Solutions to Homework 3

Solve problem 11.5, 11.6, 20.2 and the following problems:

Solution to 11.5 Let x, y, z be congruence classes of line segments such that $x+y > z$, $x+z > y$, $y+z > x$. Consider a line l and a point A on it. There is D on l such that $|\overline{AD}| = x$. Take B on opposite side of A on l than the point D and such that $|\overline{AB}| = y$. Thus we have $D * A * B$. Take E on l on opposite side of B than A and such that $|\overline{BE}| = z$. Thus $A * B * E$. Consider the circles $\Gamma_1(A, x)$ and $\Gamma_2(B, z)$. Clearly $D \in \Gamma_1$. Since $\overline{BD} = \overline{BA} + \overline{AD} = y + x > z$, the point D is in the exterior of the circle Γ_2. The circle Γ_1 intersects the line l at D and another point H such that $D * F * H$. We have $|\overline{AH}| = x$. If $A * H * B$ then $|\overline{HB}| = y - x < z$ so H is in the interior of the circle Γ_2. If $A * B * H$ then $|\overline{HB}| = x - y < z$ so again H is in the interior of the circle Γ_2. Finally, if $H = B$ then clearly H is in the interior of the circle Γ_2. We see that the circle $Gamma_1$ has the point D in the exterior of Γ_2 and the point H in the interior of Γ_2. It follows that these circles can not be tangent. By the circle-circle intersection property (E), the circles Γ_1 and Γ_2 intersect at some point C, and C is not on the line l (otherwise the circles would be tangent). Thus ABC is a triangle, $|\overline{AB}| = y$, $|\overline{AC}| = x$, $|\overline{BC}| = z$.

Solution to 11.6 Let $\triangle ABC$ be a triangle. For reasons which will be clear later, we assume that $\overline{AB}$ is the shortest of the sides of the triangle ABC. Let p_A, p_B, p_C be the angle bisectors of the angles $\angle BAC$, $\angle ABC$, $\angle ACB$ respectively.

Our first step is to prove that the lines p_A and p_B intersect. By the definition of the angle bisector, the line p_A contains a point X in the interior of the angle $\angle BAC$. By the crossbar theorem, the ray $\overrightarrow{AX}$ intersects the interior of the segment $\overline{BC}$ at some point Y. Consider now the triangle $\triangle BYA$. Note that p_B is the angle bisector of $\angle YBA$. Thus, using the same reasoning as above, we see that p_B intersects the interior of the segment $\overline{AY}$ at some point O. Thus the lines p_A and p_B intersect at O. Note that O is in the interior of the triangle $\triangle ABC$ (as O is on the A-side of the line BC, and it is in the interior of the angle $\angle BAC$).

Let M_A, M_B, M_C be the perpendicular projections of the point O on the lines BC, AC, AB respectively. As the angle $\angle OAB$ is acute (being half of some angle), the projection of O on the line AB belongs to the ray $\overrightarrow{AB}$ (if it was on the opposite ray, then $\angle OAM_C$ would be supplementary to the angle $\angle OAB$ and therefore $\angle OAM_C$ would be obtuse; however the right triangle $\triangle OM_CA$ can not have an obtuse angle.) Similarly, as the angle $\angle OBA$ is acute, the projection of O on the line AB belongs to the ray $\overrightarrow{BA}$. It follows that M_C belongs to the intersection of the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$, i.e. M_C belongs to the interior of the segment $\overline{BC}$ (since M_C is neither A nor B). We can use similar argument to show that M_B is in the interior of the ray $\overrightarrow{AC}$ and M_A in the interior of the ray $\overrightarrow{BC}$.

Consider now the triangles $\triangle AOM_C$ and $\triangle AOM_B$. We have $\angle OAM_C \equiv \angle OAM_B$ (as AO is the angle bisector of $\angle M_CAM_B$). Also, $\angle OM_CA \equiv \angle OM_BA$ (both are right angles). Finally, the triangles share side $\overline{AO}$. By AAS, the triangle $\triangle AOM_C$ is congruent to the triangle $\triangle AOM_B$. It follows that $\overline{OM_B} \equiv \overline{OM_C}$ and $\overline{AM_B} \equiv \overline{AM_C}$. Note that

$$\overline{AC} \geq \overline{AB} > \overline{AM_C} \equiv \overline{AM_B}$$

(here we are using the assumption that $\overline{AB}$ is the shortest side and that M_C is inside $\overline{AB}$, which we proved earlier). It follows that M_B is inside $\overline{AC}$.

