<u>Recall</u>: Two continuous functions $f,g: X \to Y$ are <u>homotopic</u> if there is a continuous function $H: X \times I \to Y$ such that $H(x,0)=f(x)$, $H(x,1)=g(x)$ for all $x \in X$. H is a <u>homotopy</u> between f and g.

Let $f_n: S^1 \to S^1$, $f_n(z)=z^n$ (here $S^1=\{z \in \mathbb{C}: |z|=1\}$).

Last time we proved:

(a) If f_n is not homotopic to f_0 then any polynomial of degree n with complex coefficients has a root in $\mathbb{C}$.

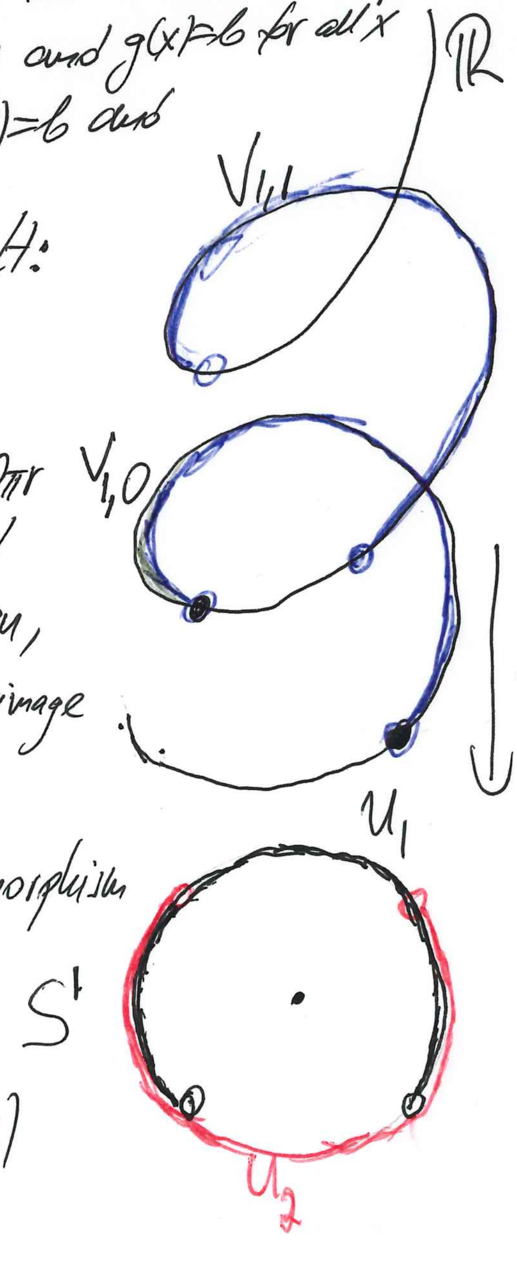
(b) If f_1 is not homotopic to f_0, then any continuous function $\varphi: D^2 \to D^2$ has a fixed point (here $D^2=\{z \in \mathbb{C}: |z| \le 1\}$ is the two dimensional disk).

<u>Remark</u>: If Y is path-connected then any two constant maps $f,g: X \to Y$ are homotopic: If $f(x)=a$ for all x and $g(x)=b$ for all x then choose a path $\gamma: I \to Y$ s.t. $\gamma(0)=a$, $\gamma(1)=b$ and define $H: X \times I \to Y$ by $H(x,t)=\gamma(t)$.

Our goal is to prove the following key result:

<u>Theorem</u>: $f_n \sim f_m$ if and only if $m=n$.

We will use the map $p: \mathbb{R} \to S^1$, $p(r)=e^{2\pi i r} = \cos 2\pi r + i \sin 2\pi r$



<u>Key fact about p</u>: Let $U_1=\{e^{2\pi i t}: -\frac{1}{8}<t<\frac{5}{8}\}$ and $U_2=\{e^{2\pi i t}: \frac{3}{8}<t<\frac{7}{8}\}$. Then U_1, U_2 are open, connected sets in S^1 and $U_1 \cup U_2 = S^1$. The preimage $p^{-1}(U_i)$ is a union of disjoint open sets $V_{i,\alpha}$, $\alpha \in \mathbb{Z}$ and $p: V_{i,\alpha} \to U_i$ is a homeomorphism for all α. We write $\boxed{q_{i,\alpha}=p^{-1}: U_i \to V_{i,\alpha}}$

Thus $p \circ q_{i,\alpha}=$ identity on U_i.

