Solutions to Exam 1, Math 465

Problem 1. The following 10 questions ask to state a theorem, a definition, or to explain a concept. In order to get full credit your answer must be written clearly, using complete sentences (and, of course, be correct).

1) Who is the author of the first comprehensive text on geometry? When was it written? (3 points)

Answer: The first comprehensive text on geometry is called *The Elements* and it was written by *Euclid* in Alexandria (Egypt) around 300 BC.

2) State three theorems from the *Elements* which you find important and which speak about **both** lines and circles. (3 points each)

There are many possible choices. Perhaps the most interesting are the following:

Book III, Proposition 32. *Let A be a point of a circle Γ, let l be the line tangent to Γ at A and let m be a line through A cutting Γ at another point B. If P is a point on l different from A and if C is a point on Γ such that C and P are on the opposite sides of m then $\angle PAB \equiv \angle ACB$.*

Book III, Proposition 35. *Let P be a point inside a circle Γ. Let a line through P cut Γ at A and B and let another line through P cut Γ at C and D. Then $AP \cdot PB = CP \cdot PD$ (i.e. the rectangle with sides AP, PB has content equal to the content of the rectangle with sides CP, PD).*

Book III, Proposition 36. *Let P be a point outside a circle Γ. Let a line through P cut Γ at A and B and let another line through P be tangent to Γ at C. Then $PA \cdot PB = PC^2$.*

Remark. By the above propositions, if P is a point not on a circle Γ and if a line through P cuts Γ in two points A, B ($A = B$ when the line is tangent to Γ) then the quantity $PA \cdot PB$ is the same for all lines through P. This quantity is called **the power of P with respect to Γ**. When P is inside the circle, the power is taken with negative sign (i.e. it is $-PA \cdot PB$).

3) Define the following concepts (2 points each):

a) altitudes, b) medians, c) centroid, d) orthocenter, e) incenter, f) circumcenter

An **altitude** of a triangle is a line passing through a vertex of the triangle and perpendicular to the side subtended by the vertex.

A **median** of a triangle is a line passing through a vertex of the triangle and the midpoint of the side subtended by the vertex.

The **centroid** of a triangle is the point where all three medians intersect.

The **orthocenter** of a triangle is the point where all three altitudes intersect.

The **incenter** of a triangle (rectilinear figure) is the center of a circle tangent to all sides of the triangle (rectilinear figure), i.e. the inscribed circle. Every triangle has its incenter; it is the point where the angle bisectors of the angles of the triangle intersect.

The **circumcenter** of a triangle (rectilinear figure) is the center of the circle containing all the vertices of the triangle (rectilinear figure), i.e. the circumscribed circle. Every triangle has its circumcenter; it is the point where the perpendicular bisectors of the three sides of the triangle intersect.

4) Define Euler line. What can you say about the position of the three points defining this line? (4 points)

The **Euler line** of a triangle is the line passing through the orthocenter H, the centroid G and the circumcenter O of the triangle. The point G is always between the points H and O and HG is twice OG. Note that Euler line is not defined for equilateral triangles since then the three points coincide.

5) Define the nine point circle. Explain what are the nine points. What can you say about the nine-point center? (5 points)

Let ABC be a triangle, let H be the orthocenter and let H_A, H_B, H_C be the feet of the altitudes through A, B, C respectively (i.e. H_A is on BC and AH_A is an altitude, etc.). Then the following nine points are on one circle, called the **nine-point circle** of ABC: the midpoints of sides of ABC, the feet of the altitudes H_A, H_B, H_C and the midpoints of the segments $\overline{AH}$, $\overline{BH}$, $\overline{CH}$. The center of the nine-point circle is called **the nine-point center** of ABC. The nine point center is on the Euler line and it coincides with the midpoint of the segment joining the orthocenter and the circumcenter.

6) State Pasch axiom. (3 points)

Let l be a line which does not contain any of the vertices of a given triangle and which intersects one of the sides of the triangle. Then l intersects one (and only one) of the remaining two sides.

7) State the circle-circle intersection axiom (E) and the circle-line intersection property (CLIP). (4 points)

(E) (circle-circle intersection). Let Γ and Δ be two circles. If Γ contains a point inside Δ and a point outside Δ then Γ and Δ have a common point.

Circle-line intersection property (CLIP). If a line l contains a point inside a circle Γ then l and Γ have a common point (intersect).

8) State the plane separation theorem and explain how the sides of a line are defined. (4 points)

Plane separation. Let l be a line in a geometry which satisfies the incidence axioms and the betweenness axioms. The set of all points not on l can be divided into two disjoint subsets, called the sides of l such that:

- two points A, B are in the same side of l if and only if either $A = B$ or l does not intersect the segment $\overline{AB}$ (i.e. no point on l is between A and B).
- two points A, B are in different side of l if and only if l intersects the segment $\overline{AB}$ (i.e. there is a point on l which is between A and B).

