Homework
due on Wednesday, February 4

Read carefully sections 6, 7 of Hartshorne's book. Solve problems 7.1, 7.10 and the following problems.

Problem 1. I moved this problem to the end and made it extra credit as it is significantly harder than all the other problems in this homework.

Problem 2. An incidence geometry is called a **projective plane** if it satifies the following 2 axioms:

(PP1) any two lines intersect.

(PP2) there exist 4 points, no three of which are on one line.

Let Π be a projective plane.

a) Prove that every line in Π has at least 3 points.

b) Let l, m be two lines in Π. Prove that there is a point A not belonging to either l or m. Use lines through A to construct a bijection between points on l and points on m.

c) Suppose that Π is finite. Then, by a) and b), there is $n \geq 2$ such that every line has exactly $n+1$ points. Prove that Π has n^2+n+1 points and n^2+n+1 lines.

Choose a line l in Π.

d) Let Π_1 be the set of all points in Π which are not on l. Call a subset of Π_1 a line if it is equal to an intersection of a line in Π with Π_1. Prove that Π_1 is an affine plane. (This allows to prove c) by using results about finite affine planes)

The following two parts are for extra credit.
e) Do problem 6.7 a) from Hartshorn's book (which shows that conversly, any affine plane can be extended to a projective plane by adding "the line at infinity").

f) Let Π be a projective plane. Consider the set Π^* of all lines in Π. A line in Π^* is defined as a set of all lines in Π passing through a given point of Π (so lines in Π^*

are naturally identified with points in Π). Prove that Π^* is also a projective plane (called the dual plane to Π).

Problem 3. Let Π be an incidence geometry which satrisfies (PP1) but does not satisfy (PP2). What can you say about it?

Problem 4. Consider an incidence geometry satisfying the betweenness axioms $B1 - B4$. Let A, B, C be points on a line l such that $A * B * C$ (i.e. B is between A and C). Let t_A, t_B, t_C be lines through A, B, C respectively such that t_A and t_B are parallel and distinct and t_C and t_B are parallel and distinct. Prove that t_A and t_C are parallel. Let m be a line intesecting lines t_A, t_B, t_C at points X, Y, Z respectively. Prove that $X * Y * Z$. Hint: Use sides of the line t_B.

The following problem is optional, for extra credit.

Problem 5. Let Π be an incidence geometry with finitely many points. Prove that the number of lines cannot be smaller than the number of points.