<u>Exercise</u>: $V_{1,\alpha}=(-\frac{1}{8}+\alpha, \frac{5}{8}+\alpha)$, $V_{2,\alpha}=(\frac{3}{8}+\alpha, \frac{7}{8}+\alpha)$ for $\alpha \in \mathbb{Z}$.

Def: A continuous map $p: E \to B$ is called a *covering* if E is path connected, p is surjective and B has an open cover $\{U_i : i \in I\}$ such that $p^{-1}(U_i)$ is a union of disjoint open sets $V_{i,\alpha}$, $\alpha \in J_i$ such that $p: V_{i,\alpha} \to U_i$ is a homeomorphism for all $\alpha \in J_i$. Thus our $p: \mathbb{R} \to S^1$ is an example of a covering, with $I = \{1,2\}$ and $J_1 = J_2 = \mathbb{Z}$. Most of the discussion below can be extended to any covering, but we focus only on $p: \mathbb{R} \to S^1$.

Exercise: If $p: E \to B$ is a covering then $p^{-1}(b)$ is a discrete subset of E for every $b \in B$.

Theorem 1: Given any continuous function $f: I \to S^1$ and $r \in \mathbb{R}$ such that $p(r) = f(0)$, there is unique $\hat{f}: I \to \mathbb{R}$ such that $\hat{f}$ is continuous, $p\hat{f} = f$ and $\hat{f}(0) = r$. We call $\hat{f}$ *the lift of f starting at r*.

Proof: Let $\hat{f}_1, \hat{f}_2$ be two continuous functions $I \to \mathbb{R}$ such that $p\hat{f}_1 = p\hat{f}_2$. Then for every $t \in I$ we have $\hat{f}_1(t) - \hat{f}_2(t) \in \mathbb{Z}$ (recall that $p(s_1) = p(s_2)$ iff $s_1 - s_2$ is an integer). Thus $\hat{f}_1 - \hat{f}_2 : I \to \mathbb{Z}$ is a continuous function. Since I is connected and $\mathbb{Z}$ is totally disconnected (discrete), $\hat{f}_1 - \hat{f}_2$ must be constant. This shows that any two lifts of f differ by an integer. In particular, if $\hat{f}_1(0) = \hat{f}_2(0)$ then $\hat{f}_1 = \hat{f}_2$.

This shows that $\hat{f}$, if it exists, is unique.

Exercise: Prove that if $p: E \to B$ is a covering and, X a connected space and $f_1, f_2: X \to E$ continuous functions such that $p \circ f_1 = p \circ f_2$ and $f_1(c) = f_2(c)$ for some $c \in X$, then $f_1(x) = f_2(x)$ for all x.

We now prove the existence of $\hat{f}$. The open sets $f^{-1}(U_i)$, $i=1,2$ cover I. Let ε be a Lebesgue number of this covering and pick a natural number k s.t. $\frac{1}{k}<\varepsilon$. Then f maps each of the intervals $[0,\frac{1}{k}],[\frac{1}{k},\frac{2}{k}],\dots,[\frac{k-1}{k},\frac{k}{k}]$ to one of the open sets U_i.

Suppose we have already constructed a continuous function $\hat{f}:[0,\frac{m}{k}]\longrightarrow\mathbb{R}$ such that $p\hat{f}=f$ on $[0,\frac{m}{k}]$ and $\hat{f}(0)=r$ (the case $m=0$ is a starting point). Now f maps $[\frac{m}{k},\frac{m+1}{k}]$ to U_i for some i. Since $p\hat{f}(\frac{m}{k})=f(\frac{m}{k})\in U_i$, $\hat{f}(\frac{m}{k})\in p^{-1}(U_i)$, so $\hat{f}(\frac{m}{k})\in V_{i,\alpha}$ for some α. Thus the function $q_{i,\alpha}f:[\frac{m}{k},\frac{m+1}{k}]$ is continuous, $p\circ(q_{i,\alpha}f)=(p\circ q_{i,\alpha})f=id\circ f=f$ and $q_{i,\alpha}\circ f(\frac{m}{k})=\hat{f}(\frac{m}{k})$. Thus, we can extend $\hat{f}$ to $[0,\frac{m+1}{k}]$ by defining $\hat{f}$ to be $q_{i,\alpha}\circ f$ on $[\frac{m}{k},\frac{m+1}{k}]$ and we get a continuous function $\hat{f}:[0,\frac{m+1}{k}]\to\mathbb{R}$ s.t. $\hat{f}(0)=r$, $p\hat{f}=f$. Repeating this process several times we arrive at $\hat{f}:[0,1]\to\mathbb{R}$ s.t. $\hat{f}(0)=r$ and $p\hat{f}=f$. □

<u>Exercise:</u> Prove Theorem 1 for any covering $p:E\longrightarrow B$.

