Solutions to Homework 4

Problem 1. he altitudes of a triangle ABC intersect at a point H. Let O_A be the circumcenter of the triangle BCH. Similarly define O_B and O_C. Prove that the segments AO_A, BO_B, CO_C share a common midpoint. What is this point? Conclude that the triangles ABC and $O_AO_BO_C$ are congruent.

Hint: 1) What is the orthocenter of BCH?

2) What can you say about the nine-point circles of the triangles ABC, ABH, BCH, ACH?

b) Prove that H is the circumcenter of $O_AO_BO_C$ and that the circumcenter of ABC is the orthocenter of $O_AO_BO_C$.

c) Prove that the Euler lines of the triangles ABC, ABH, BCH, ACH intersect at one point.

Solution. Let Γ be the 9-point circle of the triangle $\triangle ABC$ and let T be its center. Then T is the midpoint of $\overline{OH}$, where O is the circumcenter of $\triangle ABC$. The midpoints of segments $\overline{AH}$, $\overline{BH}$, $\overline{CH}$, $\overline{AB}$, $\overline{AC}$, $\overline{BC}$ are on Γ.

The 9-point circle of triangle $\triangle ABH$ contains the midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{AB}$ so it coincides with Γ. Similarly Γ is the 9-point circle of $\triangle ACH$ and $\triangle BCH$.

Note that the line AC is perpendicular to BH so AC is an altitude of $\triangle ABH$. Similarly BC is an altitude of $\triangle ABH$. Tus C is the orthocenter of $\triangle ABH$. Since T is the 9-point center of $\triangle ABH$, T is the midpoint of $\overline{CO_C}$. In the same way we show that T is the midpoint of $\overline{BO_B}$ and $\overline{AO_A}$.

Let us make the following general observation: if M is the midpoint of $\overline{XY}$ and $\overline{UW}$ then $\overline{XU} \equiv \overline{YW}$ and the lines XU, YW are parallel (since the triangles $\triangle XUM$ and $\triangle YWM$ are congruent).

From the above observation we see that $\overline{AB} \equiv \overline{O_AO_B}$, $\overline{AC} \equiv \overline{O_AO_C}$, and $\overline{BC} \equiv \overline{O_BO_C}$. Thus triangles $\triangle ABC$ and $\triangle O_AO_BO_C$ are congruent by sss. This proves part a)

Using our observation again, we see that $\overline{OA} \equiv \overline{HO_A}$, $\overline{OB} \equiv \overline{HO_B}$, and $\overline{OC} \equiv \overline{HO_C}$. Since $\overline{OA} \equiv \overline{OB} \equiv \overline{OC}$, we see that $\overline{HO_A} \equiv \overline{HO_B} \equiv \overline{HO_C}$. Thus H is the circumcenter of $\triangle O_AO_BO_C$.

Note that both O and O_A are on the perpendicular bisector of $\overline{BC}$. Since BC and O_BO_C are parallel (by our general observation above), the perpendicular bisector of $\overline{BC}$ passes through O, O_A and is perpendicular to O_BO_C. Thus the perpendicular bisector of $\overline{BC}$ is the altitude of the triangle $\triangle O_AO_BO_C$ from vertex O_A. Similarly, the perpendicular bisector of $\overline{AC}$ is the altitude of the triangle $\triangle O_AO_BO_C$ from vertex O_B and the perpendicular bisector of $\overline{AB}$ is the altitude of the triangle $\triangle O_AO_BO_C$ from vertex O_C. It follows that O is the orthocenter of $\triangle O_AO_BO_C$ (another way to see this is to observe that Γ is also the 9-point circle of $\triangle O_AO_BO_C$). This proves part b).

Since the Euler line contains the 9-point center, we see that T is on the Euler lines of triangles $\triangle ABC$, $\triangle ABH$, $\triangle BCH$, $\triangle ACH$ (and $\triangle O_AO_BO_C$). This proves part c).

