Last time we proved that if x has a periodic continued fraction expansion then $x = \frac{A+\sqrt{D}}{C}$ for some integers A, D, C such that D is not a square. Numbers of the above form are exactly roots of polynomials ax^2+bx+c with a, b, c integers and $b^2-4ac>0$ not a square (x is a root of $C^2x^2-2ACx+A^2-D=0$). We will prove now the converse.

<u>Theorem:</u> Real number x has a periodic simple continued fraction expansion if and only if $Ax^2+Bx+C=0$ for some integers A, B, C s.t. $\Delta = B^2-4AC$ is not a square.

<u>Proof:</u> We already proved that if x has a periodic continued fraction then x is a root of a quadratic polynomial with integer coefficients. Now we prove the converse. Suppose $Ax^2+Bx+C=0$ for some integers A, B, C s.t. B^2-4AC is not a square. This implies that x is irrational. Define $\alpha_0 = x$, $\alpha_{k+1} = \frac{1}{\alpha_k - \lfloor \alpha_k \rfloor}$ for $k=0,1,\dots$. Let $m_k = \lfloor \alpha_k \rfloor$. Then $x = [m_0, m_1, \dots]$ and $\alpha_k = [m_k, m_{k+1}, \dots]$. As we observed last time, the sequence $m_0, m_1, \dots$ is periodic if and only if $\alpha_k = \alpha_l$ for some $k < l$. In order to prove that $\alpha_k = \alpha_l$ for some $k<l$ we will show that $A_k\alpha_k^2 + B_k\alpha_k + C_k = 0$ for some integers A_k, B_k, C_k. Then we will show that the sequences

$(A_k), (B_k), (C_k)$ are bounded. Since A_k, B_k, C_k are integers, this implies that among the polynomials $A_k x^2 + B_k x + C_k$, $k = 0, 1, \ldots$ there are only finitely many different polynomials. Since each α_k is a root of one of the finitely many polynomials, we only have finitely many different numbers among $\alpha_0, \alpha_1, \ldots$. This clearly implies that $\alpha_k = \alpha_l$ for some $k < l$.

Now we justify the above claims. Note that

$$x = [m_0, m_1, \ldots, m_{k-1}, \alpha_k] \text{ for every } k \geq 1.$$

det $\begin{bmatrix} m_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} m_k & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}$, so $\frac{p_k}{q_k}$ is the k-th convergent for x. Then

$$\begin{bmatrix} m_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} m_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{k+1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_k \alpha_{k+1} + p_{k-1}, & p_k \\ q_k \alpha_{k+1} + q_{k-1}, & q_k \end{bmatrix}.$$

Thus $x = [m_0, \ldots, m_k, \alpha_{k+1}] = \dfrac{p_k \alpha_{k+1} + p_{k-1}}{q_k \alpha_{k+1} + q_{k-1}}$. From $Ax^2 + Bx + C = 0$ we get

$$A(p_k \alpha_{k+1} + p_{k-1})^2 + B(p_k \alpha_{k+1} + p_{k-1})(q_k \alpha_{k+1} + q_{k-1}) + C(q_k \alpha_{k+1} + q_{k-1})^2 = 0$$

We will use the following useful observation:

$$[x, y] \cdot \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = Ax^2 + Bxy + Cy^2.$$

Hence, we have

$$[p_k \alpha_{k+1} + p_{k-1}, q_k \alpha_{k+1} + q_{k-1}] \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix} \begin{bmatrix} p_k \alpha_{k+1} + p_{k-1} \\ q_k \alpha_{k+1} + q_{k-1} \end{bmatrix} = 0$$

Since $\begin{bmatrix} p_k\alpha_{k+1}+p_{k-1} \\ q_k\alpha_{k+1}+q_{k-1} \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} \cdot \begin{bmatrix} \alpha_{k+1} \\ 1 \end{bmatrix}$, taking transpose

we have $[p_k\alpha_{k+1}+p_{k-1}, q_k\alpha_{k+1}+q_{k-1}] = [\alpha_{k+1}, 1] \cdot \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix}$

Thus $[\alpha_{k+1}\ 1] \cdot \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} p_k, & p_{k-1} \\ q_k, & q_{k-1} \end{bmatrix} \begin{bmatrix} \alpha_{k+1} \\ 1 \end{bmatrix} = 0$