<u>Exercise:</u> Consider the subset of $\mathbb{R}\times\mathrm{Fun}(I,S^1)$ given by $T=\{(r,f)\in\mathbb{R}\times\mathrm{Fun}(I,S^1):f(0)=p(r)\}$. Using the compact-open topology on $\mathrm{Fun}(I,S^1)$, T is a topological space with the subset topology. Theorem 1 defines a function $T\to\mathrm{Fun}(I,\mathbb{R})$, $(r,f)\longmapsto\hat{f}$, where $\hat{f}:I\to\mathbb{R}$ and $\hat{f}(0)=r$, $p\hat{f}=f$. Prove that this function is continuous. Then state and prove an analogous result for any covering $p:E\to B$

We apply now Theorem 1 to functions $f: S^1 \to S^1$ as follows.
Let $e: I \to S^1$, $e(t) = e^{2\pi i t}$. We will identify f with $fe: I \to S^1$ where $fe(1) = fe(0)$ (in other words, we think of S^1 as interval with the end-points glued). Pick any $r \in \mathbb{R}$ s.t. $fe(0) = p(r)$, and consider the lift $\hat{f}$ from Thm 1. Since $fe(0) = fe(1)$, we have $\hat{f}(1) - \hat{f}(0) \in \mathbb{Z}$. (as $p\hat{f}(0) = fe(0) = fe(1) = p\hat{f}(1)$). If we choose another $r_1 \in \mathbb{R}$ s.t. $fe(0) = p(r_1)$ and corresponding lift $\hat{f}_1$ then $\hat{f}_1 - \hat{f} = $ constant (see the beginning of our proof of Thm 1). Thus $\hat{f}_1(1) - \hat{f}_1(0) = \hat{f}(1) - \hat{f}(0)$.
So the difference $\hat{f}(1) - \hat{f}(0)$ does not depend on the choice of $\hat{f}$.
We define degree of f ($\deg(f)$) to be the integer $\hat{f}(1) - \hat{f}(0)$.
Intuitively, $\deg(f)$ is the number of times f winds around S^1.

Def: $f: S^1 \to S^1$ continuous. Lift $f \circ e: I \to S^1$ to $\hat{f}: I \to \mathbb{R}$ and set $\deg(f) = \hat{f}(1) - \hat{f}(0)$. This does not depend on the choice of $\hat{f}$.

Example: Consider $f_n: S^1 \to S^1$, $f_n(z) = z^n$. Then $f_n \circ e: I \to S^1$, $f_n \circ e(t) = e^{2\pi i n t}$.
It is easy to see that $\hat{f}_n: I \to \mathbb{R}$, $\hat{f}_n(t) = nt$ is a continuous lift of f_n, since $p\hat{f}_n(t) = p(nt) = e^{2\pi i n t} = fe(t)$. Thus

$$\deg(f_n) = \hat{f}_n(1) - \hat{f}_n(0) = n \cdot 1 - n \cdot 0 = n$$

If we prove that homotopic maps $S^1 \to S^1$ have the same degree, we will prove that f_n and f_m are not homotopic for $n \neq m$.

To get such a result we extend Theorem 1 as follows:

<u>Theorem 2</u>: Suppose $H: X\times I \to S^1$ is a continuous function and $\hat{h}: X \to \mathbb{R}$ is such that $\hat{h}$ is continuous and $p\hat{h}(x) = H(x,0)$ for all $x \in X$ (i.e. $\hat{h}$ is a lift of $H(-,0)$). Then there is unique $\hat{H}: X\times I \to \mathbb{R}$ such that $p\hat{H} = H$ and $\hat{H}(x,0) = \hat{h}(x)$ for $x\in X$, and $\hat{H}$ is continuous.

<u>Proof</u>: Theorem 1 tells us what $\hat{H}$ has to be: for any $x\in X$, $H(x,-): \{x\}\times I \to S^1$ is continuous (and $\{x\}\times I \simeq I$) and $p\hat{h}(x) = H(x,0)$ so there is unique lift $\hat{H}(x,-): \{x\}\times I \to \mathbb{R}$ s.t. $\hat{H}(x,0) = \hat{h}(x)$ and $p\hat{H}(x,t) = H(x,t)$ for all $t\in I$. The only problem is to prove that so defined $\hat{H}$ is continuous on $X\times I$ (it is continuous on each $\{x\}\times I$ but we need to control how it varies with x).

