

Quizzes for Math 465

QUIZ 1. a) What does it mean that Π is an incidence geometry?

b) State Pasch's Axiom.

c) Define an interval \overline{AB} in an incidence geometry with betweenness.

Solution: a) When we say that Π is an incidence geometry, we mean that Π is a set (whose elements are called points) and there is a family \mathcal{L} of subsets of Π , whose elements are called lines, which satisfies the following conditions:

I1. Any two distinct points belong to a unique line.

I2. Any line contains at least 2 points.

I3. There exist three points which do not belong to the same line.

b) **B4 (Pasch's Axiom).** Let A, B, C be three distinct point in an incidence geometry with betweenness and let l be a line not containing any of A, B, C . If l contains a point between A and B then l also contains a point between A and C or a point between B and C .

c) Let A, B be two distinct points in an incidence geometry with betweenness. The segment \overline{AB} consists of all the points between A and B together with the points A and B :

$$\overline{AB} = \{P : P = A \text{ or } P = B \text{ or } A * P * B\}.$$

QUIZ 2. a) State the Plane Separation Theorem.

b) State the Crossbar Theorem

c) Define the interior of a triangle $\triangle ABC$.

Solution. a) **Plane Separation Theorem.** Let Π be an incidence geometry with betweenness and let l be a line in Π . Then the set $\Pi - l$ is divided into two non-empty, disjoint subsets, called the sides of l , such that:

i) Two points $A, B \in \Pi - l$ are on the same side of l if and only if either $A = B$ or the segment \overline{AB} does not have points on the line l .

ii) Two points $A, B \in \Pi - l$ are on opposite sides of l if and only if $A \neq B$ and there is a point $P \in l$ such that $A * P * B$ (in other words, the the segment \overline{AB} has a points on the line l).

b) **Crossbar Theorem.** Let $\angle BAC$ be an angle and let P be a point in the interior of the angle $\angle BAC$. Then the ray \overrightarrow{AP} intersects the segment \overline{BC} .

c) The interior of a triangle $\triangle ABC$ consists of all points P such that P and A are on the same side of the line BC , P and B are on the same side of the line AC , and P and C are on the same side of the line AB .

QUIZ 3. a) What does it mean that $\angle BAC < \angle LKM$?

b) What does it mean that $\angle BAC$ is a right angle?

c) State the Exterior Angle Theorem. Can a triangle have two right angles?

Solution: a) We say that $\angle BAC < \angle LKM$ if there is a point N in the interior of the angle $\angle LKM$ such that the angles $\angle LKN$ and $\angle BAC$ are congruent.

b) We say that an angle $\angle BAC$ is a right angle if it is congruent to its supplementary angle $\angle DAC$, where D is a point such that $D * A * B$.

c) **Exterior Angle Theorem.** Let $\triangle ABC$ be a triangle. Then the angle supplementary to the angle $\angle BAC$ is greater than each of the other two angles $\angle ABC$, $\angle ACB$ in the triangle.

The answer to the last question is No. Suppose that the angle $\angle BAC$ in a triangle $\triangle ABC$ is a right angle. Then the supplementary angle to $\angle BAC$ is also a right angle. By the Exterior Angle Theorem, the remaining two angles in the triangle are smaller than a right angle, hence they cannot be right.

QUIZ 4. a) State the Circle-Circle Intersection Property.

b) Complete the statement of the following result:

A line l is tangent to a circle $\Gamma(O, r)$ at a point P if and only if ...

c) Let $\Gamma(O, r)$ be a circle in a Hilbert's plane which satisfies the Playfair's axiom (P). Suppose that A, B, C are distinct points on Γ and \overline{AB} is not a diameter of Γ . What are the possible relations between $\angle ACB$ and $\angle AOB$? State your answer as follows:

Half of the angle $\angle AOB$ is ...

Solution. a) **Circle-Circle Intersection Property.** If Γ_1, Γ_2 are two circles in a Hilbert's plane and Γ_1 has points both in the interior of Γ_2 and in the exterior of Γ_2 then Γ_1 and Γ_2 intersect.

b) A line l is tangent to a circle $\Gamma(O, r)$ at a point P if and only if the lines l and OP meet at P and are perpendicular.

c) Let $\Gamma(O, r)$ be a circle in a Hilbert's plane which satisfies the Playfair's axiom (P). Suppose that A, B, C are distinct points on Γ and \overline{AB} is not a diameter of Γ . Then half of the angle $\angle AOB$ is congruent to the angle $\angle ACB$ if O and C are on the same side of line AB and half of the angle $\angle AOB$ is congruent to the angle supplementary to $\angle ACB$ if O and C are on opposite sides of line AB . If \overline{AB} is a diameter then the angle $\angle ACB$ is right.