9) Define an angle and the interior of an angle. (4 points)

A **ray** $\overrightarrow{AB}$ consists of all points C of the line AB such that A is not between B and C. In other words, $\overrightarrow{AB} = \{A\} \cup \{B\} \cup \{X : A * X * B\} \cup \{X : A * B * X\}$.
An **angle** $\angle BAC$ is the union of the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ under the assumption that A, B, C are not collinear.
A point P is in the interior of an angle $\angle BAC$ if and only if P and B are on the same side of the line AC and P and C are on the same side of the line AB.

10) What is Hilbert's plane? What is Euclid's plane? (4 points)

Hilbert's plane is a set Π, whose elements are called points such that:

- Some subsets of Π, called lines, are selected and the incidence axioms are satisfied.
- A notion of betweenness is defined for some triples of points in Π and the axioms of betweenness are satisfied.
- A notion of congruence is defined for segments in Π and the congruence axioms C1-C3 are satisfied.
- A notion of congruence is defined for angles and congruence axioms C4-C6 are satisfied.

Euclid's plane is a Hilbert's plane which satisfies the Playfair's Axiom (P) and the circle-circle intersection axiom (E).

Problem 2. a) In a triangle the orthocenter coincides with the circumcenter. Prove that the triangle is equilateral. (6 points)

b) In a triangle the nine-point center and the circumcenter coincide. Prove that the triangle is equilateral. (6 points)

Solution. a) Let $\triangle ABC$ be a triangle in which the orthocenter H and the circumcenter coincide. The line AH is an altitude, so it is perpendicular to BC. Since H coincides with the circumcenter, the perpendicular bisector of $\overline{BC}$ passes through H. Since there is unique line through H which is perpendicular to BC, we conclude that AH is the perpendicular bisector of $\overline{BC}$. Thus the midpoint M_A of $\overline{BC}$ belongs to AH. Since $\angle AM_AB$ is right we have $\angle AM_AB \equiv \angle AM_AC$. By SAS, the triangles $\triangle AM_AB$ and $\triangle AM_AC$ are congruent. In particular, $\overline{AC} \equiv \overline{AB}$.

In the same way we show that $\overline{BA} \equiv \overline{BC}$. This proves that ABC is equilateral.

b) Suppose that the circumcenter and the nine-point center of a triangle coincide. Recall that the nine-point center is the midpoint between the orthocenter and the circumcenter. Thus the orthocenter coincides with the circumcenter and the triangle is equilateral by part a).

Problem 3. (10 points) Two circles Γ and Γ' in a Euclid's plane are internally tangent at a point A (say Γ is inside Γ'). A ray emanating from A intersects Γ and Γ' at points B, B' respectively. Another ray emanating from A intersects Γ and Γ' at points C, C' respectively. Prove that BC and $B'C'$ are parallel. Carefully explain your reasoning. Hint: consider the line tangent to the circles at A.

Solution. Let l be the line tangent to Γ' at A. Then all points of l except A are outside of Γ'. But all points of Γ except A are inside of Γ'. Thus A is the only point of intersection of l and Γ, i.e. l is tangent to Γ too (there are other ways to justify this. For example, let O, O' be the centers of Γ and Γ' respectively. Since l is tangent to Γ'. the line AO' is perpendicular to l. Since Γ and Γ' are tangent at A, the line AO coincides with the line AO', so it is perpendicular to l, hence l is tangent to Γ.).

Note that the points C, C' are on the same side of the line AB (since $A{*}C{*}C'$). Let P be a point on l which is on the opposite side of the line AB than the side where C, C' are. By one of the results from class (Proposition 32 from Book III of the *Elements*; see the solution to question 2 of Problem 1) , we get that $\angle PAB \equiv \angle ACB$ and $\angle PAB' \equiv \angle AC'B'$. Since

$\angle PAB = \angle PAB'$, we see that $\angle ACB \equiv \angle AC'B'$. It follows that the lines BC and $B'C'$ are parallel (Proposition 27 in Book I).

Second solution. The triangle $\triangle BOA$ is isosceles ($\overline{BO} \equiv \overline{AO}$). Similarly triangle $\triangle B'O'A$ is isosceles. Thus we have $\angle ABO \equiv \angle BAO = \angle B'AO' \equiv \angle AB'O'$. It follows that the triangles $\triangle ABO$ and $\triangle AB'O'$ are similar by aaa. Therefore $|\overline{AB}|/|\overline{AB'}| = |\overline{AO}|/|\overline{AO'}|$. In a similar way we show that $|\overline{AC}|/|\overline{AC'}| = |\overline{AO}|/|\overline{AO'}|$. By sas, the triangles $\triangle ABC$ and $\triangle AB'C'$ are similar. In particular, $\angle ACB \equiv \angle AC'B'$, which implies that the lines BC and $B'C'$ are parallel .