Remark. Since the triangles $\triangle ABC$ and $\triangle O_AO_BO_C$ are congruent, they have the same circumradius, so $\overline{O_AH} \equiv \overline{OB} \equiv \overline{OC}$. Since $\overline{O_AB} \equiv \overline{O_AC} \equiv \overline{O_AH}$, we see that O_A is the reflection of O in line BC. Similarly, O_B is the reflection of O in line AC and O_C is the reflection of O in the line AB.

Problem 2. Consider a convex quadrilateral $ABCD$ inscribed in a circle with center O such that AD is the diameter, $\angle AOB = 2y$, $\angle AOC = 2x$, $\angle COD = 2z$. Explain how to use Ptolemy's theorem to prove that

$$\sin(x-y) = \sin x \cos y - \sin y \cos x$$

and

$$\cos(y+z) = \cos y \cos z - \sin y \sin z.$$

Solution. The key observation to relate trigonometry to the chords in a circle is as follows: If points S, T are on a circle with center O and radius R then $2R \sin\alpha = |\overline{ST}|$, where α is half the angle $\angle SOT$. To see this let M be the midpoint of $\overline{ST}$. Then $\triangle OMS$ is right and $\angle MOS = \alpha$. Thus $\overline{OS} \sin\alpha = \overline{SM}$.

We apply the above observation to the quadrilateral $ABCD$. Note that $\overline{AD} = 2R$ and $x+z$ is the right angle. Thus $\cos x = \sin z$ and $\cos z = \sin x$ and $\cos(y+z) = \sin(x-y)$. By our observation above:

$$\overline{BD} = \overline{AD} \sin(\frac{1}{2}\angle BOD) = \overline{AD} \cos y,$$

$$\overline{AB} = \overline{AD} \sin y,$$

$$\overline{AC} = \overline{AD} \sin x = \overline{AD} \cos z,$$

$$\overline{CD} = \overline{AD} \sin z = \overline{AD} \cos x,$$

$$\overline{BC} = \overline{AD} \sin(x-y) = \overline{AD} \cos(y+z).$$

By Ptolemy's theorem: $\overline{AB} \cdot \overline{DC} + \overline{AD} \cdot \overline{BC} = \overline{AC} \cdot \overline{BD}$. Thus

$$\left(\overline{AD} \sin y\right) \left(\overline{AD} \cos x\right) + \overline{AD} \left(\overline{AD} \sin(x-y)\right) = \left(\overline{AD} \sin x\right) \left(\overline{AD} \cos y\right)$$

which simplifies to

$$\sin y \cos x + \sin(x-y) = \sin x \cos y, \text{ i.e. } \sin(x-y) = \sin x \cos y - \cos x \sin y.$$

Since $\cos(y+z) = \sin(x-y)$, $\sin x = \cos z$, $\cos x = \sin z$, we get

$$\cos(y+z) = \cos y \cos z - \sin y \sin z.$$

Problem 3. a) The lines AA_1, BB_1 and CC_1 intersect in one point O. Let the lines AB and A_1B_1 intersect at C_2, the lines AC and A_1C_1 intersect at B_2, and the lines BC and B_1C_1 intersect at A_2. Prove that the points A_2, B_2, C_2 are collinear. This is often called Desarques Theorem. Hint. Apply Menelaus' Theorem to triangles OAB, OBC, OAC and appropriate lines. Then apply its converse to the triangle ABC.

b) Points A_1, B_1, C_1 are collinear and so are points A_2, B_2, C_2. The lines A_1B_2 and A_2B_1 intesect at a point C, the lines A_1C_2 and A_2C_1 intesect at a point B, and the lines B_1C_2 and B_2C_1 intesect at a point A. Prove that the points A, B, C are collinear. This is Pappus' Theorem. Hint. Let A_0, B_0, C_0 be the vertices of the triangle determined by the lines A_1B_2, B_1C_2, and C_1A_2 (where A_0 is the point of intersection of A_1B_2 and A_2C_1, etc.). Apply Menalaus' Theorem to the triangle $A_0B_0C_0$ and five appropriate lines.

