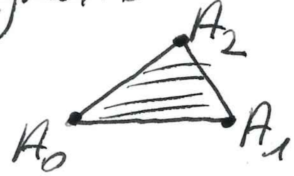
__Definition:__ We say that points $P_0, P_1, \dots, P_k$ in $\mathbb{R}^n$ are __in general position__ if the vectors $\overrightarrow{P_0P_1}, \dots, \overrightarrow{P_0P_k}$ are linearly independent (recall: if $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_n)$ then $\overrightarrow{AB}=(b_1-a_1,\dots,b_n-a_n)$).

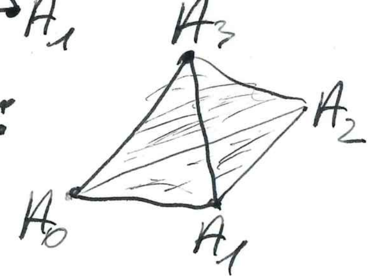
__Excerise:__ If $P_0, P_1, \dots, P_k$ are in general position the for any i the vectors $\overrightarrow{P_iP_j}$, $j=1,\dots,k$, $j\neq i$ are linearly independent so this defintion does not depend on how the points are listed.

Given points $A_0, A_1, \dots, A_k$ in $\mathbb{R}^n$ which are in general position. we define $\Delta(A_0, A_1, \dots, A_k)=\{t_0A_0+t_1A_1+\dots+t_kA_k : t_i\in[0,1]$ and $t_0+t_1+\dots+t_k=1\}$. Any set of the form $\Delta(A_0,A_1,\dots,A_k)$ is called a __k-dimensional simplex__.

__Ex:__ 0-dimensional sinlex is just a point 

1-dimensional simplex is just a segment: 

2-dimensional simplex is a triangle: 

3-dimensional simplex is a tetrahedron: 

__etc.__

__Lemma:__ Any point $P\in\Delta(A_0,A_1,\dots,A_k)$ has unique expression as $t_0A_0+t_1A_1+\dots+t_kA_k$.

__Pf:__ Note that $t_0A_0+t_1A_1+\dots+t_kA_k = t_1\overrightarrow{A_0A_1}+t_2\overrightarrow{A_0A_2}+\dots+t_k\overrightarrow{A_0A_k}+t_0A_0+(t_1+t_2+\dots+t_k)A_0 = t_1\overrightarrow{A_0A_1}+\dots+t_k\overrightarrow{A_0A_k} = A_0+t_1\overrightarrow{A_0A_1}+\dots+t_k\overrightarrow{A_0A_k}$.

Thus if $t_0 A_0 + \dots + t_k A_k = s_0 A_0 + \dots + s_k A_k$ then $t_1 \overrightarrow{A_0A_1} + \dots + t_k \overrightarrow{A_0A_k} = s_1 \overrightarrow{A_0A_1} + \dots + s_k \overrightarrow{A_0A_k}$. By linear independence of $\overrightarrow{A_0A_1}, \dots, \overrightarrow{A_0A_k}$ we conclude that $t_1 = s_1, t_2 = s_2, \dots, t_k = s_k$ so $t_0 = 1-(t_1 + \dots + t_k) = 1-(s_1 + \dots + s_k) = s_0$. $\boxtimes$

<u>Def</u>: The unique numbers $t_0, \dots, t_k$ s.t. $t_i \in [0,1]$, $t_0 + \dots + t_k = 1$ and $P = t_0 A_0 + \dots + t_k A_k$ are called the <u>barycentric coordinates</u> of P.

<u>Remark</u>: As a subset of $\mathbb{R}^n$, $\Delta(A_0, \dots, A_k)$ is a convex, closed, bounded subset of $\mathbb{R}^n$, hence it is compact.

$\Delta(A_0, \dots, A_k)$ is the smallest convex set which contains $A_0, \dots, A_k$.

The points $A_0, \dots, A_k$ are called <u>vertices</u> of $\Delta(A_0, \dots, A_k)$.

Each function $t_i : \Delta(A_0, \dots, A_k) \to [0,1]$, $t_i(P) =$ i-th barycentric coordinate of P is a continuous function.

<u>Cor</u>: Any two k-dimensional simplices are homeomorphic:

$$\begin{aligned} \Delta(A_0, \dots, A_k) &\longrightarrow \Delta(B_0, \dots, B_k) \\ \sum_{i=0}^{k} t_i A_i &\longmapsto \sum_{i=0}^{k} t_i B_i \end{aligned}$$ is a homeomorphism.