Let $\begin{bmatrix} A_k, & B_k/2 \\ B_k/2, & C_k \end{bmatrix} = \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} p_k, & p_{k-1} \\ q_k, & q_{k-1} \end{bmatrix}$ (*)

Then $[\alpha_{k+1}, 1] \begin{bmatrix} A_k & B_k/2 \\ B_k/2 & C_k \end{bmatrix} \begin{bmatrix} \alpha_{k+1} \\ 1 \end{bmatrix} = 0$, i.e.

$$A_k(\alpha_{k+1})^2 + B_k\alpha_k + C_k = 0 \quad (**)$$

Taking determinants of (*) and using $\det \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} = (-1)^{k+1}$, we get

$$A_kC_k - \frac{B_k^2}{4} = AC - \frac{B^2}{4}, \text{ i.e. } B_k^2 - 4A_kC_k = B^2 - 4AC$$

Thus all the quadratic polynomials $A_kx^2 + B_kx + C_k$ have the same discriminant B^2-4AC. Multiplying the matrices in (*) we get

$$A_k = Ap_k^2 + Bp_kq_k + Cq_k^2$$
$$B_k = 2Ap_kp_{k-1} + B(p_kq_{k-1} + q_kp_{k-1}) + 2Cq_kq_{k-1}$$
$$C_k = Ap_{k-1}^2 + Bp_{k-1}q_{k-1} + Cq_{k-1}^2 = A_{k-1}$$

We have $A_k = q_k^2\left[A\left(\frac{p_k}{q_k}\right)^2 + B\left(\frac{p_k}{q_k}\right) + C\right]$. Recall now that $Ax^2+Bx+C=0$. Since x is between consecutive convergents $\frac{p_k}{p_{k+1}}$ and $\frac{p_{k+1}}{q_{k+1}}$, we have $\left|\frac{p_k}{q_k} - x\right| < \left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \frac{1}{q_k q_{k+1}}$.

Thus: $\frac{A_k}{q_k^2} = \frac{A_k}{q_k^2} - 0 = \left[A\left(\frac{p_k}{q_k}\right)^2 + B\left(\frac{p_k}{q_k}\right) + C\right] - \left[Ax^2+Bx+C\right] =$

$= \left(\frac{p_k}{q_k} - x\right)\cdot\left[A\left(\frac{p_k}{q_k}+x\right)+B\right]$, so

$\frac{|A_k|}{q_k^2} \le \left|\frac{p_k}{q_k} - x\right| \cdot \left(|A|\cdot\left(\left|\frac{p_k}{q_k}\right| + |x|\right) + |B|\right)$. Since $\lim\limits_{k\to\infty} \frac{p_k}{q_k} = x$, the sequence $|A|\cdot\left(\left|\frac{p_k}{q_k}\right|+|x|\right)+|B|$ is convergent, hence it is bounded (as a matter of fact, $\left|\frac{p_k}{q_k}\right| \le \left|\frac{p_1}{q_1}\right| = |m_1| = m_1$) so there is $M>0$ s.t. $|A|\cdot\left(\left|\frac{p_k}{q_k}\right|+|x|\right)+|B| \le M$ for all k.

Thus $\frac{|A_k|}{q_k^2} \le M\left|\frac{p_k}{q_k} - x\right| \le \frac{M}{q_k q_{k+1}}$, i.e. $|A_k| \le \frac{q_k}{q_{k+1}} M \le M$.

(as $q_{k+1} \ge q_k$). We see that $|A_k| \le M$ for all k. Since $C_k = A_{k-1}$, we have $|C_k| \le M$ for all k. Now $B_k^2 = 4A_kC_k + B^2 - 4AC \le 4M^2 + B^2 - 4AC$ so the sequence (B_k) is bounded. We proved that the sequences (A_k), (B_k), (C_k) are bounded, which according to our earlier discussion implies that $\alpha_k = \alpha_l$ for some $k < l$. $\boxtimes$

Let $x = [m_0, m_1, m_2, \dots]$. We will show that the convergents $\frac{p_k}{q_k}$ are very good approximations to x. To make this more precise, recall first that:

$$\frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} > \dots > x > \dots \frac{p_6}{q_6} > \frac{p_4}{q_4} > \frac{p_2}{q_2} > \frac{p_0}{q_0},$$

and
$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{(-1)^{k+2}}{q_k q_{k+1}}; \quad \frac{p_{k+2}}{q_{k+2}} - \frac{p_k}{q_k} = \frac{(-1)^k m_{k+2}}{q_k q_{k+2}}$$

<u>Proposition 1</u>: If $\frac{c}{d}$ is a rational number strictly between x and $\frac{p_k}{q_k}$ (so $\frac{c}{d} \neq \frac{p_k}{q_k}$, $\frac{c}{d} \neq x$) then $d > q_{k+1}$ (<u>we assume that $d > 0$</u>).