Solution. a) We start with perhaps the hardest part of the argument: showing that either none or exactly two of the points A_2, B_2, C_2 are on the sides of the triangle $\triangle ABC$. To this end, we may assume that C_2 is between A and B. Since C_2 is on the line A_1B_1, the points A and B are on opposite sides of the line A_1B_1. The point O is either on the A-side of A_1B_1 or on the B-side. In the former case, B_1 is between O and B and A_1 is not between O and A. In the latter case, A_1 is between O and A and B_1 is not between O and B. Thus we see that exactly one holds: either A_1 is between O and A or B_1 is between O and B. We assume that A_1 is between O and A and B_1 is not between O and B. (the other case is handled in the same way).

Assume that B_2 is between A and C and A_2 is between B and C. The same argument as above gives us that exactly one holds: either A_1 is between O and A or C_1 is between O and C. Since A_1 is between O and A, the point C_1 is not between O and C. Thus neither C_1 is between O and C nor B_1 is between O and B. But then A_2 is not between B and C, a contradiction. This shows that we can not have both: B_2 between A and C and A_2 between B and C . Therefore if B_2 is between A and C then A_2 is not between B and C and exactly two of the points A_2, B_2, C_2 are on the sides of the triangle $\triangle ABC$.

Assume B_2 is not between A and C. Since B_2 is on the line A_1C_1, The points A and C are on the same side of the line A_1C_1. Since A_1 is between O and A, the points O and A are on opposite sides of the line A_1C_1. Thus the points O and C are on opposite sides of the line A_1C_1. This means that C_1 is between O and C. Thus O and C are on opposite sides of the line B_1C_1. Since B_1 is not between O and B, the points O and B are on the same side of the line B_1C_1. Thus B and C are on opposite side of B_1C_1. This means that A_2 is between B and C. We see that again exactly two of the points A_2, B_2, C_2 are on the sides of the triangle $\triangle ABC$. This completes our argument.

Remark. There is a conceptually nicer and faster way to prove that either none or exactly two of the points A_2, B_2, C_2 are on the sides of the triangle $\triangle ABC$. For any two points K, L and a point X on the line KL define $\epsilon_{KL}(X) = 1$ if X is between K and L and $\epsilon_{KL}(X) = -1$ if X is not on the segment $\overline{KL}$. Then if $\triangle KLM$ is a triangle abd X, Y, Z are points on the lines KL, KM, LM respectively and different from the vertices, then $\epsilon_{KL}(X)\epsilon_{KM}(Y)\epsilon_{LM}(Z) = -1$ if and only if either none or exactly two of the point X, Y, Z are on the sides of the triangle $\triangle KLM$. Thus, from Menelaus Theorem applied to the triangle $\triangle OAB$ and the collinear points A_1, C_2, B_1 we have

$$\epsilon_{OA}(A_1)\epsilon_{AB}(C_2)\epsilon_{OB}(B_1) = -1.$$

Similarly, we have

$$\epsilon_{OA}(A_1)\epsilon_{AC}(B_2)\epsilon_{OC}(C_1) = -1$$

and

$$\epsilon_{OB}(B_1)\epsilon_{BC}(A_2)\epsilon_{OC}(C_1) = -1.$$

Multiplying these three equalities, we get

$$\epsilon_{AB}(C_2)\epsilon_{BC}(A_2)\epsilon_{AC}(B_2) = -1.$$

We can now finish our solution. By Menelaus Theorem, in order to prove that A_2, B_2, C_2 are collinear, it suffices now to prove that

$$\frac{|\overline{AC_2}|}{|\overline{C_2B}|}\frac{|\overline{BA_2}|}{|\overline{A_2C}|}\frac{|\overline{CB_2}|}{|\overline{B_2A}|} = 1. \tag{1}$$

By Menelaus Theorem applied to the triangle $\triangle OAB$ and the collinear points A_1, C_2, B_1 we have

$$\frac{|\overline{OA_1}|}{|\overline{A_1A}|}\frac{|\overline{AC_2}|}{|\overline{C_2B}|}\frac{|\overline{BB_1}|}{|\overline{B_1O}|} = 1. \tag{2}$$

