Recall from last lecture: X a topological space, $\hat{X} = X \cup \{\infty\}$ (we add one point to X, denoted ∞; we think of ∞ as "the point at infinity"). We define a topology on $\hat{X}$ as follows:
U is an open set in $\hat{X}$ iff either $U \subseteq X$ is an open set in X or $U = \{\infty\} \cup (X-K)$ for some closed, compact subset K of X.
We write U_K for $\{\infty\} \cup (X-K)$.
Last time we proved that:
(1) the above description indeed is a topology on $\hat{X}$.
(2) X is closed in $\hat{X}$ if and only if X is compact. In other words, X is dense in $\hat{X}$ if and only if X is not compact (if X is not closed then $\overline{X} = \hat{X}$).
(3) $\hat{\mathbb{R}}^n \simeq S^n$ for all $n \geq 1$.
Now we prove that
(4) $\hat{X}$ is compact.
Proof: Let $\{U_i : i \in I\}$ be a covering of $\hat{X}$ by open sets. Then there is $j \in I$ such that $\infty \in U_j$, so $U_j = U_K = \{\infty\} \cup (X-K)$ for some closed compact subset K of X. Since the open sets $\{U_i : i \in I\}$ cover K, there exist $i_1, i_2, \ldots, i_m \in I$ such that $K \subseteq U_{i_1} \cup \ldots \cup U_{i_m}$. Thus
$\hat{X} = \{\infty\} \cup (X-K) \cup K = U_j \cup K \subseteq U_j \cup U_{i_1} \cup \ldots \cup U_{i_m} \subseteq \hat{X}$, i.e.
$U_j, U_{i_1}, \ldots, U_{i_m}$ is a finite subcovering of the covering $\{U_i : i \in I\}$. ☒
We see that if X is not compact then $\hat{X}$ is a compactification of X (X is dense in $\hat{X}$ by (2)). We call $\hat{X}$ the one-point compactification of X.

Here are some simple excersises:

(1) If X is homeomorphic to Y then $\hat{X}$ is homeomorphic to $\hat{Y}$

(2) $\widehat{(0,1)} \simeq S^1$ (as $(0,1)$ is homeomorphic to $\mathbb{R}$).

(3) $\widehat{[0,1)} \simeq [0,1]$

Suppose now that X is Hausdorff. When is $\hat{X}$ Hausdorff?
It is easy to see that $\hat{X}$ is Hausdorff if and only if for any $a \in X$ there are open sets U, V in $\hat{X}$ such that $a \in U$, $\infty \in V$ and $U \cap V = \phi$. Thus $V = U_K = \{\infty\} \cup (X-K)$ for some closed, compact $K \subseteq X$, and U is open in X, $a \in U$, $U \cap (X-K) = \phi$.
The last condition is equivalent to $U \subseteq K$. Since K is compact and closed, $\overline{U} \subseteq \overline{K} = K$ and $\overline{U}$ is compact (note that since X is Hausdorff, any compact set is closed). Putting all these together: $\hat{X}$ is Hausdorff iff X is Hausdorff and for any $a \in X$ there is U open such that $a \in U$ and $\overline{U}$ is compact.

<u>Definition</u>: A topological space X is called <u>locally-compact</u> if for every $a \in X$ there is open set $U \subseteq X$ such that $a \in U$ and the closure $\overline{U}$ of U is compact.

<u>Warning</u>: The above definition is commonly used but it is different than the one in the book. For Hausdorff spaces though both definitions are equivalent.

We can now state our result as:

<u>Theorem</u>: $\hat{X}$ is Hausdorff iff X is Hausdorff and locally compact.

Example ① Since the closure of any open ball in $\mathbb{R}^n$ is compact, $\mathbb{R}^n$ is locally compact.

② Any discrete space is locally compact and Hausdorff.

Exercise: Study $\hat{X}$ when X is discrete.

Proposition: A locally compact Hausdorff space is regular.

Proof: Let X be locally compact and Hausdorff. Then $\hat{X}$ is compact and Hausdorff, hence $\hat{X}$ is normal. Every normal space is regular, hence $\hat{X}$ is regular. Since a subspace of a regular space is regular, we have X is regular. $\boxtimes$

Remark: A subspace of a normal space does not need to be normal. Can you give an example?