<u>Proof</u>: Since $\frac{c}{d}$ is strictly between x and $\frac{p_k}{q_k}$, we have
$$\left|\frac{c}{d} - \frac{p_k}{q_k}\right| < \left|x - \frac{p_k}{q_k}\right| < \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| = \frac{1}{q_k q_{k+1}}$$

Also $\left|\frac{c}{d} - \frac{p_k}{q_k}\right| = \frac{|cq_k - dp_k|}{dq_k} \geq \frac{1}{dq_k}$ (a positive integer is ≥ 1)

Thus $\frac{1}{dq_k} < \frac{1}{q_k q_{k+1}}$, i.e. $d > q_{k+1}$. $\boxtimes$

<u>Proposition 2</u>: If $\frac{c}{d}$ is a rational number on the opposite side of x than $\frac{p_k}{q_k}$ and $\frac{c}{d} \neq \frac{p_{k+1}}{q_{k+1}}$ and $d \leq q_{k+2}$ then

$$|dx - c| > |p_k x - q_k|.$$

Proof: $\frac{c}{d}$ is on the same side of x as $\frac{p_{k+1}}{q_{k+1}}$. Since $\frac{c}{d} \neq \frac{p_{k+1}}{q_{k+1}}$ and $d \leq q_{k+2}$, Proposition 1 tells us that $\frac{c}{d}$ can not be between x and $\frac{c}{d}$. It follows that $\frac{p_{k+1}}{q_{k+1}}$ is between x and $\frac{c}{d}$ and therefore:

$$\left|x - \frac{c}{d}\right| > \left|\frac{p_{k+1}}{q_{k+1}} - \frac{c}{d}\right| = \frac{|p_{k+1}d - cq_{k+1}|}{dq_{k+1}} \geq \frac{1}{dq_{k+1}}$$

i.e. $|dx - c| > \frac{1}{q_{k+1}}$.

On the other hand, $\left|x - \frac{p_k}{q_k}\right| < \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| = \frac{1}{q_k q_{k+1}}$, so $\frac{1}{q_{k+1}} > |q_k x - p_k|$. Thus $|dx - c| > \frac{1}{q_{k+1}} > |q_k x - p_k|$. ☒

Corollary 1: $|q_{k-1}x - p_{k-1}| > |q_k x - p_k|$ for all k.

Proof: This follows from proposition 2 with $\frac{c}{d} = \frac{p_{k-1}}{q_{k-1}}$. ☒

Theorem: For any rational number $\frac{c}{d}$ such that $d \leq q_{k+1}$ and $\frac{c}{d} \neq \frac{p_k}{q_k}$, $\frac{c}{d} \neq \frac{p_{k+1}}{q_{k+1}}$ we have $|dx - c| > |q_k x - p_k|$.

Pf: If $\frac{c}{d}$ is on opposite side of x than $\frac{p_k}{q_k}$, the Theorem follows from Proposition 2. If $\frac{c}{d}$ is on the same side of x as $\frac{p_k}{q_k}$ then $\frac{p_{k-1}}{q_{k-1}}$ and $\frac{c}{d}$ are on opposite sides of x and Proposition 2 tells us that $|dx - c| > |q_{k-1}x + p_{k-1}|$. The theorem in this case follows now from Corollary 1. ☒

Theorem 1 leads to the following curious observation. For any integer $d \geq 1$ define

$$F(d) = \min\{|dx-c| : c \in \mathbb{Z}\} = \text{distance from } dx \text{ to the closest integer.}$$

From Corollary 1, we have $F(q_0) > F(q_1) > \dots$. By Theorem 1, we have $F(d) > F(q_n)$ for all $d < q_{n+1}$, $d \neq q_n$.

Theorem 1 leads to the following useful characterization of the convergents $\frac{p_k}{q_k}$, $k = 0,1,2,3,\dots$. First a definition.