<u>Definition</u>: The set of all points P in $\Delta(A_0, \dots, A_k)$ whose all barycentric coordinates are non-zero is called the <u>interior</u> of $\Delta(A_0, \dots, A_k)$. Interior is an open subset of $\Delta(A_0, \dots, A_k)$.

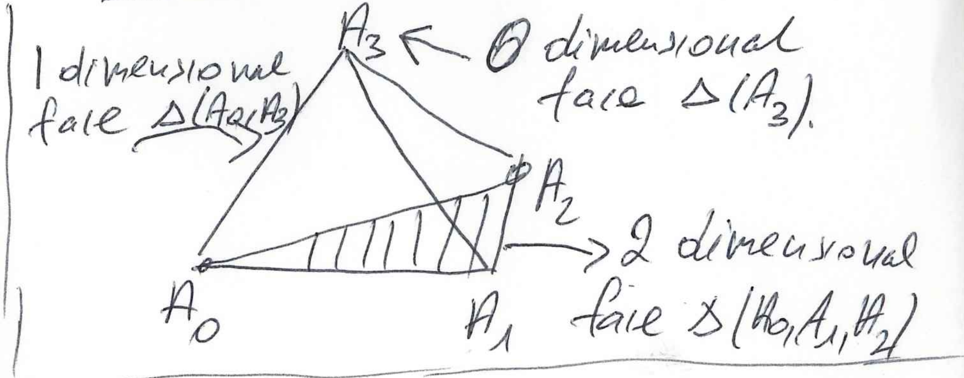
The set of all points in $\Delta(A_0, \dots, A_k)$ whose barycentric coordinate t_i is 0 is called a <u>$(k-1)$-dimensional face of $\Delta(A_0, \dots, A_k)$ opposite vertex A_i</u>.

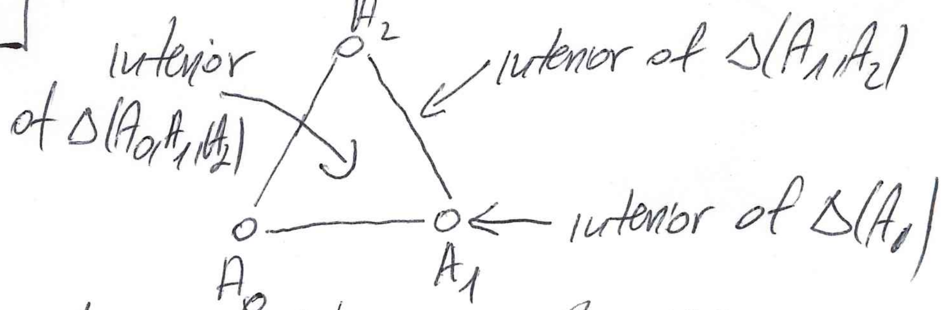
Clearly, the $k-1$ dimensional face of $\Delta(A_0,\dots,A_k)$ opposite vertex A_i is the $k-1$ dimensional simplex with vertices $A_0,\dots,\cancel{A_i},\dots,A_k$ (i.e. we remove A_i from the vertices).

In general, for any subset $I \subseteq \{0,1,\dots,k\}$ the points $P \in \Delta(A_0,\dots,A_k)$ such that $t_j(P)=0$ for all $\underline{j \notin I}$ form an $(|I|-1)$-dimensional simplex with vertices A_i, $i \in I$. Any such simplex is called a <u>face</u> of $\Delta(A_0,\dots,A_k)$.

It will be useful to observe the following:

The simplex $\Delta(A_0,\dots,A_k)$ is a disjoint union of interiors of all its faces





The following simple observation is of key importance:

<u>Observation</u>: Let $\Delta(A_0,\dots,A_k) \subseteq \mathbb{R}^n$ be a k-dimensional simplex. Given any points $B_0, B_1,\dots,B_k$ in $\mathbb{R}^m$ (for some m) the function

$$\phi: \Delta(A_1,\dots,A_k) \to \mathbb{R}^m, \quad \phi(t_0A_0+\dots+t_kA_k) = t_0B_0+\dots+t_kB_k$$ is

a continuous, linear function s.t. $\phi(A_i)=B_i$.

