Last time we proved the following results:

Theorem: A metric space X is compact if and only if every sequence in X has a convergent subsequence.

Theorem: A metric compact space is separable and 2nd countable.

Let us mention without proof the following classical results:

Theorem A: A compact, seperable, Hausdorff space is metrizable.

Theorem B: Every compact metric space X is a continuous image of the Cantor set, i.e. there is a continuous surjective function $f: C \to X$.

Theorem C: Every compact metric space is homeomorphic to a closed subset of the Hilbert cube $I^{\mathbb{N}} = I \times I \times I \times \dots$

Last time we also proved the following result:

Theorem: Let (X,d) be a compact metric space and let $\{U_i : i \in I\}$ be an open cover of X. There is $\varepsilon > 0$ such that for every $x \in X$ the open ball $B(x,\varepsilon)$ is contain is some member of the covering, i.e. $B(x,\varepsilon) \subseteq U_i$ for some $i \in I$.

Definition: Any ε as in the theorem is called a Lebesgue number of the covering $\{U_i : i \in I\}$.

Application: Let X be a compact metric space and $f: X \to \mathbb{R}$ a continuous function. Then f is continuous at every $a \in X$. Recall that this means that given any $\varepsilon > 0$ one can find $\delta > 0$ such that if $d(x,a) < \delta$ then $|f(x) - f(a)| < \varepsilon$. The δ usually depends on the point a (for a fixed ε).

In many applications one would like to know if there is $\delta > 0$ which works for all $a \in X$. The following example shows that this is not always the case:

<u>Example:</u> Let $f: (0, +\infty) \to \mathbb{R}$ be given by $f(x) = \frac{1}{x}$. Note that $f(\frac{1}{n}) - f(\frac{2}{n}) = \frac{n}{2}$ and $|\frac{1}{n} - \frac{2}{n}| = \frac{1}{n}$. This shows that there is no $\varepsilon > 0$ for which one can find $\delta > 0$ which works for all $a \in (0, +\infty)$ (take n s.t. $\frac{1}{n} < \delta$ and $n > 2\varepsilon$, $a = \frac{1}{n}$, $x = \frac{2}{n}$).

<u>Definition:</u> A continuous function $f: X \to \mathbb{R}$ on a metric space (X, d) is called <u>uniformly continuous</u> if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x,y) < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Our example above shows that $f(x) = \frac{1}{x}$ is NOT uniformly continuous on the interval $(0, +\infty)$. However we have the following important result:

<u>Theorem:</u> Let (X, d) be a compact metric space. Any continuous function $f: X \to \mathbb{R}$ is uniformly continuous.

<u>Proof:</u> Fix $\varepsilon > 0$. The open intervals $\{(t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}) : t \in \mathbb{R}\}$ cover $\mathbb{R}$. Thus $\{f^{-1}((t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2})) : t \in \mathbb{R}\}$ is an open cover of X. Let δ be a Lebesgue number of this covering. If $d(x,y) < \delta$ then $y \in B(x, \delta) \subseteq f^{-1}((t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}))$ for some $t \in \mathbb{R}$. Thus $f(x), f(y) \in (t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2})$, hence $|f(x) - f(y)| < \varepsilon$. $\boxtimes$

This result is key for many theorems in analysis, for example to prove that continuous functions are Riemann integrable.

Since compact spaces have many nice properties, it is often desirable to realize a given topological space as a subspace of a compact space. Note that if $X \subseteq Y$ and Y is compact then $\overline{X}$ is also compact and X is dense in $\overline{X}$. This leads us to the following concept.

<u>Definition</u>: A <u>compactification</u> of a topological space X is any topological space Y which is compact and contains X as a dense subset. Slightly more generally, Y is a compactification of X if Y has a dense subset homeomorphic to X and Y is <u>compact</u>.