<u>Definition</u>: A fraction $\frac{c}{d}$ $(d>0)$ is called a best approximation to x if for any fraction $\frac{a}{b}$ with $0 < b \leq d$ we have $|bx-a| > |dx-c|$ unless $a=c$, $b=d$

<u>Note</u>: if $\frac{c}{d}$ is a best approximation to x then

- $\gcd(c,d)=1$ (if $n|c$ and $n|d$ then $c=c_1 n$, $d=d_1 n$ and $|d_1 x - c_1| = \frac{1}{n}|dx-c|$; if $n>1$ then $|d_1 x - c_1| < |dx-c|$, which is not possible
- $|x-\frac{c}{d}| < |x-\frac{a}{b}|$ for any $\frac{a}{b} \neq \frac{c}{d}$ with $0 < b \leq d$.

<u>Theorem 2</u>: A fraction $\frac{c}{d}$ is a best approximation to x if and only if $c=p_k$, $d=q_k$ for some convergent $\frac{p_k}{q_k}$ to x.

<u>Proof</u>: If $c=p_k$, $d=q_k$ then for any $\frac{a}{b}$ with $b<q_{k+1}$ we have $|bx-a| > |q_k x - p_k|$ unless $\frac{a}{b} = \frac{p_k}{q_k}$ (note that $\frac{a}{b} \neq \frac{p_{k+1}}{q_{k+1}}$ since $b<q_{k+1}$). Thus if $b \leq q_k$ and $\frac{a}{b} \neq \frac{p_k}{q_k}$ then $|bx-a| > |q_k x - p_k|$. If $\frac{a}{b} = \frac{p_k}{q_k}$ and $b \leq q_k$ then $a=p_k$, $b=q_k$, since $\gcd(p_k,q_k)=1$. Thus $\frac{p_k}{q_k}$ is a best approximation to x.

Conversely, let $\frac{c}{d}$ be a best approximation to x. There is k such that $q_k \leq d < q_{k+1}$. If $\frac{c}{d} \neq \frac{p_k}{q_k}$ then $|dx-c| > |q_k x - p_k|$ by Theorem 1. On the other hand, since $\frac{c}{d}$ is a best approximation to x and $q_k \le d$ and $\frac{c}{d} \neq \frac{p_k}{q_k}$, ~~then~~ we have $|dx-c| \not< |q_k x - p_k|$, a contradiction. Thus $\frac{c}{d} = \frac{p_k}{q_k}$. Since $\gcd(c,d)=1$, we get $c=p_k$, $d=q_k$. $\boxtimes$

Another application of Theorem 1 is the following

<u>Theorem 3</u>: If x is an irrational number and $|x-\frac{a}{b}| \le \frac{1}{2b^2}$ for some rational number $\frac{a}{b}$ with $b>0$ and $\gcd(a,b)=1$, then $a=p_k$, $b=q_k$ for some convergent $\frac{p_k}{q_k}$ to x.

<u>Proof</u>. There is k s.t. $q_k \le b < q_{k+1}$. Suppose that $\frac{a}{b} \neq \frac{p_k}{q_k}$. Then $|\frac{a}{b} - \frac{p_k}{q_k}| = \frac{|aq_k - bp_k|}{bq_k} \geq \frac{1}{bq_k}$. Furthermore, from Theorem 1, we have $|bx-a| > |q_k x - p_k|$. Thus

$|q_k x - p_k| < |bx-a| = b \cdot |x-\frac{a}{b}| \le b \cdot \frac{1}{2b^2} = \frac{1}{2b}$, so $|x - \frac{p_k}{q_k}| < \frac{1}{2bq_k}$. It follows that

$$\frac{1}{bq_k} \le \left|\frac{a}{b} - \frac{p_k}{q_k}\right| = \left|\left(\frac{a}{b}-x\right) - \left(\frac{p_k}{q_k}-x\right)\right| \le \left|\frac{a}{b}-x\right| + \left|\frac{p_k}{q_k}-x\right| < \frac{1}{2b^2} + \frac{1}{2bq_k}.$$

This implies that $\frac{1}{2b^2} > \frac{1}{2bq_k}$ so $q_k > b$, a contradiction.

Since the assumption $\frac{a}{b} \neq \frac{p_k}{q_k}$ leads to a contradiction, we have $\frac{a}{b} = \frac{p_k}{q_k}$. Since $\gcd(a,b)=1$, we get $a=p_k$, $b=q_k$. $\boxtimes$