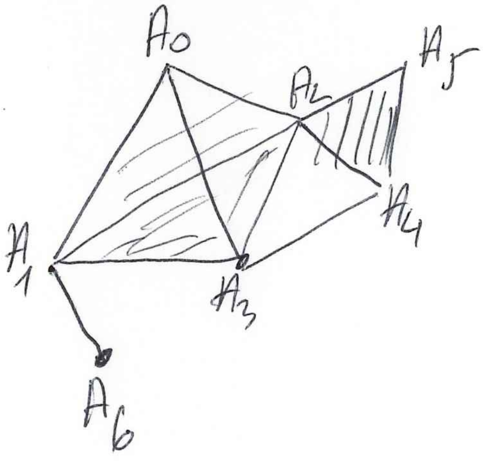
<u>Definition</u>: The <u>barycenter</u> of a simplex $\Delta(A_0,A_1,\dots,A_k)$ is the point $P = \frac{1}{k+1}A_0 + \frac{1}{k+1}A_1 + \dots + \frac{1}{k+1}A_k$.

<u>Definition</u>: A <u>simplicial complex</u> is a finite collection $\mathcal{C}$ of simplices in $\mathbb{R}^n$ (for some n) with the following properties:

1. If a simplex Δ belongs to $\mathcal{C}$ then every face of Δ is also in $\mathcal{C}$
2. If $\Delta_1, \Delta_2 \in \mathcal{C}$ then $\Delta_1 \cap \Delta_2 = \emptyset$ or $\Delta_1 \cap \Delta_2$ is a common face of Δ_1 and Δ_2.

The set $|\mathcal{C}|$ = union of all simplices in $\mathcal{C}$ is called the <u>realization of $\mathcal{C}$</u>. $|\mathcal{C}|$ is a compact topological space (in the induced topology from $\mathbb{R}^n$). Intuitively, $|\mathcal{C}|$ is obtained by "glueing" the simplices in $\mathcal{C}$ along some of their faces.

We write $V_{\mathcal{C}}$ for the finite set of all vertices of simplices in $\mathcal{C}$ (i.e. all 0-dimensional simplices in $\mathcal{C}$).



$$\begin{aligned}\mathcal{C} = \{ & \Delta(A_0,A_1,A_2,A_3), \Delta(A_0,A_1,A_2), \Delta(A_0,A_2,A_3), \\ & \Delta(A_0,A_1,A_3), \Delta(A_1,A_2,A_3), \Delta(A_0,A_1), \\ & \Delta(A_0,A_2), \Delta(A_0,A_3), \Delta(A_1,A_2), \Delta(A_1,A_3), \Delta(A_2,A_3), \\ & \Delta(A_0), \Delta(A_1), \Delta(A_2), \Delta(A_3), \Delta(A_2,A_4,A_5), \Delta(A_2,A_4), \\ & \Delta(A_2,A_5), \Delta(A_4,A_5), \Delta(A_4), \Delta(A_5), \Delta(A_3,A_4) \\ & \Delta(A_1,A_6), \Delta(A_6) \}\end{aligned}$$

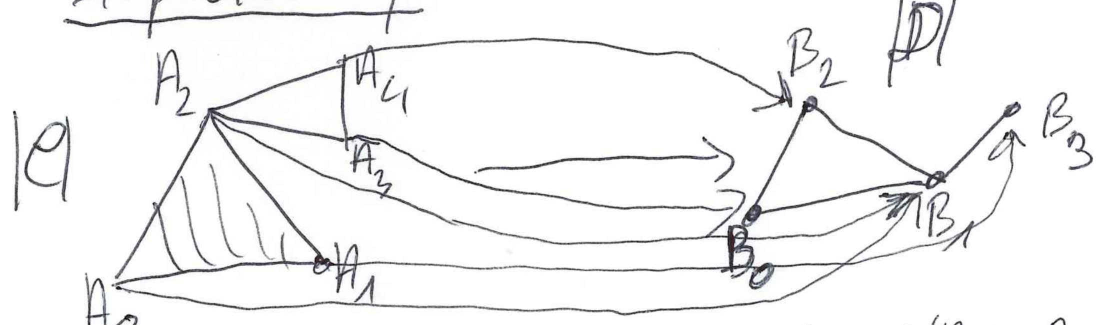
$$V_{\mathcal{C}} = \{A_0, A_1, A_2, A_3, A_4, A_5, A_6\}$$

We can extend our observation ~~about~~ about simplices to simplicial complexes as follows:

If $\mathcal{C}$ is a simplicial complex with vertices $V_{\mathcal{C}}$ then any function $\varphi: V_{\mathcal{C}} \to \mathbb{R}^m$ determines unique continuous function $\varphi: |\mathcal{C}| \to \mathbb{R}^m$ which is linear on each simplex in $\mathcal{C}$ (so φ is piecewise-linear) as follows: if $P \in |\mathcal{C}|$ then $P \in \Delta(A_0, \dots, A_k)$ for some simplex $\Delta(A_0, \dots, A_k) \in \mathcal{C}$, so $P = t_0 A_0 + \dots + t_k A_k$, t_i - barycentric coordinates of P; define

$$\varphi(P) = t_0 \varphi(A_0) + \dots + t_k \varphi(A_k).$$

Suppose now that $\mathcal{D}$ is a simplicial complex in $\mathbb{R}^m$ and $\varphi: V_{\mathcal{C}} \to V_{\mathcal{D}}$ is a function with the following property: if $\Delta(A_{i_0}, A_{i_1}, \dots, A_{i_k}) \in \mathcal{C}$ then $\{\varphi(A_{i_0}), \dots, \varphi(A_{i_k})\}$ are vertices of a simplex in $\mathcal{C}$. Then, by the remark above, the function $\varphi: |\mathcal{C}| \to \mathbb{R}^m$ maps $|\mathcal{C}|$ into $|\mathcal{D}|$, hence we get a continuous function $\varphi: |\mathcal{C}| \to |\mathcal{D}|$. Any such function is called a <u>simplicial map</u>.



$\varphi(A_0) = B_1, \varphi(A_1) = B_3, \varphi(A_2) = B_1, \varphi(A_3) = B_0, \varphi(A_4) = B_2$

φ extends to continuous piecewise-linear map $|\mathcal{C}| \to |\mathcal{D}|$.

Note that a simplicial map $\varphi : |\mathcal{C}| \to |\mathcal{D}|$ maps any simplex Δ in $\mathcal{C}$ onto a simplex in $\mathcal{D}$ of dimension smaller or equal to dimension of Δ.

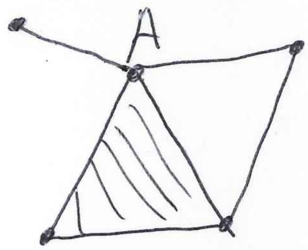
<u>Definition</u>: The <u>dimension</u> of a simplicial complex $\mathcal{C}$ is the largest dimension of the simplices in $\mathcal{C}$.

<u>Corollary</u>: If $\varphi : |\mathcal{C}| \to |\mathcal{D}|$ is simplicial and $\dim \mathcal{C} < \dim \mathcal{D}$ Then φ is not surjective.

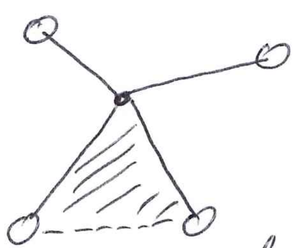
In fact if $\Delta \in \mathcal{D}$ and $\dim \Delta > \dim \mathcal{C}$ then no point in the interior of Δ is in the image of φ.

Let $\mathcal{C}$ be a simplicial complex and $A \in V_{\mathcal{C}}$ a vertex of $\mathcal{C}$.

<u>Definition</u>: The <u>star $St(A)$</u> of the vertex A is the union of interiors of all simplices with A as a vertex.



$St(A) =$



Note that $St(A)$ is an open subset of $|\mathcal{C}|$ and the sets $\{St(A) : A \in V_{\mathcal{C}}\}$ cover $|\mathcal{C}|$.

<u>Remark</u>: As in the case of a simplex the realization $|\mathcal{C}|$ is the disjoint union of the interiors of simplices in $\mathcal{C}$.

<u>Thm</u>: Let $A_0, A_1, \dots, A_k \in V_{\mathcal{C}}$. Then $\Delta(A_0, \dots, A_k) \in \mathcal{C}$ if and only if $St(A_0) \cap \dots \cap St(A_k) \neq \emptyset$.