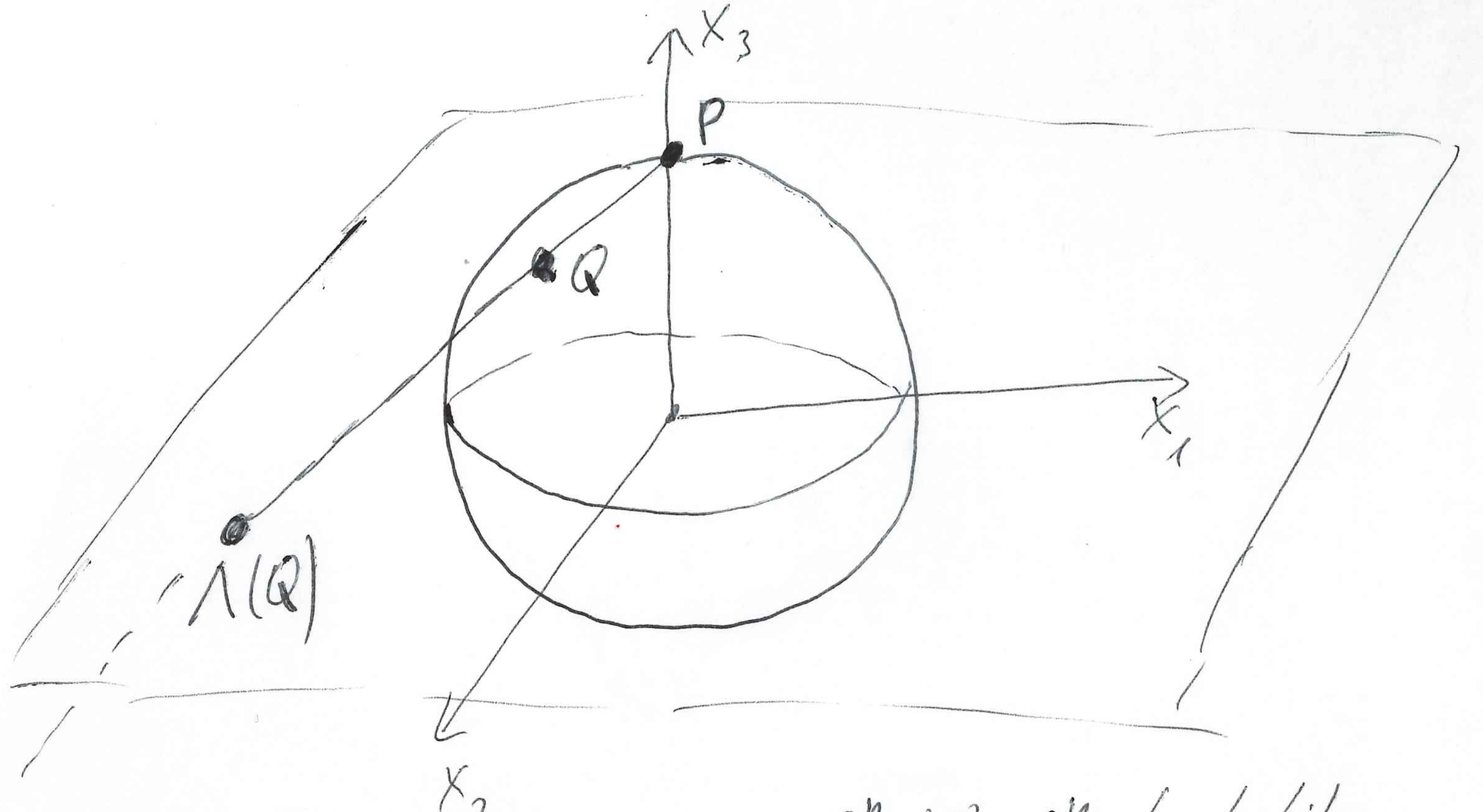
<u>Example</u>: S^n is a compactification of $\mathbb{R}^n$.

To see this we use $S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$, and $\mathbb{R}^n = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Let $P = (0, 0, \dots, 0, 1) \in S^n$, we call P the "north pole" of S^n. We define $\Lambda : S^n - \{P\} \to \mathbb{R}^n$ as follows: for a point $Q = (x_1, \dots, x_{n+1}) \in S^n - \{P\}$ the line joining P and Q intersects $\mathbb{R}^n$ at a point which we denote $\Lambda(Q)$. The line PQ consists of all points of the form $(0, 0, \dots, 0, 1) + t(x_1, \dots, x_n, x_{n+1} - 1) = (tx_1, \dots, tx_n, tx_{n+1} + 1 - t)$. Such point is in $\mathbb{R}^n$ if $tx_{n+1} - t + 1 = 0$, i.e. $t = \frac{1}{1 - x_{n+1}}$. Thus $\Lambda(x_1, x_2, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}, 0\right)$.