Similarly, by Menelaus Theorem applied to the triangle $\triangle OBC$ and the collinear points B_1, A_2, C_1 we have

$$\frac{|\overline{OB_1}|}{|\overline{B_1B}|}\frac{|\overline{BA_2}|}{|\overline{A_2C}|}\frac{|\overline{CC_1}|}{|\overline{C_1O}|} = 1. \tag{3}$$

By Menelaus Theorem applied to the triangle $\triangle OCA$ and the collinear points C_1, B_2, A_1 we have

$$\frac{|\overline{OC_1}|}{|\overline{C_1C}|}\frac{|\overline{CB_2}|}{|\overline{B_2A}|}\frac{|\overline{AA_1}|}{|\overline{A_1O}|} = 1. \tag{4}$$

Multiplying the equalities (2), (3), (4) we get (1). This completes our proof that A_2, B_2, C_2 are collinear.

Remark. If the lines AA_1, BB_1, CC_1 are parallel to each other, the conclusion of the problem also holds (one can use the degenerate Menelaus theorem discussed in the solution to part b) below). Also, if AB and A_1B_1 are parallel, then the line B_2C_2 is parallel to them (if B_2, C_2 are well defined). If also AC and A_1C_1 are parallel, then we must have BC parallel to B_1C_1.

b) Let us assume that no two of the lines A_1B_2, B_1C_2, C_1A_2 are parallel. Let A_0 be the point where A_1B_2 and C_1A_2 intersect, B_0 the point where A_1B_2 and B_1C_2 intersect, C_0 the point where C_1A_2 and B_1C_2 intersect. Let T be the triangle $\triangle A_0B_0C_0$. For any point X on the line A_0B_0 different from A_0 and B_0 set $\epsilon(X) = 1$ if X is between A_0 and B_0 and set $\epsilon(X) = -1$ otherwise. Similarly define ϵ for points Y of B_0C_0 and points Z on A_0C_0. Then Menelaus theorem can be stated as follows:

points X, Y, Z are collinear if and only if $\epsilon(X)\epsilon(Y)\epsilon(Z) = -1$ and

$$\frac{|\overline{A_0X}|}{|\overline{XB_0}|}\frac{|\overline{B_0Y}|}{|\overline{YC_0}|}\frac{|\overline{C_0Z}|}{|\overline{ZA_0}|} = 1.$$

We will apply this result to 5 lines:

line through C, B_1, A_2:

$$\epsilon(C)\epsilon(B_1)\epsilon(A_2) = -1 \text{ and } \frac{|\overline{A_0C}|}{|\overline{CB_0}|}\frac{|\overline{B_0B_1}|}{|\overline{B_1C_0}|}\frac{|\overline{C_0A_2}|}{|\overline{A_2A_0}|} = 1$$

line through A_1, C_2, B:

$$\epsilon(A_1)\epsilon(C_2)\epsilon(B) = -1 \text{ and } \frac{|\overline{A_0A_1}|}{|\overline{A_1B_0}|}\frac{|\overline{B_0C_2}|}{|\overline{C_2C_0}|}\frac{|\overline{C_0B}|}{|\overline{BA_0}|} = 1$$

line through B_2, A, C_1:

$$\epsilon(B_2)\epsilon(A)\epsilon(C_1) = -1 \text{ and } \frac{|\overline{A_0B_2}|}{|\overline{B_2B_0}|}\frac{|\overline{B_0A}|}{|\overline{AC_0}|}\frac{|\overline{C_0C_1}|}{|\overline{C_1A_0}|} = 1$$

line through A_1, B_1, C_1:

$$\epsilon(A_1)\epsilon(B_1)\epsilon(C_1) = -1 \text{ and } \frac{|\overline{A_1B_0}|}{|\overline{A_0A_1}|}\frac{|\overline{B_1C_0}|}{|\overline{B_0B_1}|}\frac{|\overline{C_1A_0}|}{|\overline{C_0C_1}|} = 1$$

line through B_2, C_2, A_2:

$$\epsilon(B_2)\epsilon(C_2)\epsilon(A_2) = -1 \text{ and } \frac{|\overline{B_2B_0}|}{|\overline{A_0B_2}|}\frac{|\overline{C_2C_0}|}{|\overline{B_0C_2}|}\frac{|\overline{A_2A_0}|}{|\overline{C_0A_2}|} = 1$$