<u>Pf</u>: If $\Delta(A_0, \dots, A_k) \in \mathcal{C}$ then interior of $\Delta(A_0, \dots, A_k)$ belongs to $St(A_i)$ for every i, hence $St(A_0) \cap \dots \cap St(A_k) \neq \emptyset$. Conversely, suppose

$P \in St(A_0) \cap \dots \cap St(A_k)$. There is unique simplex Δ in $\mathcal{C}$ such that $P \in$ interior of Δ. Thus interior of Δ belongs to $St(A_i)$ for all A_i, i.e. each A_i is a vertex of Δ. It follows that $\Delta(A_0, \dots, A_k)$ is a face of Δ, so $\Delta(A_0, \dots, A_k) \in \mathcal{C}$. $\boxtimes$

We make now several key observations:

(A) If $f, g : |\mathcal{C}| \to |\mathcal{D}|$ are continuous functions (here $\mathcal{C}$ is a simplicial complex in $\mathbb{R}^n$, $\mathcal{D}$ a simplicial complex in $\mathbb{R}^m$) such that for every $P \in |\mathcal{C}|$, $f(P)$ and $g(P)$ belong to the same simplex in $\mathcal{D}$, then $f \sim g$ (f, g are homotopic).

Pf. The function $H : |\mathcal{C}| \times I \to \mathbb{R}^m$, $H(P,t) = (1-t)f(P) + tg(P)$ is continuous. Since $f(P), g(P)$ are in the same simplex of $\mathcal{D}$, $(1-t)f(P) + tg(P)$ stays in that simplex for all $t \in [0,1]$. Thus H takes values in $|\mathcal{D}|$, i.e. $H : |\mathcal{C}| \times I \to |\mathcal{D}|$ is a homotopy between f and g.

(B) Suppose $f : |\mathcal{C}| \to |\mathcal{D}|$ is a continuous function and for every vertex $A \in V_{\mathcal{C}}$ the image $f(St(A))$ is contained in $St(B)$ for some $B \in V_{\mathcal{D}}$. Then f is homotopic to a simplicial map $\varphi : |\mathcal{C}| \to |\mathcal{D}|$.

Proof: Define $\varphi : V_{\mathcal{C}} \to V_{\mathcal{D}}$ as follows: for $A \in V_{\mathcal{C}}$ there is $B \in V_{\mathcal{D}}$ such that $f(St(A)) \subseteq St(B)$. Set $\varphi(A) = B$ (there may be several such B; just pick one). Suppose $\Delta(A_{i_0}, \dots, A_{i_k}) \in \mathcal{C}$. Then $St(A_{i_0}) \cap \dots \cap St(A_{i_k}) \neq \emptyset$. Since $f(St(A_i)) \subseteq St(\varphi(A_i))$ for every i, we conclude that

~~$\varphi(St(\varphi(A_{i_0})) \cap \varphi(St(\varphi(A_{i_1})) \cap \dots \cap \varphi($~~

$St(\varphi(A_{i_0})) \cap St(\varphi(A_{i_1})) \cap \dots \cap St(\varphi(A_{i_k})) \neq \emptyset$. By the Thm on page 6, the set $\{\varphi(A_{i_0}), \dots, \varphi(A_{i_k})\}$ consists of all vertices of a simplex in $\mathcal{D}$. This property implies that φ extends to a simplicial map $\varphi: |\mathcal{C}| \to |\mathcal{D}|$. If $P \in |\mathcal{C}|$ then P belongs to the interior of a unique simplex $\Delta(A_{j_0}, \dots, A_{j_k})$ of $\mathcal{C}$. Thus $P \in St(A_{j_0}) \cap \dots \cap St(A_{j_k})$. Since $f(St(A_j)) \subseteq St(\varphi(A_j))$, we see that $f(P) \in St(\varphi(A_{j_0})) \cap \dots \cap St(\varphi(A_{j_k}))$. This means that the unique simplex Δ in $\mathcal{D}$ whose interior contains $f(P)$ has each of the points $\varphi(A_{j_0}), \dots, \varphi(A_{j_k})$ as a vertex. Also, by the definition of φ,

$\varphi(P) \in \varphi(\Delta(A_{j_0}, \dots, A_{j_k})) =$ simplex with vertices $\varphi(A_{j_0}), \dots, \varphi(A_{j_k}) \subseteq \Delta$

Thus $f(P)$ and $\varphi(P)$ are both in Δ. By observation (A) we get $f \sim \varphi$. $\boxtimes$

(C) <u>Definition</u>: A <u>diameter</u> of a bounded subset of $\mathbb{R}^n$ is the supremum of all the distances between any 2 points in the subset.

<u>Easy fact</u>: The diameter of a simplex is the largest distance between its vertices.

<u>Exercise</u>: The distance from a barycenter of $\Delta(A_0, \dots, A_k)$ to any other point in the simplex is no longer than $\frac{k}{k+1} \cdot$ diameter of $\Delta(A_0, \dots, A_k)$.

