Homework

due on Wednesday, March 4

Read carefully sections 11,12, 19, 20 of Hartshorne's book. Solve problem 11.5, 11.6, 20.2 and the following problems:

Problem 1. A subset S of a plane satisfying incidence and betweenness axioms is called **convex** if for any two points X, Y in S the whole segment $\overline{XY}$ is contained in S. In clas we proved that a side of any line is a convex set.

a) Prove that intersection of (any family of) convex sets is convex.

b) Prove that the interior of an angle and the interior of a triangle are convex (use a)).

Problem 2. In any Hilbert plane, prove that the interior of a circle is convex. Hint: Prove first that if ABC is a triangle and X is between B and C then either $\overline{AX} < \overline{AB}$ or $\overline{AX} < \overline{AC}$. You may use propositions 2-27 from book 1 of Euclid.

Problem 3. Let P be the set of congruence classes of segments in a Hilbert Plane which satisfies Playfair's axiom (P). We define multiplication on P by selecting first one element and call it 1. Suppose that we select a different element c and define another multiplication in the same way but with c playing a role of one. Denote this new multiplication by $*$. Prove that for any $a, b \in P$ we have $a * b = abc^{-1}$, where on the right we have the original multiplication.

Problem 4. Let Π be a Hilbert plane which does not satisfy the Archimedes axiom (A). Thus there exist segments $\overline{AB}$ and $\overline{AQ}$ such that $n\overline{AB} < \overline{AQ}$ for every natural number n. Consider the set Π_A which consists of A and all points P in Π for which there exists a natural number m such that $\overline{AP} < m\overline{AB}$ (we call such points finitely bounded from A). Call a subset l of Π_A a line if it is non-empty and there is a line L in Π such that $l = L \cap \Pi_A$.

a) Prove that Π_A with the lines defined above is an incidence geometry.

b) Prove that Π_A does not satisfy the parallel postulate (P) (hint: note first that for any $P \in \Pi_A$ the line PQ in Π intersected with Π_A is a line in Π_A through P and all these lines are parallel to each other.

c) Define betweenness in Π_A as follows: Y is between X and Z in Π_A if the same holds when we consider them as points in Π. Prove thet the betweenness axioms are satisfied for Π_A.

d) Define two segments $\overline{XY}$ and $\overline{KL}$ in Π_A to be congruent if the segments $\overline{XY}$ and $\overline{KL}$ in Π are congruent. Similarly, two angles $\angle XYZ$ and $\angle KLM$ are congruent in Π_A if the angles $\angle XYZ$ and $\angle KLM$ are congruent in Π. Prove that Π_A satisfies the congruence axioms. It follows that Π_A is a Hilbert plane which does not satisfy the parallel axiom.