Recall some key results from last time:

① If $\mathcal{C}, \mathcal{D}$ are simplicial complexes and $f, g: |\mathcal{C}| \to |\mathcal{D}|$ are two continuous functions such that for every point $P \in |\mathcal{C}|$, the points $f(P)$ and $g(P)$ belong to the same simplex in $\mathcal{D}$. Then $f \sim g$ (f, g are homotopic).

② A continuous function $\varphi: |\mathcal{C}| \to |\mathcal{D}|$ is *simplicial* if for any simplex $\Delta \in \mathcal{C}$, φ maps Δ linearly onto a simplex in $\mathcal{D}$. Given any continuous function $f: |\mathcal{C}| \to |\mathcal{D}|$ there is a simplicial set $\mathcal{C}_1$ (obtained from $\mathcal{C}$ by dividing its simplices into smaller simplices) such that $|\mathcal{C}_1| = |\mathcal{C}|$ and a simplicial map $\varphi: |\mathcal{C}_1| \to |\mathcal{D}|$ such that $f \sim \varphi$ (f is homotopic to a simplicial map with respect to a complex obtained from $\mathcal{C}$ by dividing its simplices into smaller simplices)

③ If $\dim \mathcal{C} < \dim \mathcal{D}$ then no simplicial map $\varphi: |\mathcal{C}| \to |\mathcal{D}|$ can be surjective. Thus any continuous map $f: |\mathcal{C}| \to |\mathcal{D}|$ is homotopic to a continuous map which is not surjective.

④ If $n < m$ then any continuous map $f: S^n \to S^m$ is homotopic to a constant function.

Pf: $S^{n-1} \cong |\mathcal{C}_n|$, where $\mathcal{C}_n$ is a simplicial complex which consists of all faces of $\Delta(A_0, \dots, A_n)$ of dimension less than n (i.e. all faces except the n-dimensional face).

Since $\dim \mathcal{C}_n = n-1$, f is homotopic to some $g: S^n \to S^m$ which is not surjective, i.e. $g: S^n \to S^m - \{P\}$ for some point P. Since $S^m - \{P\} \cong \mathbb{R}^m$ is contractible, g is homotopic to a constant function. ∎

<u>Theorem (Sperner's Lemma)</u>: Let $\Delta = \Delta(A_0, \dots, A_n)$ be an n-dimensional simplex and let $\mathcal{C}$ be a simplicial complex such that $|\mathcal{C}| = \Delta$. Suppose that to every vertex P of $\mathcal{C}$ we assign a number $v(P) \in \{0, 1, \dots, n\}$ such that if P belongs to a face $\Delta(A_{i_0}, A_{i_1}, \dots, A_{i_s})$ of Δ then $v(P) \in \{i_0, \dots, i_s\}$ (the number assigned to P is equal to the number of one of the vertices of the face of Δ which contains P in its interior). Then there is an n-dimensional simplex in $\mathcal{C}$ whose vertices have assigned every of the numbers $0, 1, \dots n$.

<u>Proof</u>: We will prove that the number of n-dimensional simplices in $\mathcal{C}$ whose vertices have assigned every of the numbers $0, 1, \dots, n$ is <u>odd</u> (so it is $\neq 0$) using induction on n.

For $n=0$ the result is clear, as $\Delta = \Delta(A_0)$ is just a point. Suppose the result holds for simplices of dimension less than n. Consider an n-dimensional simplex $\Delta = \Delta(A_0, \dots, A_n)$, a simplicial complex $\mathcal{C}$ s.t. $|\mathcal{C}| = \Delta$ and an assignment $v : V_{\mathcal{C}} \longrightarrow \{0, 1, \dots, n\}$ as in the lemma (here $V_{\mathcal{C}}$ is the set of all vertices of $\mathcal{C}$).

Since $A_i \in \Delta(A_i)$, we have $v(A_i) = i$.

Let $\mathcal{F}$ be the collection of all $(n-1)$-dimensional simplices in $\mathcal{C}$ whose vertices have assigned every number $0, 1, \dots, n-1$. Let $\mathcal{F}^*$ be the subset of $\mathcal{F}$ of those simplices which are contained in $\Delta(A_0, \dots, A_{n-1})$. All the simplices in $\mathcal{C}$ which are contained in $\Delta(A_0, \dots, A_{n-1})$ form a simplicial complex $\mathcal{C}^*$ s.t. $|\mathcal{C}^*| = \Delta(A_0, \dots, A_{n-1})$. The restriction of v to the vertices of $\mathcal{C}^*$ satisfies the assumptions of Sperner's Lemma. By the inductive assumption, $|\mathcal{F}^*|$ is odd.