Multiplying these 5 equalities we get

$$\epsilon(C)\epsilon(B)\epsilon(A) = -1 \text{ and } \frac{|\overline{A_0C}|}{|\overline{CB_0}|}\frac{|\overline{B_0A}|}{|\overline{AC_0}|}\frac{|\overline{C_0B}|}{|\overline{BA_0}|} = 1$$

Menelaus theorem implies now that A, B, C are collinear. This completes our proof of Pappus' Theorem in the case when no two of the lines A_1B_2, B_1C_2, C_1A_2 are parallel.

Suppose now that the lines B_1C_2 and C_1A_2 are parallel and the line A_1B_2 intersects them at B_0, A_0 respectively. In this case the following "degenerate" version of Menalaus theorem can be used in the same way as above.

Degenerate Menelaus Theorem. Let l_A, l_B be parallel lineas passing through points A_0, B_0 respectively. Pick a side of the line A_0B_0 and set $\epsilon(Y) = 1$ if Y is on the chosen side of A_0B_0 and $\epsilon(Y) = -1$ if Y is on the opposite side. Then points X on A_0B_0, Y on l_B, Z on l_A are collinear if and only if $\epsilon(X)\epsilon(Y)\epsilon(Z) = -1$ and

$$\frac{|\overline{A_0X}|}{|\overline{XB_0}|}\frac{|\overline{B_0Y}|}{|\overline{ZA_0}|} = 1.$$

We leave the details to the reader.

If all three lines A_1B_2, B_1C_2, C_1A_2 are parallel to each other then consider the lines A_1C_2, C_1B_2, B_1A_2 instead. Note that they can not be all parallel to each other.

Problem 4. In class we proved the following theorem.

Theorem. Let ABC be a triangle and let P be a point whose orthogonal projections on the sides of the triangle are K, L, M. Then the points K, L, M are collinear if and only if P is on the circumscribed circle of the triangle ABC. The line through K, L, M is then called **the Simson line of** P.

Using this result solve the following problem.

Let A, B, C be three collinear points and let P be a point outside the line through A, B, C. Prove that the circumcenters of the triangles PAB, PAC, PBC and the point P lie on a circle. Hint: Note that if two circles intesect at two points X, Y then the line joining the centers of the circles is the perpendicular bisector of XY. Consider the triangle with vertices at the circumcenters. What are the projections of P on the sides of this triangle?

Solution. Let Γ_A, Γ_B, Γ_C be the circumcircles of the triangles $\triangle BCP$, $\triangle ACP$, $\triangle ABP$ respectively. Let O_A, O_B, O_C be the circumcenters of $\triangle BCP$, $\triangle ACP$, $\triangle ABP$ respectively. The circles Γ_A and Γ_B intersect at two points P and C. Thus the line O_AO_B is the perpendicular bisector of $\overline{PC}$. It follows that the perpendicular projection of P on the line O_AO_B is the midpoint M_C of $\overline{PC}$. Similarly, the perpendicular projection of P on the line O_AO_C is the midpoint M_B of $\overline{PB}$ and the perpendicular projection of P on the line O_BO_C is the midpoint M_A of $\overline{PA}$ (this in particular shows that the points O_A, O_B, O_C can not be collinear). The line M_AM_B is parallel to AB and the line M_AM_C is parallel to AC. Since $AB = AC$, we see that M_A, M_B, M_C are collinear. By the Simson's line theorem, P is on the circumcircle of the triangle $O_A O_B O_C$.