Given a simplicial complex $\mathcal{C}$ in $\mathbb{R}^n$ whose every simplex has diameter bounded above by b, we can subdivide each simplex in $\mathcal{C}$ into finer simplices to create a simplicial complex $\mathcal{C}_1$ such that $|\mathcal{C}| = |\mathcal{C}_1|$ and every simplex in $\mathcal{C}_1$ has diameter $\leq \frac{n}{n+1} b$.

Pf: The idea is simple: for every simlex $\Delta \in \mathcal{C}$ let A_Δ be the barycenter of Δ. For every sequence $\Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_k$ in $\mathcal{C}$ consider a simplex $\Delta(A_{\Delta_0}, \dots, A_{\Delta_k})$. The collection of all such simplices forms a simplicial complex $\mathcal{C}_1$ such that $|\mathcal{C}| = |\mathcal{C}_1|$.

$\mathcal{C}_1$ is called a <u>barycentric subdivision of $\mathcal{C}$</u>. By our excersie, diameter of $\Delta(A_{\Delta_0}, \ldots, A_{\Delta_K}) \le \frac{m}{m+1}$ diameter of Δ_K, where m is the dimension of Δ_K. Since $m \le n$, we get diameter of $\mathcal{C}_1 \le \frac{n}{n+1}$ · diameter of $\mathcal{C}$. We leave the details as an excercise.

<u>Corollary</u>: For any $\varepsilon > 0$ there is simplicial complex $\mathcal{C}_\varepsilon$ such that $|\mathcal{C}| = |\mathcal{C}_\varepsilon|$ and diameter of $\mathcal{C}_\varepsilon < \varepsilon$.

Just apply the barycentric subdivision many times and note that $\left(\frac{n}{n+1}\right)^N$ tends to 0 when $N \to \infty$.

(D) Given simplicial complexes $\mathcal{C}$ and $\mathcal{D}$ and a continuous function $f: |\mathcal{C}| \to |\mathcal{D}|$ there is a simplicial complex $\mathcal{C}_1$ such that $|\mathcal{C}| = |\mathcal{C}_1|$ and a simplicial map $\varphi: |\mathcal{C}_1| \to |\mathcal{D}|$ such that $f \sim \varphi$.

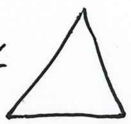
<u>Pf</u>: The sets $St(B_i)$, $B_i \in V_{\mathcal{D}}$ form an open cover of $|\mathcal{D}|$. Then $\{f^{-1}(St(B_i)) : B_i \in V_{\mathcal{D}}\}$ is an open cover of $|\mathcal{C}|$. Since $|\mathcal{C}|$ is compact, take Lebegue number ε of this cover. It follows that any subset of $|\mathcal{C}|$ of diameter $< \varepsilon$ is contained in one of the sets $f^{-1}(St(B_i))$. Take $\mathcal{C}_1$ such that $|\mathcal{C}_1| = |\mathcal{C}|$ and every simplex in $\mathcal{C}_1$ has diameter $< \varepsilon$. Then for any vertex A of $\mathcal{C}_1$ the set $St(A)$ is contained in the ball with center A and radius ε. Thus $St(A) \subseteq f^{-1}(St(B_i))$ for some i, ie. $f(St(A)) \subseteq St(B_i)$. By part (B) there is a simplicial map $\varphi: |\mathcal{C}_1| = |\mathcal{C}| \to |\mathcal{D}|$ such that $\varphi \sim f$. ☒

<u>Cor</u>: If dimension of $\mathcal{C}$ < dimension of $\mathcal{D}$, then any continuous map $f: |\mathcal{C}| \to |\mathcal{D}|$ is homotopic to a function which is not surjective.

Proof: This follows by the corollary on page 6.

Thm: If $n < m$ then any continuous function $f: S^n \to S^m$ is homotopic to a constant function.

Pf: S^n is ~~homotop~~ homeomorphic to the boundary of an n-dimensional simplex: $S^n \simeq \Delta(A_0, \dots, A_n) -$ interior of $\Delta(A_0, \dots, A_n)$.

$S^1 \simeq$  , $S^2 \simeq$ 

$S^n =$ union of all the faces of $\Delta(A_0, \dots, A_n)$ of dimension $< n$.

This is a simplicial complex. As simplicial complexes, S^n has dimension n. Thus by the last corollary, $f: S^n \to S^m$ is homotopic to some φ which is not surjective. We know that S^m without a point is homeomorphic to $\mathbb{R}^m$, which is contractible. Thus φ is homotopic to a constant map.