To prove that $\hat{H}$ is continuous we modify our proof of Theorem 1. We show that for every $x\in X$ there is open set V_x s.t. $x\in V_x$ and a continuous function $\hat{H}_x: V_x\times I \to \mathbb{R}$ s.t. $p\hat{H}_x = H$ on $V_x \times I$ and $\hat{H}_x(y,0) = \hat{h}(y)$ for $y\in V_x$. By uniqueness from Thm 1, $\hat{H}_x = \hat{H}$ on $V_x\times I$, so $\hat{H}$ is continuous on $V_x \times I$, and hence $\hat{H}$ is continuous (as $\{V_x\times I\}$ cover $X\times I$). Fix $x\in X$. Suppose we have open set V_m s.t. $x\in V_m$ and a function $\hat{H}_m: V_m\times[0,\frac{m}{k}]$ which is continuous, $p\hat{H}_m = H$ on $V_m\times[0,\frac{m}{k}]$ (here k is as in the proof of Thm 1, where ~~f(t)~~ $H(x,t)$) and $\hat{H}_m(u,0) = \hat{h}(u)$ for $u\in V_m$. The function $H(x,-)$ maps $\{x\}\times[\frac{m}{k},\frac{m+1}{k}]$ into U_i (as in the proof of Thm 1) so $\{x\}\times[\frac{m}{k},\frac{m+1}{k}] \subseteq H^{-1}(U_i)$. Using compactness of $[\frac{m}{k},\frac{m+1}{k}]$ we can find an open set $\tilde{V}_m$ s.t. $x\in \tilde{V}_m \subseteq V_m$ and H maps $\tilde{V}_m\times[\frac{m}{k},\frac{m+1}{k}]$ to U_i (i.e. $\tilde{V}_m\times[\frac{m}{k},\frac{m+1}{k}] \subseteq H^{-1}(U_i)$; see problem from homework)

Also, $\hat{H}_m(x,\frac{m}{k}) \in p^{-1}(U_i)$, as $p\hat{H}_m(x,\frac{m}{k}) = H(x,\frac{m}{k}) \in U_i$), so $\hat{H}_m(x,\frac{m}{k}) \in V_{i,\alpha}$ for some α (as in the proof of Thm 1).

Since $\hat{H}_m : \tilde{V}_m \times [0,\frac{m}{k}] \to \mathbb{R}$ is continuous, there is open set $\tilde{\tilde{V}}_m$ s.t. $x \in \tilde{\tilde{V}}_m \subseteq \tilde{V}_m$ and $\hat{H}_m$ maps $\tilde{\tilde{V}}_m \times \{\frac{m}{k}\}$ into $V_{i,\alpha}$.

Set $V_{m+1} = \tilde{\tilde{V}}_m$ (so $V_{m+1} \subseteq V_m$, $x \in V_{m+1}$). Define $\hat{H}_{m+1} : V_{m+1} \times [0,\frac{m+1}{k}] \to \mathbb{R}$ by $\hat{H}_{m+1} = \begin{cases} \hat{H}_m \text{ on } V_{m+1} \times [0,\frac{m}{k}] \\ q_{i,\alpha} \circ H \text{ on } V_{m+1} \times [\frac{m}{k},\frac{m+1}{k}] \end{cases}$. Since $\hat{H}_m(u,\frac{m}{k}) \in V_{i,\alpha}$ and

$p\hat{H}_m(u,\frac{m}{k}) = H(u,\frac{m}{k}) \in U_i$ we have $\hat{H}_m(u,\frac{m}{k}) = q_{i,\alpha} H(u,\frac{m}{k})$ for all $u \in V_{m+1}$. In other words, the top and bottom functions in the definition of $\hat{H}_{m+1}$ agree on $V_{m+1} \times \{\frac{m}{k}\}$ and hence $\hat{H}_{m+1}$ is continuous (we use here that $V_{m+1} \times [0,\frac{m}{k}]$, $V_{m+1} \times [\frac{m}{k},\frac{m+1}{k}]$ are closed subsets of $V_{m+1} \times [0,\frac{m+1}{k}]$, $\hat{H}_m$ is continuous on $V_{m+1} \times [0,\frac{m}{k}]$, $q_{i,\alpha} \circ H$ is continuous on $V_{m+1} \times [\frac{m}{k},\frac{m+1}{k}]$ and the functions agree on $V_{m+1} \times [0,\frac{m}{k}] \cap V_{m+1} \times [\frac{m}{k},\frac{m+1}{k}] = V_{m+1} \times \{\frac{m}{k}\}$, hence $\hat{H}_{m+1}$ is continuous). Thus $\hat{H}_{m+1} : V_{m+1} \times [0,\frac{m+1}{k}] \to \mathbb{R}$ satisfies $p\hat{H}_{m+1} = H$, $\hat{H}_{m+1}(u,0) = \hat{h}(u)$ for $u \in V_{m+1}$.