Solution to problem 4.7. Let F_1 be the second point where the line DE intersects the circumcircle of $\triangle ABC$. Since $\triangle ADE$ is equilateral we have

$$a = |\overline{DE}| = |\overline{AE}| = |\overline{EC}| = |\overline{AD}| = |\overline{DB}|.$$

Let $x_1 = |\overline{DF_1}|$, $x = |\overline{EF}|$. Then

$$x_1(|\overline{DE}| + x) = a^2 = x(|\overline{DE}| + x_1).$$

This implies that $x = x_1$ and $x(|\overline{DE}| + x) = a^2$. Since $a = |\overline{DE}|$, $x = |\overline{EF}|$ and $|\overline{DE}| + x = |\overline{DF}|$, the result follows.

Solution to problem 5.5. Let the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{AD}$ be K, L, M, N respectively. Then KL is parallel to AC and $2\overline{KL} \equiv \overline{AC}$. Similarly, Then MN is parallel to AC and $2\overline{MN} \equiv \overline{AC}$. It follows that the segments $\overline{KL}$ and $\overline{MN}$ are congruent and parallel, In the same way we show that the segments $\overline{LM}$ and $\overline{KN}$ are congruent and parallel. This means that $KLMN$ is a parallelogram.

Solution to problem 5.19. If the points O, C, P are collinear then $C = D$ and there is nothing to prove. So we may assume that no three of the points O, C, D, P are collinear. The key observation on which we base our solution is that it suffices to show that the segments $\overline{AB}$ and $\overline{CD}$ have the same midpoint.

Let M be the midpoint of the segment $\overline{CD}$, let N be the midpoint of $\overline{CP}$ K the midpoint of $\overline{OD}$, and L the midpoint of $\overline{OP}$. The line MN joins midpoints of two sides in the triangle $\triangle CPD$, so it is parallel to the third side PD. Similarly, the line MK is parallel to OC. Since PD and OC are parallel (both being perpendicular to AB), the lines MN and MK are parallel, i.e. the points M, N, K are collinear. Similarly, the line LN is parallel to OC, so L is also on the line containing M, N, K. This shows that the line LM is perpendicular to AB (as it is parallel to OC).

Since $\angle OAP$ and $\angle OBP$ are right, the points A and B are on the circle with diameter $\overline{OP}$, whose center is L. Thus the perpendicular bisector of $\overline{AB}$ contains L. In other words, if M_1 is the midpoint of $\overline{AB}$ then the line LM_1 is perpendicular to AB, and so is the line LM. This means that $M = M_1$, which completes our argument.

Solution to problem 5.20. We will use different notation that the one in the problem. Let H be the orthocenter of $\triangle ABC$, let H_A, H_B, H_C be the feet of altitudes from A, B, C respectively. Let H_{AC}, H_{AB} be the perpendicular projections of H_A on the sides AC, AB respectively. Similarly we define H_{BA}, H_{BC}, H_{CA}, H_{CB}. Our goal is to show that the points $H_{AC}, H_{AB}, H_{BC}, H_{BA}, H_{CA}, H_{CB}$ are on one circle. We assume that the triangle $\triangle ABC$ is not right, otherwise there is nothing to prove.

Note that $\triangle AH_AB$ is right and H_AH_{AB} is the altitude in this triangle. It follows that H_{AB} is between A and B. In the same way we show that all the points $H_{AC}, H_{AB}, H_{BC}, H_{BA}$, H_{CA}, H_{CB} are on the sides of the triangle $\triangle ABC$ (the subscript is the side on which the point is).