QUIZ 5. a) Complete the statement:

Theorem (Chord-Tangent Theorem). Let l be a line tangent to a circle $\Gamma(O, r)$ at the point P and let $A \in \Gamma, B \in l$ be points different from P . Then the angle $\angle APB \dots$

b) Complete the statement:

Two circles $\Gamma_1(O_1, r_1), \Gamma_2(O_2, r_2)$ are tangent if and only if they intersect at a point P which is located on ...

c) Let Π be a Hilbert's plane satisfying Playfair's axiom. Let P be the set of congruence classes of segments in Π . Explain how multiplication $a \cdot b$ is defined for

$a, b \in P$.

Solution. a) **Theorem (Chord-Tangent Theorem).** Let l be a line tangent to a circle $\Gamma(O, r)$ at the point P and let $A \in \Gamma$, $B \in l$ be points different from P . Then the angle $\angle APB$ is congruent to the angle $\angle PCA$ for any point $C \in \Gamma$ such that points C and B are on opposite sides of the line AP .

b) Two circles $\Gamma_1(O_1, r_1)$, $\Gamma_2(O_2, r_2)$ are tangent if and only if they intersect at a point P which is located on the line O_1O_2 and $O_1 \neq O_2$.

c) Let Π be a Hilbert's plane satisfying Playfair's axiom. Let P be the set of congruence classes of segments in Π . We choose one element of P and call it 1. Given any $a \in P$ there is a right-angled triangle whose sides at the right angle are in the congruence classes 1 and a respectively. Any two such triangles are congruent. Pick one such triangle and call it T_a . Let α be the angle of T_a opposite side a . There is a right angle triangle whose one side is in the congruence class b and the angles at this side are congruent to α and a right angle. Any two such triangles are congruent. Then $a \cdot b$ is the congruence class of the side in such triangle opposite angle α .

QUIZ 6. a) State Ptolemy's Theorem.

b) Define the altitudes and orthocenter in a triangle.

c) Define the Euler line of a triangle $\triangle ABC$.

Solution. a)

Ptolemy's Theorem. Let $ABCD$ be a convex quadrilateral inscribed in a circle. Then

$$|\overline{AB}||\overline{CD}| + |\overline{AD}||\overline{BC}| = |\overline{AC}||\overline{BD}|.$$

b) An **altitude** is a line passing through a vertex and perpendicular to the side opposite the vertex. The foot of an altitude is the point where the altitude intersects the side it is perpendicular to (or its extension). It is a fact that the three altitudes in any triangle intersect at one point, which is called the **orthocenter** of the triangle.

c) The orthocenter H , the circumcenter O , and the centroid G in any triangle are

on one line, called the **Euler line** of the triangle (when the triangle is equilateral, we have $H = O = G$ and the Euler line is not defined). G is between H and O and $|\overline{HG}| = 2|\overline{OG}|$.

QUIZ 7. a) State the Miquel poin theorem.

b) Let Π be a Hilbert's plane.

- 1) Define the reflexion in a line l in Π .
- 2) What is a rigid motion of Π ?

Solution. a) **Miquel Point Theorem.** Let P_A, P_B, P_C be points on lines BC, AC, AB of a triangle $\triangle ABC$ respectively, different from the vertices of the triangle. Then the circumcircles of the trianngles $\triangle AP_B P_C, \triangle BP_A P_C,$ and $\triangle CP_A P_B$ intersect ot one point P , called the Miquel point (or pivot point). The point P is on the circumcircle of the triangle $\triangle ABC$ if and only if the points P_A, P_B, P_C are collinear.

b) 1) The reflexion in a line l is a function $R_l : \Pi \longrightarrow \Pi$ defined as follows: $R_l(A) = A$ for every point $A \in l$ and for points A not on l the point $R_l(A)$ is the unique point such that the line l is the perpendicular bisector of the segment $\overline{AR_l(A)}$.

2) A rigid motion is a bijection $R : \Pi \longrightarrow \Pi$ such that

- if $A * B * C$ then $R(A) * R(B) * R(C)$;
- $\overline{AB} \equiv \overline{R(A)R(B)}$ for any distinct points A, B .
- $\angle BAC \equiv \angle R(B)R(A)R(C)$ for any angle $\angle BAC$.

QUIZ 8. a) What is a lectilinear figure?

b) What does it mean that two rectilinear figures are equidecomposable?

c) Show that in a Hilbert's plane with (P) any triangle is equidecomposable with some rectangle.

