Solutions to Homework 2

When you read the solutions, please draw your own diagrams to visualize the discussion.

Solution to 8.2. Suppose that $\overline{AE} \not\equiv \overline{CF}$. We will show that this leads to a contradiction. Either $\overline{AE} > \overline{CF}$ or $\overline{AE} < \overline{CF}$. Without loss of generality, we may assume that $\overline{AE} > \overline{CF}$. Then there is a point K such that $A*K*E$ and $\overline{AK} \equiv \overline{CF}$. By subtraction,

$$\overline{KB} \equiv \overline{FD} \equiv \overline{CF} \equiv \overline{AK}.$$

On the other hand, we have $A*K*E$ and $K*E*B$, so $\overline{AK} < \overline{AE}$ and $\overline{EB} < \overline{KB}$. Since $\overline{AE} \equiv \overline{EB}$, we see that $\overline{AK} < \overline{KB}$. This contradicts our earlier observation that $\overline{AK} \equiv \overline{KB}$.

Solution to 8.7. Consider the cartesian plane $\mathbb{R}^2$. We have seen that it has the notion of betweenness induced from the order of the real numbers. Now for any two distinct points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ define $d(A, B) = |a_1 - b_1| + |a_2 - b_2|$. We define congruence of segments as follows: $\overline{AB} \equiv \overline{CD}$ if $d(A, B) = d(C, D)$.

The axiom C2 (congruence is an equivalence relation) is immediate: if $d(A, B) = d(C, D)$ and $d(C, D) = d(E, F)$ then clearly $d(A, B) = d(E, F)$. The reflexibility and symmetry are even easier.

For the axiom C3, we first make the following observation: if a, b, c are real numbers and b is weakly between a and c (this means that either $a \leq b \leq c$ or $a \geq b \geq c$) then $|a-c| = |a-b| + |b-c|$. Indeed, if $a \leq b \leq c$ then $|a-b| = b-a$, $|b-c| = c-b$, and $|a-c| = c-a = (c-b) + (b-a)$. Similar argument works when $a \geq b \geq c$.

Assume now that $A = (a_1, a_2)$, $B = (b_1, b_2)$ and $C = (c_1, c_2)$ satisfy $A*B*C$. This implies that A, B, C are collinear and b_1 is weakly between a_1 and c_1 and b_2 is weakly between a_2 and c_2. From the observation above we see that $d(A, C) = d(A, B) + d(B, C)$. This immediately implies the axiom C3.

To verify C1 consider a segment $\overline{AB}$, where $A = (a_1, a_2)$, $B = (b_1, b_2)$. Let $\overrightarrow{CD}$ be a ray, where $C = (c_1, c_2)$, $D = (d_1, d_2)$. Recall that the the points on the ray are exactly the points of the form $P_t = (c_1 + t(d_1 - c_1), c_2 + t(d_2 - c_2))$ for some

$t \geq 0$. Now $d(C, P_t) = td(C, D)$. It follows that there is unique $t > 0$ such that $d(C, P_t) = d(A, B)$, namely $t = d(A, B)/d(C, D)$. In other words there is unique point P_t on the ray $\overrightarrow{CD}$ such that $\overline{CP_t} \equiv \overline{AB}$.

The circle with center $(0, 0)$ and radius 1 consists of all points (a, b) such that $|a| + |b| = 1$. It is the boundary of the square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$.

Solutiuon to 10.6 Consider a triangle $\triangle ABC$ and assume that $\overline{AC} > \overline{AB}$. Then there is a point D between A and C such that $\overline{AD} \equiv \overline{AB}$. The triangle $\triangle BAD$ is then isosceles, so $\angle ABD \equiv \angle ADB$. Since the points A and C are on opposite sides of the point D on line AC, the angles $\angle ADB$ and $\angle BDC$ are supplementary. Thus $\angle ADB > \angle BDC = \angle ACB$. It follows that $\angle ABD > \angle ACD$. Now the points D and A are on the same side of the line BC since we hve $A * D * C$. Similarly, the points D and C are on the same side of the line AB. This means that D is in the interior of the angle $\angle ABC$. It folllows that $\angle ABC > \angle ABD$. Thus

$$\angle ABC > \angle ABD > \angle ACD$$

which completes our proof of Proposition 18.

Solutiuon to 10.9 Consider triangles $\triangle ABC$ and $\triangle A'B'C'$ such that the angles $\angle ABC$ and $\angle A'B'C'$ are right, $\overline{AB} \equiv \overline{A'B'}$, and $\overline{AC} \equiv \overline{A'C'}$. On the line BC there is a point C_1 such that $C_1 * B * C$ and $\overline{C_1B} \equiv \overline{C'B'}$. By the SAS, the triangles $\triangle ABC_1$ and $\triangle A'B'C'$ are congruent (we have $\overline{C_1B} \equiv \overline{C'B'}$, $\overline{AB} \equiv \overline{A'B'}$, and both angles $\angle ABC_1$ and $A'B'C'$ are right; note that $\angle ABC_1$ is suplementary to the right angle $\angle ABC$). It follows that $\angle AC_1B \equiv \angle A'C'B'$ and $\overline{AC_1} \equiv \overline{A'C'} \equiv \overline{AC}$. Thus the triangle C_1AC is isosceles and consequently $\angle ACB \equiv \angle AC_1B$. Therefore $\angle ACB \equiv \angle A'C'B'$. By the side-angle-angle property, the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Solutiuon to 10.10 This problem was not formulated precisely enough. The meaning of *quadrilateral* is not clear, though the picture suggests that it is assumed to be a convex quadrilateral, or at least that the opposite sides do not intersect. We will assume the latter. Consider the line AC. First we show that B and D are on opposite sides of the line AC. Suppose contrary, that they are on the same side. Note that $\triangle ACD \equiv \triangle CAB$ by SSS. Therefore $\angle ACB \equiv \angle CAD$ and $\angle CAB \equiv \angle ACD$.

If $\angle ACD \equiv \angle ACB$ then the lines CD and CB coincide and the lines AB and AD coincide. Thus $B = D$, which is not possible. Therefore the angles $\angle ACD$ and $\angle ACB$ are not congruent, so one is larger that the other. Withouit loss of generality, assume $\angle ACD < \angle ACB$. Then $\angle CAB < \angle CAD$. The point D is in the interior of the angle $\angle ACB$ (since $\angle ACD < \angle ACB$) so the ray $\overrightarrow{CD}$ intersects the segment $\overline{AB}$ at some point E. If we had $C * D * E$ then D would be in the interior of the triangle $\triangle ACB$. In particular, D would be in the interior of the angle $\angle CAB$, which is not possible as $\angle CAB < \angle CAD$. Thus we have $C * E * D$ or $E = D$. This however means that the opposite sides $\overline{AB}$ and $\overline{CD}$ intersect, contrary to our assumption.

We showed that B and D are on opposite sides of the line AC. Since $\triangle ACD \equiv \triangle CAB$ by SSS, $\angle BAC \equiv \angle DCA$. Now the line AC falls on the lines CD and AB and the angles $\angle BAC \equiv \angle DCA$ are alternate (since B and D are on opposite sides of the line AC) and congruent. Thus the lines CD and AB are parallel by Euclid's Proposition 27.