Solutions to Homework 1

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Solution to Problem 7.1. Let l be the line containing A, B, C, D.

a) Suppose that $A * B * C$ and $B * C * D$. Then A and C are on opposite sides of B on the line l and C and D are on the same side of B on l. Thus A and D are on opposite sides of B on l, i.e. $A * B * D$.

Similarly, A and B are on the same side of C on the line l and B and D are opposite sides of C on l. Thus A and D are on opposite sides of C on l, i.e. $A*C*D$.

b) Suppose that $A * B * D$ and $B * C * D$. Then A and D are on opposite sides of B on the line l and C and D are on the same side of B on l. Thus A and C are on opposite sides of B on l, i.e. $A * B * C$. This means that A, B are on the same side of C on l. Since B and D are on opposite sides of C on l, we see that A and D are on opposite sides of C on l. Thus we have $A * C * D$.

Solution to Problem 7.10. Since D is in the interior of $\angle BAC$, The ray $\overrightarrow{AD}$ intersects the segment $\overline{BC}$ at some point P by the crossbar theorem. Thus $B*P*C$ and $P \neq D$. Since P is on the ray $\overrightarrow{AD}$, either $A * P * D$ or $A * D * P$. Since A and D are on the same side of the line BC, $A * P * D$ is not possible. Thus we have $A * D * P$. This means that the line l has a point between A and P. From Pasch axiom applied to the triangle $\triangle APC$ we see that either l contains one of the points A, C, P or it has a point between A and C or it has a point between P and C. In each case l meets one of the sides of $\triangle ABC$.

Problem 2. An incidence geometry is called a **projective plane** if it satifies the following 2 axioms:

(PP1) any two lines intersect.

(PP2) there exist 4 points, no three of which are on one line.

Let Π be a projective plane.

a) Prove that every line in Π has at least 3 points.

b) Let l, m be two lines in Π. Prove that there is a point A not belonging to either

l or m. Use lines through A to construct a bijection between points on l and points on m.

c) Suppose that Π is finite. Then, by a) and b), there is $n \geq 2$ such that every line has exactly $n+1$ points. Prove that Π has n^2+n+1 points and n^2+n+1 lines.

Choose a line l in Π.

d) Let Π_1 be the set of all points in Π which are not on l. Call a subset of Π_1 a line if it is equal to an intersection of a line in Π with Π_1. Prove that Π_1 is an affine plane. (This allows to prove c) by using results about finite affine planes)

The following two parts are for extra credit.
e) Do problem 6.7 a) from Hartshorn's book (which shows that conversly, any affine plane can be extended to a projective plane by adding "the line at infinity").

f) Let Π be a projective plane. Consider the set Π^* of all lines in Π. A line in Π^* is defined as a set of all lines in Π passing through a given point of Π (so lines in Π^* are naturally identified with points in Π). Prove that Π^* is also a projective plane (called the dual plane to Π).

Solution. a) Let l be a line in Π and let A, B, C, D be 4 points, no three of which are collinear (they exist by PP2). One of these 4 points is not on l. We may assume that A is not on l. The three lines AB, AC, AD are distinct and each of them intersects l. Since the only point any two of these 3 lines share is A, they intersect l at different points. Thus we get at least 3 different points on l.

b) If $l = m$, we can take for A any point not on l. If $l \neq m$, take a point X on l but not on m and a point Y on m but not on l. The line XY has a third point A (by a)) which is neither on l nor on m.

We define a function $f : l \longrightarrow m$ as follows. For $P \in l$ the line AP intersects the line m at a unique point Q. We define $f(P) = Q$. Similarly, we define a function $g : m \longrightarrow l$ as follows. For $Q \in m$ the line AQ intersect the line l at a unique point P. We define $g(Q) = P$. It is clear that $gf(P) = P$ for any $P \in l$ and $fg(Q) = Q$ for ant $Q \in m$. Thus f and g are inverse of each other bijections.

c) If Π is finite, then b) implies that all lines have the same number of points $n+1$, and $n \geq 2$ by a). Now take a line l and a point A not on l. Let $B_1, \dots, B_{n+1}$ be the points on l. If $P \neq A$ is any point in Π then the line AP intersects l at some point B_i. Thus $AP = AB_i$. This shows two things:

1. There are exactly $n+1$ different lines containing A.
2. Every point in Π different from A belongs to exactly one of the lines AB_1, ..., AB_{n+1}

Now we can count the points in Π. Every line AP_i contributes n points different from A. Thus we get $n(n+1)$ points different from A and consequently Π has $1 + n(n+1) = n^2 + n + 1$ points.