Exercise: Prove the proposition directly (without any use of the space $\hat{X}$).

Exercise: Use the proposition to show that the book definition and our definition of locally compact spaces are equivalent for Hausdorff spaces.

Exercise: Show that if X has a compactification which is Hausdorff then X must be regular.

Exercise: Prove that a subspace of a regular space is regular.

An important topic in topology and analysis is to study the set $\mathrm{Hom}(X,Y)$ of all continuous functions from X to Y, where X,Y are topological spaces. One of the techniques is to put a "usefull" topology on $\mathrm{Hom}(X,Y)$, which plays well with the topologies on X and Y, and then use techniques from topology to get some understanding of $\mathrm{Hom}(X,Y)$.

There are 2 types of maps naturally connected to $\mathrm{Hom}(X,Y)$:

(A) We have the "evaluation function" ev given by:

$$ev: X\times \mathrm{Hom}(X,Y) \longrightarrow Y, \quad ev(x,f)=f(x).$$

(B) For any topological space Z and any continuous function $F: Z\times X \to Y$ we get ~~the~~ function $\tilde{F}: \cancel{Z} \to \mathrm{Hom}(X,Y)$, $\tilde{F}(z)=F(z,-)$ (i.e. $\tilde{F}(z)$ is the function which takes $x\in X$ to $F(z,x)$ in Y).

A desirable topology on $\mathrm{Hom}(X,Y)$ would have the property that both ev and $\tilde{F}$ are continuous (for any F as above). This however is not always possible and in general one has various topologies on $\mathrm{Hom}(X,Y)$ which are "close" to achieving the desirable properties. Note that if some topology on $\mathrm{Hom}(X,Y)$ makes ev continuous, then any stronger topology also has this property. Similarly if all $\tilde{F}$ are continuous in some topology on $\mathrm{Hom}(X,Y)$ then the same holds for any weaker topology.

It is not hard to show that there is at most one topology on $Hom(X,Y)$ which has the desirable properties.

<u>Proposition</u>: If ev is continuous in some topology on $Hom(X,Y)$ then any set of the form

$$\{f \in Hom(X,Y) : f(K) \subseteq U\} = S(K,U)$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open, must be open in $Hom(X,Y)$.

<u>Proof</u>: Let $K \subseteq X$ be compact and $U \subseteq Y$ be open. We know that $ev^{-1}(V)$ is open in $X \times Hom(X,Y)$, for any V open in Y. Take $f \in S(K,U)$, so $f(K) \subseteq U$. This means that $K \times \{f\}$ is contained in $ev^{-1}(U)$. For any $k \in K$ there are open sets U_k in X and V_k in $Hom(X,Y)$ s.t. $k \in U_k$, $f \in V_k$ and $U_k \times V_k \subseteq ev^{-1}(U)$ (by definition of product topology on $X \times Hom(X,Y)$). Now $\{U_k : k \in K\}$ is an open cover of K so there are $k_1, k_2, \dots k_m \in K$ such that $K \subseteq U_{k_1} \cup \dots \cup U_{k_m}$.
Now $V = V_{k_1} \cap V_{k_2} \cap \dots \cap V_{k_m}$ is open in $Hom(X,Y)$ and $f \in V$. We have $K \times V \subseteq ev^{-1}(U)$ so $V \subseteq S(K,U)$. We proved that for any $f \in S(K,U)$ there is open set V in $Hom(X,Y)$ such that $f \in V$ and $V \subseteq S(K,U)$. This means that $S(K,U)$ is open. $\boxtimes$

The last proposition motivates the following definition.

<u>Definition</u>: The <u>compact-open topology</u> on $Hom(X,Y)$ is the topology generated by sets of the form $S(K,U)$, i.e. it is the weakest topology in which all the sets $S(K,U)$ are open. A basis of this topology consists of all finite intersections of sets of the form $S(K,U)$: a set is open in the compact-open topology if it is a union of sets of the form $S(K_1,U_1) \cap S(K_2,U_2) \cap \ldots \cap S(K_m,U_m)$, with K_i compact in X and U_i open in Y.

<u>Corollary</u>: Any topology ~~for~~ on $Hom(X,Y)$ for which ev is continuous must contain the compact-open topology.