Applying similar reasoning to the triangles $\triangle BOM_C$ and $\triangle BOM_A$, we prove that $\overline{OM_A} \equiv \overline{OM_C}$ and M_A is inside $\overline{BC}$.

Consider now the triangles $\triangle COM_A$ and $\triangle COM_B$. These are right-angle triangles which share the hypotenuse $\overline{OC}$. We proved that $\overline{OM_A} \equiv \overline{OM_C} \equiv \overline{OM_B}$. Thus, by $SSRA$ (see Problem 10.9 from homework 2), the triangles $\triangle OCM_A$ and $\triangle OCM_B$ are congruent. It follows that $\angle OCM_A \equiv \angle OCM_B$, so the line CO is the angle bisector p_C. This completes our proof that the three angle bisectors intersect at one point O. Moreover, the circle with center O and radius $\overline{OM_C}$ is tangent to the sides AB, AC, BC at the points M_C, M_B, M_A respectively. This circle is called the **incircle** of the triangle $\triangle ABC$ (or the circle insribed in the triangle) and the point O is called the **incenter** of the triangle $\triangle ABC$.

Solution to 20.2 The problem is to prove the following result (in a Hilbert's plane which satisfies (P)):

Angle Bisector Theorem. Let $\triangle ABC$ be a triangle and let $D \in \overline{BC}$ be a point such that AD is an angle bisector of the angle $\angle BAC$. Then

$$\frac{|\overline{AB}|}{|\overline{AC}|} = \frac{|\overline{DB}|}{|\overline{DC}|}.$$

Note that the point D exists by the cross-bar theorem. There is a point E on line AB such that $B*A*E$ (E is on opposite side of A than B on line AB) and $\overline{AE} \equiv \overline{AC}$. The triangle $\triangle CAE$ is isosceles, so $\angle ACE \equiv \angle AEC$. Furtharmore $\angle ACE + \angle AEC \equiv \angle BAC$ (which is the angle supplementary to the angle $\angle EAC$). Since AD is the angle bisector of $\angle BAC$, we have $\angle BAD \equiv \angle CAD$ and $\angle BAC + \angle CAD \equiv \angle BAC$. We see that $\angle BAD \equiv \angle AEC = \angle BEC$ (both angles beeing half of $\angle BAC$). By the AAA criterion for similarity, the triangles $\triangle ABD$ and $\triangle EBC$ are similar. Thus

$$\frac{|\overline{EB}|}{|\overline{AB}|} = \frac{|\overline{BC}|}{|\overline{BD}|}.$$

Since $|\overline{EB}| = |\overline{AB}| + |\overline{AE}| = |\overline{AB}| + |\overline{AC}|$ and $|\overline{BC}| = |\overline{BD}| + |\overline{CD}|$, we see that

$$\frac{|\overline{AB}| + |\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{BD}| + |\overline{CD}|}{|\overline{BD}|},$$

which implies that

$$\frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{CD}|}{|\overline{BD}|}$$

which is exactly what we needed to prove.

Problem 1. A subset S of a plane satisfying incidence and betweenness axioms is called **convex** if for any two points X, Y in S the whole segment $\overline{XY}$ is contained in S. In clas we proved that a side of any line is a convex set.

a) Prove that intersection of (any family of) convex sets is convex.

b) Prove that the interior of an angle and the interior of a triangle are convex (use a)).

Solution. a) Suppose that E_i, $i \in I$ is a collection of convex subsets of Π and let $E = \bigcap_{i \in I} E_i$ be their intersection. Take any two distinct points A, B in E. Then, for every $i \in I$, points A, B are in E_i. Since E_i is convex, the whole segment $\overline{AB}$

is contained in E_i. We showed that $\overline{AB}$ is contained in E_i for every $i \in I$, so it is contained in E. Thus E is convex.

b) We know from class that a side of a line is convex. Let $\angle BAC$ be an angle. The interior of $\angle BAC$ is the intersection of B-side of the line AC and the C-side of the line AB. Since intersection of convex sets is convex by a), we see that the interior of an angle is convex.

Let now $\triangle ABC$ be a triangle. The interior of $\triangle ABC$ is the intersection of A-side of line BC, B-side of line AC and C-side of line AB, so it is convex by a).

Problem 2. In any Hilbert plane, prove that the interior of a circle is convex. Hint: Prove first that if ABC is a triangle and X is between B and C then either $\overline{AX} < \overline{AB}$ or $\overline{AX} < \overline{AC}$. You may use propositions 2-27 from book 1 of Euclid.

