QUIZ: ① What does it mean that Y is a compactification of X?

② Let Y_1 be a compactification of X_1 and Y_2 a compactification of X_2. Prove that $Y_1 \times Y_2$ is a compactification of $X_1 \times X_2$.

③ Let X be a metric space with metric ρ and let A be a compact subset of X. Fix a point $u \in X$. Prove that there is $a \in A$ such that

$\rho(a,u) \le \rho(b,u)$ for every $b \in A$.

Solutions: ① We say that Y is a compactification of X if Y contains X as a dense subset and Y is compact (or that Y is compact and has a dense subset homeomorphic to X).

② We know that Y_1 is compact and $\overline{X_1} = Y_1$ (X_1 is dense in Y_1), and Y_2 is compact, $\overline{X_2} = Y_2$ (X_2 is a dense subset of Y_2).
Then $Y_1 \times Y_2$ is compact and $\overline{(X_1 \times X_2)} = \overline{X_1} \times \overline{X_2} = Y_1 \times Y_2$, so $X_1 \times X_2$ is dense in $Y_1 \times Y_2$, i.e. $Y_1 \times Y_2$ is a compactification of $X_1 \times X_2$.

③ Consider the function $f: A \to \mathbb{R}$ given by $f(b) = \rho(b,u)$. f is a continuous function on a compact space, hence f attains the smallest value at some point $a \in A$. This means that $\rho(a,u) \le \rho(b,u)$ for all $b \in A$. $\square$