Note that the triangles $\triangle AH_{CA}H_C$ and $\triangle AH_BB$ are similar by aaa. In fact, $H_{CA}H_C$ and BH_B are both perpendicular to AC, so they are parallel. It follows that the two triangles have corresponding angles congruent. This means that $|\overline{AH_{CA}}|/|\overline{AH_B}| = |\overline{AH_C}|/|\overline{AB}|$, i.e.

$$|\overline{AH_{CA}}| = \frac{|\overline{AH_B}| \cdot |\overline{AH_C}|}{|\overline{AB}|} \tag{5}$$

Similarly, the triangles $\triangle AH_{BA}H_B$ and $\triangle AH_CC$ are similar, which gives us

$$|\overline{AH_{BA}}| = \frac{|\overline{AH_C}| \cdot |\overline{AH_B}|}{|\overline{AC}|} \tag{6}$$

From (5) and (6) we get

$$\frac{|\overline{AH_{CA}}|}{|\overline{AH_{BA}}|} = \frac{|\overline{AC}|}{|\overline{AB}|}. \tag{7}$$

By sas, the last equality means that the triangles $\triangle AH_{BA}H_{CA}$ and $\triangle ABC$ are similar. Consequently, the lines $H_{BA}H_{CA}$ and BC are parallel.

Note now that the triangles $\triangle AH_{AC}H_A$ and $\triangle AH_BH$ are similar by aaa. In fact, $H_{AC}H_A$ and HH_B are both perpendicular to AC, so they are parallel. It follows that the two triangles have corresponding angles congruent. This means that $|\overline{AH_{AC}}|/|\overline{AH_B}| = |\overline{AH_A}|/|\overline{AH}|$, i.e.

$$|\overline{AH_{AC}}| = \frac{|\overline{AH_B}| \cdot |\overline{AH_A}|}{|\overline{AH}|}. \tag{8}$$

Similarly, the triangles $\triangle AH_{AB}H_A$ and $\triangle AH_CH$ are similar and

$$|\overline{AH_{AB}}| = \frac{|\overline{AH_C}| \cdot |\overline{AH_A}|}{|\overline{AH}|}. \tag{9}$$

From (8) and (9) we get

$$\frac{|\overline{AH_{AC}}|}{|\overline{AH_{AB}}|} = \frac{|\overline{AH_B}|}{|\overline{AH_C}|}. \tag{10}$$

The triangles $\triangle AH_BB$ and AH_CC are similar by aaa. Thus $|\overline{AH_B}|/|\overline{AH_C}| = |\overline{AC}|/|\overline{AB}|$. From (10) we get

$$\frac{|\overline{AH_{AC}}|}{|\overline{AH_{AB}}|} = \frac{|\overline{AB}|}{|\overline{AC}|}. \tag{11}$$

By sas, the last equality means that the triangles $\triangle AH_{AB}H_{AC}$ and $\triangle ACB$ are similar. Thus

$$\angle AH_{AB}H_{AC} \equiv \angle ACB \text{ and } \angle AH_{AC}H_{AB} \equiv \angle ABC. \tag{12}$$

Multiplying (7) and (11) yields

$$|\overline{AH_{AC}}| \cdot |\overline{AH_{CA}}| = |\overline{AH_{AB}}| \cdot |\overline{AH_{BA}}|. \tag{13}$$

This implies that the points $H_{AC}, H_{AB}, H_{CA}, H_{BA}$ are on one circle.

So far we only focused on vertex A. Repeating the above arguments for vertex B tells us that the points $H_{BC}, H_{AB}, H_{BA}, H_{CB}$ are on one circle too and

$$\angle BH_{BA}H_{BC} \equiv \angle BCA \text{ and } \angle BH_{BC}H_{BA} \equiv \angle BAC. \tag{14}$$

Repeating the above arguments for vertex C tells us that the points $H_{CB}, H_{CA}, H_{AC}, H_{BC}$ are on one circle and the lines $H_{BC}H_{AC}$ and AB are parallel.