To count the lines, note that by 1. above we know that there are exactly $(n+1)$ lines passing through every point in Π. Counting these lines for every point and adding the results yields $(n+1)(n^2+n+1)$. Since every line has $n+1$ points, we counted every line exactly $n+1$ times. So the number of lines in Π is $(n+1)(n^2+n+1)/(n+1) = n^2 + n + 1$.

d) We define lines in Π_1 to be the sets of the form $m \cap \Pi_1$, where m is a line in Π, $m \neq l$. Let us verify the incidence axioms $I1 - I3$. If A, B are distinct points in Π_1 then there is unique line m in Π which contains A and B, hence $m \cap \Pi_1$ is the unique line in Π_1 which contains A and B. This verifies I1. Let $m \cap \Pi_1$ be a line in Π_1, where $m \neq l$ is a line in Π. Then m has at least 3 points and only one of them is on l. Thus $m \cap \Pi_1$ has at least two points. This verifies I2. Finally, if A is not on l and B, C are distinct points on l then line AB has a third point B_1, and line AC has a third point C_1. It is clear that A, B_1, C_1 three points in Π_1 and they are not collinear in Π, hence also not collinear in Π_1. This verifies I3. Thus Π_1 is an incidence geometry.

We need to show that Π_1 satisfies the strong Playfair axiom. Suppose that $m \cap \Pi_1$ is a line in Π_1 and A is a point in Π_1. The line m intersects l at some point Q by PP1. If $n \cap \Pi_1$ is a line in Π_1 passing through A then n intersects l at some point P and then $n = AP$. The lines n and m intersect at some point R. If $R \in l$ then

$R = P = Q$ and the lines $m \cap \Pi_1$ and $n \cap \Pi_1$ do not intersect, i.e. parallel in Π_1. If $R \notin l$, then R is in Π_1 and the lines $m \cap \Pi_1$, $n \cap \Pi_1$ are not parallel. Thus $AQ \cap \Pi_1$ is the only line in Π_1 parallel to $m \cap \Pi_1$. This shows that Π_1 satisfies the strong Playfair's axiom, i.e. Π_1 is an affine plane.

Using d) and the results from class about finite affine planes we can give a different solution to part c). We proved that a finite affine plane is of some order n and then it has n^2 points and $n^2 + n$ lines. Thus Π_1 has this property. Since the lines in Π_1 are in bijection with lines in Π different from l, we see that Π has n^2+n+1 lines. Also, Π is a disjoint union of Π_1 and l, so Π has n^2+n+1 points.

e) Suppose now that Π is an affine plane. Then the relation **parallel** on the set of lines is an equivalence relation. Let L be the set of equivelence classes of this relation. We define Π' to be the union of Π and L. A subset of Π' is a line if either it is equal to $l \cup [l]$ where l is a line in Π and $[l]$ is the equivalence class of l or it is euqal to L. Thus we added to each line l in Π a new "ideal" point $[l]$ and we added one new line which consists of all ideal points. We claim that Π' is a projective plane. In fact, if A, B are two distinct points in Π' then:

- if both A and B are in Π then the line $AB \cup [AB]$ is the uniqueline in Π' containing A and B,
- if both A and B are in L then L is the unique line in Π' containing both A and B.
- if $A \in \Pi$ and $B \in L$ then a line $l \cup \{[l]\}$ contains A if and only if $A \in l$ and it contains B if and only if $B = [l]$. Thus l muts be a line through A which belongs to class B and there is unique such line l by Playfair's axiom.

This shows that I1 holds. If A, B, C are three non-collinear points in Π then no two of the lines AB, AC, BC are parallel so we get three ideal points $[AB], [AC], [BC]$. So the line L has at least 3 points and every other line also has at least two points (since it is equal to $l \cup \{[l]\}$ for some line l in Π and l has at least 2 points). So I2 holds.

Now we will check PP2. Note that PP2 always implies I3. Take two distinct points A, B in Π. Since L has at least 3 points, there are 2 points on L different from $[AB]$, call them $[l]$ and $[m]$, where l, m are lines in Π. No three of the points $A, B, [l], [m]$ are collinear.