List now all n-dimensional simplices in $\mathcal{C}$ in some order: $\Delta_1, \dots, \Delta_N$. Let r be the number of such simplices whose vertices have assigned each of the numbers $0, 1, \dots, n$. Our goal is to show that r is odd. Let K_i be the set of all numbers assigned to vertices of Δ_i and let r_i be the number of $n-1$ dimensional faces of Δ_i whose vertices have assigned all the numbers $0, 1, \dots, n-1$ (i.e. number of faces which belong to $\mathcal{F}$). Note that:

(i) if $K_i = \{0, 1, \dots, n\}$ then $r_i = 1$

(ii) if $K_i = \{0, 1, \dots, n-1\}$ then $r_i = 2$

(iii) if $\{0, 1, \dots, n-1\}$ is not contained in K_i then $r_i = 0$.

Since r = the number of Δ_i s.t. $K_i = \{0, 1, \dots, n\}$, we see that

$$r \equiv r_1 + \dots + r_N \pmod 2$$

On the other hand, each $(n-1)$-dimensional simplex in $\mathcal{F}^*$ belongs to exactly one of the simplices $\Delta_1, \dots, \Delta_N$ and each $(n-1)$-dimensional simplex in $\mathcal{F} - \mathcal{F}^*$ belongs to exactly 2 of $\Delta_1, \dots, \Delta_N$.

It follows that in the sum $r_1 + \dots + r_N$ each simplex in $\mathcal{F}^*$ is counted 1 time and each simplex in $\mathcal{F} - \mathcal{F}^*$ is counted twice.

Thus $r_1 + \dots + r_N \equiv |\mathcal{F}^*| + 2|\mathcal{F} - \mathcal{F}^*| \equiv |\mathcal{F}^*| \pmod 2$.

We conclude that $r \equiv |\mathcal{F}^*| \pmod 2$. Since $|\mathcal{F}^*|$ is odd, r is odd as well. $\boxtimes$

Theorem (Brouwer's fixed point theorem): Let $f: \Delta \to \Delta$ be a continuous function, where $\Delta = \Delta(A_0, \ldots, A_n)$ is an n-simplex. Then f has a fixed point.

Proof: Suppose that f has no fixed point, i.e. $f(P) \neq P$ for all $P \in \Delta$. For $P \in \Delta$, let $t_0(P), t_1(P), \ldots, t_n(P)$ be the barycentric coordinates of P, so $P = t_0(P)A_0 + t_1(P)A_1 + \ldots + t_n(P)A_n$, $t_i(P) \in [0,1]$ and $t_0(P) + t_1(P) + \ldots + t_n(P) = 1$. Each t_i is a continuous function from Δ to $[0,1]$. Consider the sets

$$M_i = \{P \in \Delta : t_i(P) > t_i(f(P))\}, \; i = 0, 1, \ldots, n.$$

Since the function $P \mapsto t_i(P) - t_i(f(P))$ is a continuous function and M_i is the preimage of $(0, \infty)$ by this function, we see that each M_i is an open set in Δ. Suppose that there is a point $P \in \Delta$ which does not belong to any M_i, $i = 0, \ldots, n$. Then $t_i(f(P)) \leq t_i(P)$ for $i = 0, 1, \ldots, n$. Since $t_0(f(P)) + \ldots + t_n(f(P)) = t_0(P) + \ldots + t_n(P) = 1$, we conclude that $t_i(P) = t_i(f(P))$ for all i. But this means that $P = f(P)$, so P would be a fixed point of f, contrary to our assumption. It follows that every point of Δ belongs to one of the sets $M_0, \ldots, M_n$. In other words, $M_0, M_1, \ldots, M_n$ is an open cover of Δ. Let ε be a Lebesgue number of this cover, so any subset of Δ of diameter $< \varepsilon$ must be contained in one of $M_0, \ldots, M_n$. Consider now a simplicial complex $\mathcal{C}$