Solutions. a) A rectilinear figure is a union of finitely many non-overlapping tri-angles.

b) Two figures F and G are equidecomposable if F can be expressed as the union of non-overlapping triangles T_1, \dots, T_k and G can be expressed as the union of non-overlapping triangles S_1, \dots, S_k in such a way that T_i and S_i are congruent for $i = 1, \dots, k$.

c) Consider a triangle $\triangle ABC$ and assume that the angles $\angle ABC$ and $\angle ACB$ are acute. Let M, N be the midpoints of \overline{AB} and \overline{AC} respectively. The line MN is parallel to AB . Let K, L, G be points on the line MN such that BK, AG, CL are perpendicular to MN . Then $BKLC$ is a rectangle which is the union of non-overlapping triangles $\triangle BKM, \triangle CLN, \triangle BCM$ and $\triangle CMN$. The triangle $\triangle ABC$ is the union of non-overlapping triangles $\triangle AGM, \triangle AGN, \triangle BCM$ and $\triangle CMN$. Since $\triangle BKM \cong \triangle AGM, \triangle CLN \cong \triangle AGN$, we see that $\triangle ABC$ and the rectangle $BKLC$ are equidecomposable.

QUIZ 9. a) Let Γ be a circle with center O and radius r in a Euclid's plane Π . Define the inversion I_Γ in Γ .

b) State the Bolyai-Gerwien-Wallace Theorem.

c) Let Γ_1 and Γ_2 be two circles which are tangent at O and let Γ be a circle with center O . What can you say about $I_\Gamma(\Gamma_1 - \{O\})$ and $I_\Gamma(\Gamma_2 - \{O\})$?

Solution. a) The inversion I_Γ in Γ is a function $I_\Gamma : \Pi - \{O\} \rightarrow \Pi - \{O\}$ defined as follows: $I_\Gamma(P)$ is the unique point on the ray \overrightarrow{OP} such that $|\overline{OP}| |\overline{OI_\Gamma(P)}| = r^2$.

b) **Bolyai-Gerwien-Wallace Theorem.** Two rectilinear figures in a Hilbert's plane which satisfies (P) and (A) (Archimedes axiom) are equidecomposable if and only if they have the same area.

c) An inversion maps a circle passing through the center of inversion O (with O removed) to a line not passing through O . Since $\Gamma_1 - \{O\}$ and $\Gamma_2 - \{O\}$ are disjoint, their images under I_Γ are also disjoint. Thus $I_\Gamma(\Gamma_1 - \{O\})$ and $I_\Gamma(\Gamma_2 - \{O\})$ are two parallel lines.

QUIZ 10. a) What does it mean that two circles Γ_1 and Γ_2 in a Euclid's plane are perpendicular?

b) Let A, B, C, D be 4 different points in a Hilbert's plane with (P). What is the cross-ratio $(A, B; C, D)$?

c) Consider a Poincare model \mathbb{H} of a hyperbolic plane. How is the hyperbolic distance between 2 points A, B in \mathbb{H} defined.

Solution. a) Let O_i be the center of Γ_i , $i = 1, 2$. The circles Γ_1 and Γ_2 are perpendicular if they intersect at a point P and the tangent lines to the circles at P are perpendicular. This is equivalent to any of the following conditions:

the angle $\angle O_1 P O_2$ is right

$P O_1$ is tangent to Γ_2

$P O_2$ is tangent to Γ_1

$I_{\Gamma_1}(\Gamma_2) = \Gamma_2$, where I_{Γ_1} is the inversion in Γ_1 .

$I_{\Gamma_2}(\Gamma_1) = \Gamma_1$, where I_{Γ_2} is the inversion in Γ_2 .

there is a point $A \in \Gamma_2$ such that $A \notin \Gamma_1$ and $I_{\Gamma_1}(A) \in \Gamma_2$.

there is a point $B \in \Gamma_1$ such that $B \notin \Gamma_2$ and $I_{\Gamma_2}(B) \in \Gamma_1$.

b) The cross-ratio $(A, B; C, D)$ is defined by the formula

$$(A, B; C, D) = \frac{|\overline{AC}|}{|\overline{AD}|} : \frac{|\overline{BC}|}{|\overline{BD}|} = \frac{|\overline{AC}|}{|\overline{AD}|} \frac{|\overline{BD}|}{|\overline{BC}|}.$$

c) Let γ be the p -line AB and let P, Q be the ends of γ , where $P * A * B * Q$. Then the hyperbolic distance $d(A, B)$ is defined as

$$d(A, B) = (A, B; P, Q)^{-1} = (A, B; Q, P).$$