Homework
due on Wednesday, March 18

Read carefully sections 19, 20, 5 of Hartshorne's book. Solve problems 4.7 (use multiplication instead of area), 5.5, 5.19, 5.20, 20.4, and the following problems:

Problem 1. The altitudes of a triangle ABC intersect at a point H. Let O_A be the circumcenter of the triangle BCH. Similarly define O_B and O_C. Prove that the segments AO_A, BO_B, CO_C share a common midpoint. What is this point? Conclude that the triangles ABC and $O_AO_BO_C$ are congruent.

Hint: 1) What is the orthocenter of BCH?

2) What can you say about the nine-point circles of the triangles ABC, ABH, BCH, ACH?

b) Prove that H is the circumcenter of $O_AO_BO_C$ and that the circumcenter of ABC is the orthocenter of $O_AO_BO_C$.

c) Prove that the Euler lines of the triangles ABC, ABH, BCH, ACH intersect at one point.

Problem 2. Consider a convex quadrilateral $ABCD$ inscribed in a circle with center O such that AD is the diameter, $\angle AOB = 2y$, $\angle AOC = 2x$, $\angle COD = 2z$. Explain how to use Ptolemy's theorem to prove that

$$\sin(x - y) = \sin x \cos y - \sin y \cos x$$

and

$$\cos(y + z) = \cos y \cos z - \sin y \sin z.$$

Problem 3. a) The lines AA_1, BB_1 and CC_1 intersect in one point O. Let the lines AB and A_1B_1 intersect at C_2, the lines AC and A_1C_1 intersect at B_2, and the lines BC and B_1C_1 intersect at A_2. Prove that the points A_2, B_2, C_2 are collinear. This is often called Desarques Theorem. Hint. Apply Menelaus' Theorem to triangles OAB, OBC, OAC and appropriate lines. Then apply its converse to the triangle ABC.

b) Points A_1, B_1, C_1 are collinear and so are points A_2, B_2, C_2. The lines A_1B_2 and A_2B_1 intesect at a point C, the lines A_1C_2 and A_2C_1 intesect at a point B, and the lines B_1C_2 and B_2C_1 intesect at a point A. Prove that the points A, B, C are collinear. This is Pappus' Theorem. Hint. Let A_0, B_0, C_0 be the vertices of the triangle determined by the lines A_1B_2, B_1C_2, and C_1A_2 (where A_0 is the point of intersection of A_1B_2 and A_2C_1, etc.). Apply Menalaus' Theorem to the triangle $A_0B_0C_0$ and five appropriate lines.

Problem 4. In class we proved the following theorem.

Theorem. Let ABC be a triangle and let P be a point whose orthogonal projections on the sides of the triangle are K, L, M. Then the points K, L, M are collinear if and only if P is on the circumscribed circle of the triangle ABC. The line through K, L, M is then called **the Simson line of** P.

Using this result solve the following problem.

Let A, B, C be three collinear points and let P be a point outside the line through A, B, C. Prove that the circumcenters of the triangles PAB, PAC, PBC and the point P lie on a circle. Hint: Note that if two circles intesect at two points X, Y then the line joining the centers of the circles is the perpendicular bisector of XY. Consider the triangle with vertices at the circumcenters. What are the projections of P on the sides of this triangle?