It is clear that Λ is continuous. It is a simple excersice to see that $\Psi : \mathbb{R}^n \to S^n - \{P\}$ given by $\Psi(x_1, x_2, \dots, x_n, 0) =$

$$= \left(\frac{2x_1}{x_1^2 + \dots + x_n^2 + 1}, \frac{2x_2}{x_1^2 + \dots + x_n^2 + 1}, \dots, \frac{2x_n}{x_1^2 + \dots + x_n^2 + 1}, \frac{-1 + (x_1^2 + \dots + x_n^2)}{x_1^2 + \dots + x_n^2 + 1} \right)$$

is the inverse function to Λ. Thus Λ is a homeomorphism (as clearly Ψ is continuous). The function Λ is called the <u>stereographic projection</u>.



Let us discuss how open sets in $\mathbb{R}^n \cup \{P\} = S^n$ look like (where we identify $\mathbb{R}^n$ with $\Psi(\mathbb{R}^n) = \Lambda^{-1}(\mathbb{R}^n)$). If $P \notin U$ then U is open in S^n iff it is open in $\mathbb{R}^n$. If $P \in U$ then U is open in S^n iff $S^n - U$ is ~~open~~ closed in S^n. Since S^n is compact, this means that $S^n - U = K$ is compact and $K \subseteq \mathbb{R}^n$, so K is a compact subset of $\mathbb{R}^n$ (in particular, K is closed in $\mathbb{R}^n$, as $\mathbb{R}^n$ is Hausdorff). Thus U is open in this case iff $U = \{P\} \cup (\mathbb{R}^n - K)$ for some K compact in $\mathbb{R}^n$. To summarize: U is open in $S^n = \mathbb{R}^n \cup \{P\}$ if and only if U is open in $\mathbb{R}^n$ or $U = \{P\} \cup (\mathbb{R}^n - K)$ for some compact subset of $\mathbb{R}^n$.

We can try to mimic this for any X. Let X be a topological space. Let $\hat{X} = X \cup \{\infty\}$. We introduce the following topology on $\hat{X}$: U is open if either $U \subseteq X$ is open or $U = U_K = \{\infty\} \cup (X - K)$ for some compact, closed subset of X.

We need to show that this is a topology:

(1) ϕ is open in X and $\hat{X} = U_\phi = \{\infty\} \cup (X-\phi)$, so $\phi, \hat{X}$ are open

(2) If U_1, U_2 are open then either U_1, U_2 are open in X and then $U_1 \cap U_2$ is open in X, or U_1 is open in X and $U_2 = \{\infty\} \cup (X-K)$ and then $U_1 \cap U_2 = U_1 \cap (X-K) =$ open in X since $X-K$ is open in X (K is closed), or $U_1 = \{\infty\} \cup (X-K_1)$, $U_2 = \{\infty\} \cup (X-K_2)$ and then $U_1 \cap U_2 = \{\infty\} \cup \left((X-K_1) \cap (X-K_2)\right) = \{\infty\} \cup X-(K_1 \cup K_2) = U_{K_1 \cup K_2}$, and $K_1 \cup K_2$ is closed and compact in X. Thus our open sets are closed under intersection

(3) Union of any collection of open in X sets is open in X.
Union of any collection of sets of the form U_K is of that form:

$$\bigcup_{i \in I} \left[\{\infty\} \cup (X-K_i)\right] = \{\infty\} \cup \bigcup_{i \in I} (X-K_i) = \{\infty\} \cup \left(X - \bigcap_{i \in I} K_i\right) = U_{\bigcap_{i \in I} K_i}$$

and $\bigcap_{i \in I} K_i$ is compact and closed (it is closed since all K_i are closed and compact as a closed subset of K_i).

Finally $U \cup \left(\{\infty\} \cup (X-K)\right) = \{\infty\} \cup X-\left(K \cap (X-U)\right) = U_{K \cap (X-U)}$ and $K \cap (X-U)$ is closed and compact ($X-U$ closed and K closed and compact).

So our open sets are closed under unions.

Note that the closure of X in $\hat{X}$ can be either X or $\hat{X}$. If $\overline{X} = \hat{X}$, then X is dense in $\hat{X}$. If $\overline{X} = X$, then X is closed in $\hat{X}$ hence $\hat{X} - X = \{\infty\}$ is open so $\{\infty\} = \{\infty\} \cup (X-K)$ for some K which must be X. Thus X is compact. We see that X is dense in $\hat{X}$ if and only if X is not compact.