One of the most fundamental concepts in topology is the notion of <u>homotopy</u>.

<u>Definition</u>: Two functions $f,g: X \to Y$ are <u>homotopic</u> if there is a continuous function $H: X \times I \to Y$ such that $H(x,0)=f(x)$, $H(x,1)=g(x)$ for all $x \in X$. H is called a <u>homotopy</u> between f and g.

We write $f \sim g$ to say that f,g are homotopic (we write $f \underset{H}{\sim} g$ to indicate that H is a homotopy between f,g).

The relation $\sim$ is an equivalence relation:

(1) $f \sim f$ via the homotopy $H(x,t)=f(x)$

(2) If $f \underset{H}{\sim} g$ then $g \sim f$ via the homotopy $\tilde{H}(x,t)=H(x,1-t)$

(3) If $f \underset{H_1}{\sim} g$ and $g \underset{H_2}{\sim} h$ then $f \sim h$ via $H(x,t)=\begin{cases} H_1(x,2t), & t \in [0,\frac{1}{2}] \\ H_2(x,2t-1), & t \in [\frac{1}{2},1] \end{cases}$

<u>Remark</u>: A continuous function $H: X \times I \to Y$ gives rise to a continuous function $\tilde{H}: I \to \mathrm{Fun}(X,Y)$ (with the compact-open topology), where $\tilde{H}(t): x \mapsto H(x,t)$. We have $\tilde{H}(0)=f$, $\tilde{H}(1)=g$, so $\tilde{H}$ is a path in $\mathrm{Fun}(X,Y)$ joining f and g. Thus if f,g are homotopic, then f,g are in the same path component of $\mathrm{Fun}(X,Y)$. The converse is not always true, but if X is locally compact and Hausdorff then $\mathrm{Fun}(X \times I, Y)$ and $\mathrm{Fun}(I, \mathrm{Fun}(X,Y))$ are in bijective correspondence $H \leftrightarrow \tilde{H}$ and f,g are then homotopic iff they are in the same path component of $\mathrm{Fun}(X,Y)$.

Remark. Since I is compact and Hausdorff, the natural map $\mathrm{Fun}(X\times I, Y) \longrightarrow \mathrm{Fun}(X, \mathrm{Fun}(I,Y))$ is a bijection, so a homotopy $H: X\times I \to Y$ can be identified with a function $\tilde{H}: X \to \mathrm{Fun}(I,Y)$, $\tilde{H}(x): t \to H(x,t)$. From this point of view, it is important to understand $\mathrm{Fun}(I,Y)$, which is the space of paths in Y.

Def: A subset G of $\mathbb{R}^n$ is convex if for any two points x, y in G the whole segment joining x and y is contained in G. Note that the segment can be described as $\{(1-t)x+ty : t\in[0,1]\}$. Thus G is convex iff for any $x,y\in G$ all points $(1-t)x+ty$ are in G for $t\in[0,1]$.

Example: Let $G\subseteq \mathbb{R}^n$ be convex. Then any two continuous functions $f,g: X \longrightarrow G$ are homotopic. Indeed, the function $H: X\times I \to G$, $H(x,t) = (1-t)f(x) + tg(x)$ is a homotopy between f and g.

Remark: Suppose $f,g: X\to Y$ are homotopic. Then for any continuous $\varphi: Z_1 \to X$, $\psi: Y \to Z_2$ we have $\psi f \varphi \sim \psi g \varphi$. In fact, if $f \underset{H}{\sim} g$ then $H_1: Z_1\times I \to Z_2$, $H_1(z,t) = \psi H(\varphi(z), t)$ is a homotopy between $\psi f \varphi$ and $\psi g \varphi$.

Example: Recall that $S^n - \{P\}$ is homeomorphic to $\mathbb{R}^n$ (P any point of S^n) and $\mathbb{R}^n$ is convex. Thus any function $f: X \to S^n$ which misses a point (i.e. is not onto) is homotopic to a constant function.

Definition: A space X is contractible if the identity function $id: X \to X$ is homotopic to a constant function.

Corollary: Any convex set is contractible.

Cor: If Y is contractible, any two functions $f, g: X \to Y$ are homotopic. In fact, let $H: Y \times I \to Y$ be such that $H(y,0) = y$ and $H(y,1) = c$ for a fixed $c \in Y$. Then $H_1: X \times I \to Y$, $H_1(x,t) = H(f(x), t)$ is a homotopy between $H_1(x,0) = H(f(x),0) = f(x)$ and $H_1(x,1) = H(f(x),1) = c$ so $f \sim$ constant function. Same for g, so $f \sim g$.