Looking at the quadrilateral $H_{AB}H_{AC}H_{BC}H_{BA}$ we know that $H_{BC}H_{AC}$ and $H_{AB}H_{BA} = AB$ are parallel, $\angle AH_{AB}H_{AC} \equiv \angle ACB \equiv \angle BH_{BA}H_{BC}$ (by (12) and (14)). By the equality of alternate angles in parallel lines, we see that $\angle H_{AB}H_{AC}H_{BC} \equiv \angle ACB \equiv \angle H_{BA}H_{BC}H_{AC}$. We have two possibilities for the points H_{AB}, H_{BA} on the segment $\overline{AB}$: either H_{BA} is between A and H_{AB} or H_{AB} is between A and H_{BA}. In the former case, H_{BA} and H_{AC} are on the same side of the line $H_{AB}H_{BC}$ and $\angle H_{AB}H_{AC}H_{BC} \equiv \angle BH_{BA}H_{BC} = \angle H_{AB}H_{BA}H_{BC}$. Thus the points $H_{AB}, H_{AC}, H_{BC}, H_{BA}$ are on one circle. In the latter case, H_{BA} and H_{AC} are opposite sides of the line $H_{AB}H_{BC}$, $\angle H_{AB}H_{AC}H_{BC} \equiv \angle BH_{BA}H_{BC}$, and $\angle H_{AB}H_{BA}H_{BC}$ is supplementary to $\angle BH_{BA}H_{BC}$. Again, this implies that the points $H_{AB}, H_{AC}, H_{BC}, H_{BA}$ are on one circle.

Let us summarize what we proved so far:
- a) the points $H_{AB}, H_{AC}, H_{BC}, H_{BA}$ are on one circle
- b) the points $H_{AC}, H_{AB}, H_{CA}, H_{BA}$ are on one circle
- c) the points $H_{BC}, H_{AB}, H_{BA}, H_{CB}$ are on one circle.

By a) and b) all five points $H_{AB}, H_{AC}, H_{BC}, H_{BA}$ are on the circumcircle of the triangle $\triangle H_{AB}, H_{AC}, H_{BA}$. This and c) implies that the sixth point H_{CB} is also on this circle. This completes our solution.

Remark. We could avoid the argument showing that the points $H_{AB}, H_{AC}, H_{BC}, H_{BA}$ are on one circle as follows. After showing that
- a) the points $H_{AC}, H_{BC}, H_{CB}, H_{CA}$ are on one circle Γ_1
- b) the points $H_{AC}, H_{AB}, H_{CA}, H_{BA}$ are on one circle Γ_2
- c) the points $H_{BC}, H_{AB}, H_{BA}, H_{CB}$ are on one circle Γ_3

we show that these three circles coincide. Note that if two of these circles coincide then all three are the same circle. Assume that no two of them coincide. Then the radical axis of Γ_1 and Γ_2 is the line CA, and the radical axis of Γ_1 and Γ_3 is the line CB. Thus C has the same power with respect to all three circles, so C is on the radical axis of Γ_2 and Γ_3, which is the line AB. This is clearly false.

Solution to problem 20.4. Let the two circles intersect at points A and B. If P is between A and B then P is in the interior of both circles and the power of P with respect to any of the circles is $-|\overline{PA}| \cdot |\overline{PB}|$. If P is on the line AB but not in the segment $\overline{AB}$, then P is in the exterior of both circles and the power of P with respect to any of the circles is $|\overline{PA}| \cdot |\overline{PB}|$. If $P = A$ or $P = B$ then the power of P with respect to any of the circles is 0. Thus all points on the line AB have the same power with respect to each of the two circles.

Suppose now that P is a point different from A, B such that P has the same power with respect to both circles. Let the line PA intersect one of the circles at a second point B_1 and the the other circle at the second point B_2. If P has the same power with respect to both circles then $|\overline{PA}| \cdot |\overline{PB_1}| = |\overline{PA}| \cdot |\overline{PB_2}|$. Thus $|\overline{PB_1}| = |\overline{PB_2}|$. In addition, either the power of P is negative and then P is between A and B_1 and between A and B_2, or the power of P is positive and then both B_1 and B_2 are on the ray $\overrightarrow{PA}$. In both cases, the equality $|\overline{PB_1}| = |\overline{PB_2}|$ implies that $B_1 = B_2$. We can not have $B_1 = B_2 = A$, because it would imply that PA is tangent to both circles at A, and then the circles would be tangent to. Therefore we must have $B_1 = B_2 = B$. Thus P is on the line AB.