Solution. Let A, B be two distinct points in the interior of the circle $\Gamma(O, r)$ and let $A * X * B$. It is a simple observation that for any point Z on the line AB the segment $\overline{AB}$ is contained in the union $\overline{AZ} \cup \overline{BZ}$. Thus if O is on the line AB then $X \in \overline{OA}$ or $X \in \overline{OB}$. In the former case, $\overline{OX} \leq \overline{OA} < r$ so X is in the interior of Γ. Same argument applies to the latter case.

Suppose now that O is not on the line AB. Then consider the triangle $\triangle AOB$. Using the hint, we show that either $\overline{OA} > \overline{OX}$ or $\overline{OB} > \overline{OX}$. Without loss of generality we may assume that $\angle OAB \geq \angle OBA$. Then $\angle OXB > \angle OAX = \angle OAB$, since exterior angle $\angle OXB$ in the triangle $\triangle OXA$ is larger than the interior angle $\angle OAX$. Therefore $\angle OXB > \angle OBA = \angle OBX$. Therefore $r > \overline{OB} > \overline{OX}$. Thus X is in the interior of the circle Γ. We showed that every point of $\overline{AB}$ is in the interior of Γ, hence the interior of Γ is convex.

Problem 3. Let P be the set of congruence classes of segments in a Hilbert Plane which satisfies Playfair's axiom (P). We define multiplication on P by selecting first one element and call it 1. Suppose that we select a different element c and define another multiplication in the same way but with c playing a role of one. Denote this new multiplication by $*$. Prove that for any $a, b \in P$ we have $a * b = abc^{-1}$, where on the right we have the original multiplication.

Solution. Consider a right triangle ABC such that $\angle ABC$ is right, $|\overline{AB}| = c$ and $|\overline{BC}| = a$. There are right triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that:

1. $\angle A_1B_1C_1$ and $\angle A_2B_2C_2$ are right;

2. $\angle B_1A_1C_1 \equiv \angle BAC \equiv \angle B_2A_2C_2$;

3. $|\overline{A_1B_1}| = 1$ and $|\overline{A_2B_2}| = b$.

It follows from definition of $*$ that $|\overline{B_2C_2}| = a * b$. On the other hand, if $|\overline{B_1C_1}| = x$ then $|\overline{BC}| = xc$ and $|\overline{B_2C_2}| = xb$. Thus $a * b = xb$ and $a = xc$. Thus $x = bc^{-1}$ and $a * b = (bc^{-1})a = abc^{-1}$.

Problem 4. Let Π be a Hilbert plane which does not satisfy the Archimedes axiom (A). Thus there exist segments $\overline{AB}$ and $\overline{AQ}$ such that $n\overline{AB} < \overline{AQ}$ for every natural number n. Consider the set Π_A which consists of A and all points P in Π for which there exists a natural number m such that $\overline{AP} < m\overline{AB}$ (we call such points finitely bounded from A). Call a subset l of Π_A a line if it is non-empty and there is a line L in Π such that $l = L \cap \Pi_A$.

a) Prove that Π_A with the lines defined above is an incidence geometry.

b) Prove that Π_A does not satisfy the parallel postulate (P) (hint: note first that for any $P \in \Pi_A$ the line PQ in Π intersected with Π_A is a line in Π_A through P and all these lines are parallel to each other.

c) Define betweenness in Π_A as follows: Y is between X and Z in Π_A if the same holds when we consider them as points in Π. Prove thet the betweenness axioms are satisfied for Π_A.

d) Define two segments $\overline{XY}$ and $\overline{KL}$ in Π_A to be congruent if the segments $\overline{XY}$ and $\overline{KL}$ in Π are congruent. Similarly, two angles $\angle XYZ$ and $\angle KLM$ are congruent in Π_A if the angles $\angle XYZ$ and $\angle KLM$ are congruent in Π. Prove that Π_A satisfies the congruence axioms. It follows that Π_A is a Hilbert plane which does not satisfy the parallel axiom.

Solution. Our key tool will the triangle inequality:

for any three points A, B, C we have $\overline{AB} \le \overline{AC} + \overline{BC}$

which holds in any Hilbert's plane.

a) Let P, R be two distinct points in Π_A. There is unique line L in Π containing P and R and $l = L \cap \Pi_A$ is the unique line in Π_A containing P and Q. Thus I1 holds in Π_A.

Let l be a line in Π_A. This means that there is a line L in Π such that $l = L \cap \Pi_A$ and l is non-empty. Thus there is a point $P \in l$. By definition, we have $\overline{AP} < m\overline{AB}$ for some positive integer m. There is a point R on L such that $\overline{PR} \equiv \overline{AB}$ (since Π satisfies C1). By the triangle inequality,

$$\overline{AR} \leq \overline{AP} + \overline{PR} < m\overline{AB} + \overline{AB} = (m+1)\overline{AB}.$$

Thus R is in Π_A, so $R \in l$. This shows that l has at least 2 points. Thus I2 holds in Π_A.