Problem 4. (10 points) In this problem you can only use the incidence axioms, the betweenness axioms, the plane separation theorem, and the crossbar theorem. Suppose that $A*B*C$ on one line and $A*D*E$ on a different line. Prove that the **segments** $\overline{BE}$ and $\overline{CD}$ have a common point. Carefully explain your reasoning pointing to the axioms or results you are using.

Solution. This solution will only use the Pasch axiom. Consider the triangle $\triangle ACD$. The line EB does not contain any vertex of this triangle (why?). EB contains the point B, which is between A and C. It follows that EB has either a point between D and C or a point between A and D. The latter is not possible, as the only point on line EB which is also on line AD is E and E is not between A and D. We conclude that the line EB has a point between C and D, i.e the line EB intesects the segment $\overline{CD}$. Let P be the point of intersection of lines EB and CD. Thus we showed that P belongs to the segment $\overline{CD}$.

In the same way, considering the triangle $\triangle ABE$ and the line DC we show that P belongs to the segment $\overline{EB}$.

Second solution. Since $A*D*E$, points A and D are on the same side of the line BE. Since $A*B*C$, points A and C are on opposite sides of the line BE. Thus C and D are on opposite sides of the line BE and therefore the segment $\overline{CD}$ intersects line BE at some point P. In the same manner, we show that B and E are on opposite sides of the line CD, hence segment $\overline{BE}$ intersects the line CD. Thus the point P (the intersection of lines CD and BE) belongs to both segments $\overline{BE}$ and $\overline{CD}$.

Problem 5. a) (6 points) State the theorems of Ceva and Menelaus.

b) (10 points) Let $\triangle ABC$ be a triangle in a Hilbert's plane with (P). Let D_A, D_B, D_C be the points where the incircle of $\triangle ABC$ touches the sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively. Prove that the lines AD_A, BD_B, CD_C intersect at one point.

Solution. a) **Ceva's Theorem.** Let $\triangle ABC$ be a triangle and let A_1, B_1, C_1 be points on lines BC, AC, AB respectively, which are different form the vertices A, B, C. The lines AA_1, BB_1, CC_1 intersect at one point if and only if either exactly one or all three of the points A_1, B_1, C_1 are on the sides of the triangle and

$$\frac{|\overline{AB_1}|}{|\overline{B_1C}|}\frac{|\overline{CA_1}|}{|\overline{A_1B}|}\frac{|\overline{BC_1}|}{|\overline{C_1A}|} = 1.$$

Menalaus Theorem. Let $\triangle ABC$ be a triangle and let A_1, B_1, C_1 be points on lines BC, AC, AB respectively, which are different form the vertices A, B, C. The points A_1, B_1, C_1 are collinear if and only if either exactly two or none of the points A_1, B_1, C_1 are on the sides of the triangle and

$$\frac{|\overline{AB_1}|}{|\overline{B_1C}|}\frac{|\overline{CA_1}|}{|\overline{A_1B}|}\frac{|\overline{BC_1}|}{|\overline{C_1A}|} = 1.$$

b) Let O be the center of the incircle. Then AO is the angle bisector of $\angle BAC$, so $\angle OAB$ is acute (being half of the angle $\angle BAC$). Similarly $\angle OBA$ is acute. The line OD_C is perpendicular to AB, so it is an altitude in the traingle OAB. Since both angles $\angle OAB$ and $\angle OBA$ are acute, the feet of the altitude from O in triangle $\triangle OAB$ is on the side $\overline{AB}$, i.e. D_C is on $\overline{AB}$. In the same way we show that D_A is on $\overline{BC}$ and B_B is on $\overline{AC}$. By Ceva's theorem, in order to prove that lines AD_A, BD_B, CD_C intersect at one point it suffices to prove that

$$\frac{|\overline{AD_B}|}{|\overline{D_BC}|}\frac{|\overline{CD_A}|}{|\overline{D_AB}|}\frac{|\overline{BD_C}|}{|\overline{D_CA}|} = 1.$$

Since AD_B and AD_C are tangents to the incircle from the same point A, we have $\overline{AD_B} \equiv \overline{AD_C}$ (to see that note that the triangles $\triangle AD_BO$ and $\triangle AD_CO$ are similar by saa; or use that fact that

$$|\overline{AD_B}|^2 = |\overline{AO}|^2 - |\overline{OD_B}|^2 = |\overline{AO}|^2 - |\overline{OD_C}|^2 = |\overline{AD_C}|^2.)$$

In the same way we show that $\overline{BD_A} \equiv \overline{BD_C}$ and $\overline{CD_A} \equiv \overline{CD_B}$. It is clear now that

$$\frac{|\overline{AD_B}|}{|\overline{D_BC}|}\frac{|\overline{CD_A}|}{|\overline{D_AB}|}\frac{|\overline{BD_C}|}{|\overline{D_CA}|} = 1.$$