Why this concept of homotopy may be useful:

Consider a polynomial $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$ with complex coefficients. Suppose f has no roots in $\mathbb{C}$, i.e. $f(z) \neq 0$ for all z. Let $S^1 = \{z : |z| = 1\} \subseteq \mathbb{C}$. We define $H_1: S^1 \times I \to S^1$ by $H_1(z,t) = \frac{f(tz)}{|f(tz)|}$. This is a homotopy between $H_1(z,0) = \frac{f(0)}{|f(0)|}$ = constant function and $H_1(z,1) = \frac{f(z)}{|f(z)|}$.

Now consider $f(\frac{1}{t}z)=\frac{1}{t^n}z^n+a_1\frac{1}{t^{n-1}}z^{n-1}+\dots+a_{n-1}\frac{1}{t}z+a_n=$

$$=\frac{1}{t^n}(z^n+ta_1z^{n-1}+t^2a_2z^{n-2}+\dots+t^na_n).$$ Thus

$$\frac{f(\frac{1}{t}z)}{|f(\frac{1}{t}z)|}=\frac{z^n+ta_1z^{n-1}+\dots+t^na_n}{|z^n+ta_1z^{n-1}+\dots+t^na_n|}.$$

We define $H_2: S^1\times I\to S^1$, $H_2(z,t)=\dfrac{z^n+ta_1z^{n-1}+\dots+t^na_n}{|z^n+ta_1z^{n-1}+\dots+t^na_n|}$.

We see that H_2 is a homotopy between $H_2(z,0)=z^n$ and $H_2(z,1)=\dfrac{f(z)}{|f(z)|}$. Thus $z^n\sim\dfrac{f(z)}{|f(z)|}\sim$ constant.

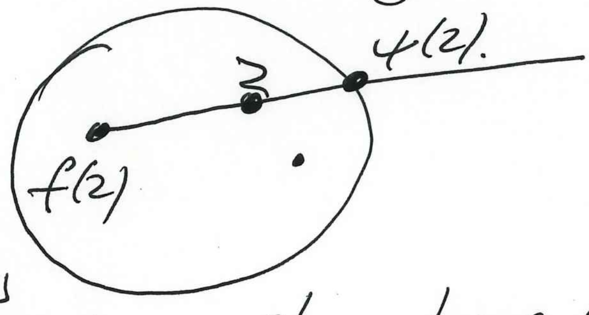
We proved that:
If a polynomial of degree n has no comlex roots then the map $f(z)=z^n: S^1\to S^1$ is homotopic to a constant map. Setting $f_m: S^1\to S^1$, $f_m(z)=z^m$ we can restate it as:

$$\boxed{\text{If a polynomial of degree } n \text{ has no roots in } \mathbb{C} \text{ then } f_n\sim f_0}$$

In other words, if we could prove that $f_n\not\sim f_0$ for $n>0$ then we would get a proof of the fundamental theorem of algebra: every polynomial with complex coefficients has a root in $\mathbb{C}$ (if its degree ≥ 1).

Another application: Suppose $f: D^2 \to D^2$ is a continuous function without fixed points, i.e. $f(z) \neq z$ (here $D^2 = \{z : |z| \leq 1\}$ is the 2-dimensional disk).

Define $\psi: D^2 \to S^1$ as follows: $\psi(z)$ is the point on S^1 which belongs to the ray (half-line) starting at $\psi(z)$ and going through z:



We claim that ψ is a continuous function (a proof is given at the end). Note that for $z \in S^1$ we have $\psi(z) = z$. Thus $\psi: D^2 \to S^1$ has the property that $\psi(z) = z$ for $z \in S^1$!

Define $H: S^1 \times I \to S^1$ by $H(z,t) = \psi(tz)$. Then $H(z,0) = \psi(0)$ is constant and $H(z,1) = \psi(z) = z = f_1(z)$

Thus: if there is a continuous function $D^2 \to D^2$ without fixed points then $f_1 \sim f_0$. If we could show that f_1 and f_0 are not homotopic, we would get a proof of the following

Theorem (Brouwer's fixed point theorem in dimension 2): Any continuous function $D^2 \to D^2$ has a fixed point.

Remark: A proof that ψ above is continuous is on page 120 in the book.