Clearly A and B are in Π_A. Take a point D in Π which is not on the line AB. There is a point C on the ray $\overrightarrow{AD}$ such that $\overline{AC} \equiv \overline{AB}$. Clearly C is in Π_A and the points A, B, C are not collinear. Thus I3 holds in Π_A.

b) We can assume that the point Q is on the line AB (by C1). There is a point Q_1 on the line AB such that $\overline{QQ_1} \equiv \overline{AB}$. We claim that Q_1 is not in Π_A. Otherwise we would have $\overline{AQ_1} < m\overline{AB}$ for some positive integer m. But then

$$\overline{AQ} \leq \overline{AQ_1} + \overline{QQ_1} < m\overline{AB} + \overline{AB} = (m+1)\overline{AB}$$

which is false. We showed in a) that there is a point $C \in \Pi_A$ not on the line AB. The lines CQ and CQ_1 in Π are different, so C is the only point they share. Thus the lines $l_1 = CQ \cap \Pi_A$ and $l_2 = CQ_1 \cap \Pi_A$ in Π_A are distinct, both contain C and both are parallel in Π_A to the line $AB \cap \Pi_A$. Thus (P) fails in Π_A.

c) Clearly $B1$ and $B3$ hold in Π_A since they hold in Π. To check $B2$ consider two points $C \neq D$ in Π_A. There is a point E in Π such that $C * D * E$ and $\overline{DE} \equiv \overline{AB}$. Since $D \in \Pi_A$, we have $\overline{AD} < m\overline{AB}$ for some positive integer m. Now

$$\overline{AE} \leq \overline{AD} + \overline{DE} < m\overline{AB} + \overline{AB} = (m+1)\overline{AB}$$

so $E \in \Pi_A$. Thus $B2$ holds in Π_A. Finally consider three non-collinear points C, D, E in Π_A and a line l in Π_A which has a point between C and D. There is a line L in Π such that $l = L \cap \Pi_A$. By $B4$ in Π, the line L has a point F between C and E or between D and E. Suppose F is between C and E (the case when F is between D and E is handled in the same way). Since $C * F * E$, we have $\overline{CF} < \overline{CE}$. There are integers $m > 0$, $n > 0$ such that $\overline{AC} < m\overline{AB}$ and $\overline{AE} < n\overline{AB}$. Now

$$\overline{AF} \leq \overline{AC} + \overline{CF} < \overline{AC} + \overline{CE} \leq \overline{AC} + \overline{AC} + \overline{AE} < (2m + n)\overline{AB}.$$

This proves that $F \in \Pi_A$ so F is on l. We showed that l has a point between C and E. Thus B4 holds in Π_A.

d) Consider a segment $\overline{CD}$ and a ray $\overrightarrow{EF}$ in Π_A. There is unique point G on the ray $\overrightarrow{EF}$ in Π such that $\overline{EG} \equiv \overline{CD}$. Since C, D, E are in Π_A, we have

$$\overline{AC} < m\overline{AB},\ \overline{AD} < n\overline{AB},\ \overline{AE} < k\overline{AB}$$

for some positive integers m, n, k. Now

$$\overline{AG} \leq \overline{AE} + \overline{EG} = \overline{AE} + \overline{CD} \leq \overline{AE} + \overline{AC} + \overline{AD} < (m + n + k)\overline{AB}.$$

Thus $G \in \Pi_A$, which proves that C1 holds in Π_A. Since C2, C3, C5, C6 hold in Π, they also hold in Π_A. To verify C4, consider an angle $\angle DCE$, a ray $\overrightarrow{FG}$ and a side of the line FG in Π_A. Note that the sides of a line $L \cap \Pi_A$ in Π_A are just intersections of the sides of L in Π with the set Π_A. There is unique ray $\overrightarrow{FX}$ in the chosen side of FG in Π such that $\angle DCE \equiv \angle XFG$. The line $XF \cap \Pi_A$ has a point Y different from X. If $Y * F * X$, there is $H \in \Pi_A$ such that $Y * F * H$. Otherwise, set $H = Y$. Thus $H \in \Pi_A$ and $\overrightarrow{FX} = \overrightarrow{FH}$ in Π. We see that $\overrightarrow{FH}$ is the unique ray on the chosen side of FG in Π_A such that $\angle DCE \equiv \angle HFG$. Thus C4 holds in Π_A.