p_k q_{k-1}+4p_k q_{k-1}+4(-1)^k=(q_{k-1}+p_k)^2+4(-1)^k$$

Thus $$x=\frac{(p_k-q_{k-1})\pm\sqrt{(q_{k-1}+p_k)^2+4(-1)^k}}{2q_k}$$

Since $x>0$, we must have $+$ in the above formula so $x=\frac{A+\sqrt{D}}{C}$. Since x is irrational, D is not a square

$\boxtimes$

Thm: If $x=[m_0,m_1,\dots,m_s,a_0,a_1,\dots,a_k,a_0,a_1,\dots,a_k,\dots]$ and $y=[a_0,a_1,\dots,a_k,a_0,a_1,\dots,a_k,\dots]=\frac{A+\sqrt{D}}{C}$ then $x=\frac{A_1+B_1\sqrt{D}}{C_1}$ for some integers A_1,B_1,C_1.

Proof: $x=[m_0,m_1,\dots,m_s,y]=\frac{p_s y+p_{s-1}}{q_s y+q_{s-1}}$ (same reasoning as in the last proof).

$$=\frac{p_s\frac{A+\sqrt{D}}{C}+p_{s-1}}{q_s\frac{A+\sqrt{D}}{C}+q_{s-1}}=\frac{A_1+B_1\sqrt{D}}{C_1}$$ (Exercise: Justify the last equality).

Examples: Period 1:

$x=[a,a,a,\dots]=[a,x]$ so $x=a+\frac{1}{x}$, i.e. $x^2-ax-1=0$

Thus $x=\frac{a\pm\sqrt{a^2+4}}{2}$ and since $x>0$, $x=\frac{a+\sqrt{a^2+4}}{2}$

$$\boxed{[a,a,a,\dots]=\frac{a+\sqrt{a^2+4}}{2}}$$

Taking $a=2k$ we get $[2k,2k,\dots]=k+\sqrt{k^2+1}$, so

$$\boxed{\sqrt{k^2+1}=[k,2k,2k,2k,\dots]}$$

Period 2: $x = [a, b, a, b, \dots] = [a, b, x]$, so

$$x = a + \cfrac{1}{b + \cfrac{1}{x}} = a + \frac{x}{bx+1} = \frac{abx + a + x}{bx+1}$$

so $bx^2 + x = abx + a + x$, i.e. $bx^2 - abx - a = 0$

$$\Delta = a^2b^2 + 4ab, \quad x = \frac{ab \pm \sqrt{a^2b^2 + 4ab}}{2b}$$

Since $x > 0$, we have $+$:

$$\boxed{[a, b, a, b, \dots] = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2b}}$$

Take $a = 2k$, $b = 1$, so $[2k, 1, 2k, 1, \dots] = \dfrac{2k + \sqrt{4k^2 + 8k}}{2} = k + \sqrt{k^2 + 2k} =$
$= k + \sqrt{(k+1)^2 - 1}$

Thus $\boxed{[k, 1, 2k, 1, 2k, \dots] = \sqrt{(k+1)^2 - 1}}$

Now take $b = k$, $a = 2k$, so $[2k, k, 2k, k, \dots] = \dfrac{2k^2 + \sqrt{4k^4 + 8k^2}}{2k} =$
$= k + \sqrt{k^2 + 2}$

so $\boxed{[k, k, 2k, k, 2k, \dots] = \sqrt{k^2 + 2}}$

<u>Thm:</u> If x satisfies $Ax^2 + Bx + C = 0$ for some integers A, B, C s.t. $B^2 - 4AC$ is not a square then x has periodic continued fraction.

Before proving the result look at an example.

<u>Example:</u> $\sqrt{7}$ satisfies $(\sqrt{7})^2-7=0$.

Let $\alpha_0=\sqrt{7}$, $\alpha_{k+1}=\frac{1}{\alpha_k-\lfloor\alpha_k\rfloor}$. Then $\sqrt{7}=[m_0,m_1,\ldots]$, $m_i=\lfloor\alpha_i\rfloor$.

Note: $\alpha_k=[m_k,m_{k+1},\ldots]$. So (m_k) is periodic if and only if $\alpha_k=\alpha_l$ for some $k<l$.

$$\alpha_0=\sqrt{7},\quad m_0=\lfloor\alpha_0\rfloor=2$$

$$\alpha_1=\frac{1}{\sqrt{7}-2}=\frac{\sqrt{7}+2}{7-4}=\frac{\sqrt{7}+2}{3},\quad m_1=\lfloor\alpha_1\rfloor=1$$

$$\alpha_2=\frac{1}{\frac{\sqrt{7}+2}{3}-1}=\frac{3}{\sqrt{7}-1}=\frac{3(\sqrt{7}+1)}{6}=\frac{\sqrt{7}+1}{2},\quad m_2=\lfloor\alpha_2\rfloor=1$$

$$\alpha_3=\frac{1}{\frac{\sqrt{7}+1}{2}-1}=\frac{2}{\sqrt{7}-1}=\frac{2(\sqrt{7}+1)}{7-1}=\frac{\sqrt{7}+1}{3},\quad \lfloor\alpha_3\rfloor=1$$

$$\alpha_4=\frac{1}{\frac{\sqrt{7}+1}{3}-1}=\frac{3}{\sqrt{7}-2}=\frac{3(\sqrt{7}+2)}{7-4}=\sqrt{7}+2,\quad \lfloor\alpha_4\rfloor=4$$

$$\alpha_5=\frac{1}{(\sqrt{7}+2)-4}=\frac{1}{\sqrt{7}-2}=\frac{\sqrt{7}+2}{7-4}=\frac{\sqrt{7}+2}{3}=\alpha_1$$

We have $\alpha_5=\alpha_1$ so the process repeats and

$$\sqrt{7}=[2,\overline{1,1,1,4}]$$

<u>Exercise:</u> Do the same for $\sqrt{13}$ to get

$$\sqrt{13}=[3,\overline{1,1,1,1,6}]$$