Finally, we need to shaw that Π' satisfies PP1. Every line in Π' has a point in L, so L intersects every other line. If $l \cup [l]$ and $m \cup [m]$ are two lines in Π' different from L then

- if l and m intersect in Π then $l \cup [l]$ and $m \cup [m]$ intersect in Π';
- if l and m do not intersect in Π, then they are parallel and $[l] = [m]$. Thus $l \cup [l]$ and $m \cup [m]$ intersect in Π';

Thus any two lines in Π' intersect, i.e. Π' satisfies PP1. This completes our proof that Π' is a projective space.

f) Take any two distinct points l, m in Π^*. This means that l, m are lines in Π so they intersect in a unique point A. The set of all lines in Π passing through A is the unique line in Π^* which contains l and m. Thus $\Pi*$ satisfies I1. Since in Π thare are at least 3 lines passing through any given point, every line Π^* has at least 3 points. In particuler, I2 holds in Π^*. Let A, B, C, D be 4 points in Π such that no three of them are collinear. Then the lines AB, BC, CD, AD are four points in Π^* and since no 3 of these lines share a point, this means that no 3 of the points AB, BC, CD, AD are collinear in Π^*. This shows that Π^* satisifes PP2, so also I3. Finall consider two lines L_1, L_2 in Π^*. This means that there are two points A, B in Π such that L_1 consists of all lines in Π through A and L_2 consists of all lines in Π through B. If $A = B$ then $L_1 = L_2$. If $A \neq B$, then the line AB is a point in Π^* which is on L_1 and L_2. Thus L_1 and L_2 intersect. This proves that Π^* satisfies PP1. Thus Π^* is a projective plane.

Remark. Note that $(\Pi^*)^*$ is naturally identified with Π.

Problem 3. Let Π be an incidence geometry which satisfies (PP1) but does not satisfy (PP2). What can you say about it?

Solution. Suppose that Π does not satisfy the axiom PP2. Let A_1, A_2, A_3 be three non-collinear points in Π (they exist by the axiom I3). If there are no other points in Π then $\Pi = \{A_1, A_2, A_2\}$ and the family of lines in Π is

$$\mathcal{L} = \{\{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}\}.$$

Suppose that there is another point A_4. If A_4 is not on any of the lines A_1A_2, A_1A_3, A_2A_3 then no three of the points A_1, A_2, A_3, A_4 are collinear, so PP2 holds, contrary to our assumption. Thus any point in Π must be on one of the lines A_1A_2, A_1A_3, A_2A_3. Without any loss of generality, we may assume that A_4 is on the line A_2A_3, $A_4 \neq A_2$ and $A_4 \neq A_3$. If there is another point A_5 which is not on the line A_2A_3, then A_5 is on the line A_1A_2 or the line A_1A_3. Suppose A_5 is on A_1A_3. Thus A_5 is not on A_1A_4 and not on A_1A_2. Thus no three of the points A_1, A_2, A_4, A_5 are collinear, contrary to our assumption that Π does not satisfy PP2. Similarly, we get a contradiction if A_5 is on A_1A_2. Thus A_5 can not exist, i.e. all other points in Π must be on the line A_2A_3. If $A_2, A_3, \ldots, A_k$ are all the points on the line A_2A_3, then $\Pi = \{A_1, A_2, \ldots, A_k\}$ and the family $\mathcal{L}$ of lines in Π is

$$\mathcal{L} = \{\{A_1, A_2\}, \{A_1, A_3\}, \ldots, \{A_1, A_k\}, \{A_2, A_3, \ldots, A_k\}\}.$$

Problem 4. Consider an incidence geometry satisfying the betweenness axioms $B1 - B4$. Let A, B, C be points on a line l such that $A * B * C$ (i.e. B is between A and C). Let t_A, t_B, t_C be lines through A, B, C respectively such that t_A and t_B are parallel and distinct and t_C and t_B are parallel and distinct. Prove that t_A and t_C are parallel. Let m be a line intesecting lines t_A, t_B, t_C at points X, Y, Z respectively. Prove that $X * Y * Z$. Hint: Use sides of the line t_B.

Solution. We consider the sides of the line t_B. Since $A * B * C$ and $B \in t_B$, the points A and C are on opposite sides of the line t_B. Let $P \neq A$ be any point on the line t_A. The segment $\overline{AP}$ is contained in the line t_A and t_A does not intersect t_B. It follows that $\overline{AP}$ and t_B have no points in common. Thus A and P are on the same side of the line t_B. This shows that the line t_A is contained in the A-side of the line t_B. In exactly the same way we show that the line t_C is contained in the C-side of the line t_B. Since t_A, t_C are contained in different sides of t_B, lines t_A, t_C can not have any points in common, i.e. t_A and t_C are parallel.

Now the point X is on the line t_A, so X and A are on the same side of t_B. Similarly, Z and C are on the same side of t_B. Thus X and Z are on the opposite sides of

t_B. Thus the segment $\overline{XZ}$ must intersect the line t_B. Since Y is the only point of intersection of the lines $XZ = m$ and t_B, Y belongs to the segment $\overline{XZ}$. Thus $X * Y * Z$.