such that $|\mathcal{C}|=\Delta$ and each simplex in $\mathcal{C}$ has diameter $<\varepsilon$ (we know that Δ can be divided into small simplices to form $\mathcal{C}$).
For every vertex P of $\mathcal{C}$ we have at least one i such that $t_i(P)>t_i(f(P))$. Indeed, otherwise we would have $t_i(P)\leq t_i(f(P))$ for all i, which as before implies that $t_i(P)=t_i(f(P))$ for all i, hence $P=f(P)$, a contradiction. Choose any such i and call it $v(P)$. Note that if P belongs to a face $\Delta(A_{i_0},\dots,A_{i_s})$ then $t_j(P)=0$ for $j\notin\{i_0,\dots,i_s\}$. Since $t_{v(P)}(P)>t_{v(P)}(f(P))$, we have $t_{v(P)}(P)\neq 0$, hence $v(P)\in\{i_0,\dots,i_s\}$. It follows that v satisfies the assumptions of Sperner's Lemma. Thus there is an n-dimensional simplex $\bar{\Delta}$ in $\mathcal{C}$ whose vertices have assigned all numbers $0,1,\dots,n$. Since $\bar{\Delta}$ has diameter $<\varepsilon$, $\bar{\Delta}\subseteq U_i$ for some i. Let P be the vertex of $\bar{\Delta}$ s.t. $v(P)=i$. Then, by definition of v, $t_i(P)>t_i(f(P))$. On the other hand, since $P\in U_i$, we have $t_i(P)<t_i(f(P))$. Clearly these two inequalities contradict each other. The assumption that f has no fixed points leads to a contradiction, so f must have a fixed point. $\boxtimes$

Note that $\Delta(A_0,\dots,A_n)$ is homeomorphic to the n dimensional disk $D^n=\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_1^2+\dots+x_n^2\leq 1\}$.

Thus we conclude that every continuous function $f: D^n \to D^n$ has a fixed point.

<u>Theorem</u>: S^n is not contractible, for $n \geq 1$.

<u>Proof</u>: $S^n = \{(x_1, \dots, x_{n+1}) : x_1^2 + \dots + x_{n+1}^2 = 1\} \subseteq D^{n+1}$, S^n is the boundary of D^{n+1} in $\mathbb{R}^{n+1}$. Suppose that S^n is contractible. Then we have a homotopy $H: S^n \times I \to S^n$ such that $H(x,0) = x$ and $H(x,1) = a \in S^n$ for all $x \in S^n$.

Recall that $S^n \times I / S^n \times \{1\} \simeq CS^n$ (cone on S^n) $\simeq D^{n+1}$, where $p: S^n \times I \to D^{n+1}$, $p(x,t) = t \cdot x$ induces a homeomorphism $S^n \times I / S^n \times \{1\} \to D^{n+1}$. The homotopy $H: S^n \times I \to S^n$ maps $S^n \times \{1\}$ to a point, hence it induces a continuous function $\overline{H}: D^{n+1} = S^n \times I / S^n \times \{1\} \to S^n$. The equality $H(x,0) = x$ implies that $\overline{H}(x) = x$ for all $x \in S^n \subseteq D^{n+1}$. Consider now the function $F: D^{n+1} \to S^n \subseteq D^{n+1}$, $F(x) = -\overline{H}(x)$. By Brouwer's thm, $F(u) = u$ for some u. Since F has values in S^n, we have $u \in S^n$. But then $F(u) = -\overline{H}(u) = -u \neq u$, a contradiction. This shows that H does not exist, i.e. S^n is not contractible.

As a corollary we get the following result.

Theorem: If $n<m$ then S^n and S^m are not homeomorphic.

Pf: Suppose $f: S^n \to S^m$ is a homeomorphism. Since $n<m$, we know that f is homotopic to a constant function $g: S^n \to S^m$: $f \sim g$. It follows that $f^{-1} \circ f$ and $f^{-1} \circ g$ are homotopic, i.e. $f^{-1} \circ f = id: S^n \to S^n$ is homotopic to the constant function $f^{-1} \circ g: S^n \to S^n$. This contradicts our previous result that S^n is not contractible.

Theorem: If $n<m$ then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.

Pf: Recall that S^n is the one-point compactification of $\mathbb{R}^n$. One point compactification is a topological construction, i.e. homeomorphic spaces have homeomorphic one point compactifications. Thus if $\mathbb{R}^n$ and $\mathbb{R}^m$ were homeomorphic, then S^n and S^m would be homeomorphic, which is false. ⊠