It is not always true that ev is continuous in the compact-open topology. However we have:

<u>Proposition</u>: When $Hom(X,Y)$ has the compact-open topology all functions of the form $\tilde{F}$ are continuous (see B).

<u>Proof</u>: Let Z be a topological space and $F: Z\times X \to Y$ a continuous function. We want to show that $\tilde{F}: Z \to Hom(X,Y)$ is continuous (when $Hom(X,Y)$ has the compact-open topology). It suffices to show that $\tilde{F}^{-1}(S(K,U))$ is open in Z for any K compact in X, U open in Y. Let $z \in \tilde{F}^{-1}(S(K,U))$.

This means that $\tilde{F}(z) \in S(K, U)$, i.e. $F(z, -)$ takes K to U. In other words, $F(\{z\} \times K) \subseteq U$, i.e. $\{z\} \times K \subseteq F^{-1}(U)$. For any ~~op~~ $k \in K$ there are open sets U_k in Z, V_k in K such that $z \in U_k$, $k \in V_k$ and $U_k \times V_k \subseteq F^{-1}(U)$. The set $\{V_k : k \in K\}$ covers K, so there are $k_1, \dots, k_m$ in K such that $V_{k_1} \cup \dots \cup V_{k_m} \supseteq K$. Now $U_{k_1} \cap \dots \cap U_{k_m} = W$ is open in Z and $z \in W$. We have $W \times K \subseteq F^{-1}(U)$ which is the same as $W \subseteq \tilde{F}^{-1}(S(K, U))$. We proved that for any $z \in \tilde{F}^{-1}(S(K, U))$ there is open set W such that $z \in W$ and $W \subseteq \tilde{F}^{-1}(S(K, U))$. This means that $\tilde{F}(S(K, U))$ is open in Z. $\boxtimes$

It turns out that when X is locally compact and Hausdorff the compact-open topology on $\mathrm{Hom}(X, Y)$ has all the desired properties for any topological space Y, i.e. we have

<u>Theorem</u>: If X is locally compact and Hausdorff then ev is continuous when $\mathrm{Hom}(X, Y)$ has the compact open topology.

<u>Proof</u>: Let $U \subseteq Y$ be open. We need to show that $ev^{-1}(U)$ is open in $X \times \mathrm{Hom}(X, Y)$. Consider $(x, f) \in ev^{-1}(U)$, so $f(x) \in U$. Now $f^{-1}(U)$ is open in X and contains x, so there is open set V in X s.t. $x \in V \subseteq \overline{V} \subseteq f^{-1}(U)$ and $\overline{V}$ is compact (since X is locally compact and Hausdorff). This means that $V \times S(\overline{V}, U) \subseteq ev^{-1}(U)$ and $V \times S(\overline{V}, U)$ is open in ~~X~~ $\mathrm{Hom}(X, Y)$ and $(x, f) \in V \times S(\overline{V}, U)$. We proved that for any

$(x,f) \in ev^{-1}(\mathcal{U})$ there open set $V \times S(\overline{V}, \mathcal{U})$ in $X \times Hom(X,Y)$ which contains (x,f) and is contained in $ev^{-1}(\mathcal{U})$. This means that $ev^{-1}(\mathcal{U})$ is open in $X \times Hom(X,Y)$. In other words, ev is continuous. $\boxtimes$.

<u>Exercise:</u> Consider a topology on $Hom(X,Y)$. Prove that ev is continuous if and only if for every topological space Z and every continuous function $G: Z \to Hom(X,Y)$, the function $G^*: Z \times X \to Y$ given by $G^*(z,x) = G(z)(x)$, is continuous.

<u>Remark:</u> One can prove that if X has a Hausdorff compactification (such spaces are called <u>completely regular</u>) and $Hom(X,\mathbb{R})$ has topology with all desired properties (as discussed on page 4) then X is locally-compact.

<u>Exercise:</u> Let X be a locally compact Hausdorff space, Y, Z topological spaces. Prove that the map

$$\begin{array}{ccc} Hom(Z \times X, Y) & \longrightarrow & Hom(Z, Hom(X,Y)) \\ F & \longrightarrow & \tilde{F} \end{array}$$

is a homeomorphism when we use the compact-open topologies on Hom.