Repeating this process we get $\hat{H}_k : V_k \times [0,\frac{k}{k}=1] \to \mathbb{R}$ so $V_x = V_k$, $\hat{H}_x = \hat{H}_k$ work. This completes the proof that $\hat{H}$ is continuous.

Remark: The excercise on the bottom of page 3 provides a different proof of Theorem 2: We can consider $H: X\times I \to S^1$ as a continuous function $\tilde{H}: X \to \mathrm{Fun}(I, S^1)$. We get a continuous function $X \to T \subseteq \mathbb{R} \times \mathrm{Fun}(I, S^1)$ given by $x \mapsto (\hat{h}(x), \tilde{H}(x))$. Composing it with the function $T \to \mathrm{Fun}(I, \mathbb{R})$ from the excersise we get a continuous function $X \to \mathrm{Fun}(I, \mathbb{R})$ which is the same as continuous function $X\times I \to \mathbb{R}$ and thus is our $\hat{H}$. This approach is more abstract, but it illustrates well the usefullness of the function spaces.

Using Theorem 2 we can now prove our main result

Thm 3: If $f, g: S^1 \to S^1$ are homotopic then $\deg(f) = \deg(g)$.

Pf: Consider a homotopy $H: S^1 \times I \to S^1$, $H(x,0) = f(x)$, $H(x,1) = g(x)$. It induces $H^*: I\times I \to S^1$ where $H^*(s,t) = H(e(s), t)$ ($e: I \to S^1$ as before). The function $H^*(-,0): I \times \{0\} \to S^1$ can be lifted to $\hat{h}: I \to \mathbb{R}$ s.t. $p\hat{h}(s) = H^*(s,0)$ by Theorem 1. By Theorem 2, there is $\hat{H}^*: I\times I \to \mathbb{R}$ s.t. $p\hat{H}^* = H^*$ and $\hat{H}^*(-,0) = \hat{h}$.

Note that $H^*(0,t) = H(e(0), t) = H(e(1), t) = H^*(1,t)$ for all $t \in I$. Thus $\hat{H}^*(1,t) - \hat{H}^*(0,t) = \deg(H(-,t))$. In particular, $\hat{H}^*(1,0) - \hat{H}^*(0,0) = \deg(f)$, $\hat{H}^*(1,1) - \hat{H}^*(0,1) = \deg(g)$, and $\hat{H}^*(1,t) - \hat{H}^*(0,t) \in \mathbb{Z}$ for all $t \in I$. Thus $t \mapsto \hat{H}^*(1,t) - \hat{H}^*(0,t)$ is a continuous function from I to $\mathbb{Z}$, hence it is contant. It follows that $\deg(f) = \deg(g)$. $\boxtimes$

Excercise: Prove that if $\deg(f)=\deg(g)$ then f and g are homotopic.

Hint: First show that g is homotopic to g_1 s.t. $f(1)=g_1(1)$. Lift $f\circ e$ and $g_1\circ e$ to $\hat{f}, \hat{g}_1 : I \to \mathbb{R}$ s.t. $\hat{f}(0)=\hat{g}_1(0)$. Then $\hat{f}(1)=\hat{g}_1(1)$ since $\deg(f)=\deg(g)=\deg(g_1)$. Show that $H^*(s,t)=p\big((1-t)\hat{f}(s)+t\hat{g}_1(s)\big)$ is a continuous map s.t. $H^*(0,t)=H^*(1,t)$ for all $t\in I$, hence H yields a continuous map $H: S^1\times I \to S^1$ which is a homotopy between f and g_1.

Since $\deg(f_n)=n$, we conclude that f_n and f_m are not homotopic for $m\neq n$. By the above excercise, every continuous map $f: S^1\to S^1$ is homotopic to f_n, where $n=\deg(f)$.