

# 1 Sample problems.

I. Let  $\mathbf{v}_1 = (1, 1, 2, 2, 5)$ ,  $\mathbf{v}_2 = (-1, 1, -2, 0, -1)$ ,  $\mathbf{v}_3 = (0, 0, 1, 0, 1)$ ,  $\mathbf{v}_4 = (0, 1, 0, 1, 2)$ ,  $\mathbf{v}_5 = (1, 1, 5, 2, 8)$ . Set  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ . Among the vectors in  $S$  find a basis of  $\text{span}(S)$  which contains  $\mathbf{v}_1$  and  $\mathbf{v}_4$ . Express the vectors in  $S$  as linear combinations of vectors in the basis found.

**Solution:** We arrange the vectors from  $S$  as columns of a matrix  $M$  such that  $\mathbf{v}_1$  and  $\mathbf{v}_4$  are the first columns of  $M$ :

$$M = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & -2 & 1 & 5 \\ 2 & 1 & 0 & 0 & 2 \\ 5 & 2 & -1 & 1 & 8 \end{pmatrix}$$

Next we find the reduced row-echelon form  $N$  of  $M$ :

$$N = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The columns of  $M$  corresponding to pivot columns of  $N$  form a required basis, i.e.  $\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_3$  is a basis of  $\text{span}(S)$ . To express  $\mathbf{v}_2$  as a linear combination of  $\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_3$  we look at the third column of  $N$  (which corresponds to  $\mathbf{v}_2$ ). It tells us that

$$\mathbf{v}_2 = -\mathbf{v}_1 + 2\mathbf{v}_4.$$

Similarly, from the fifth column of  $N$  we read that

$$\mathbf{v}_5 = \mathbf{v}_1 + 3\mathbf{v}_3.$$

II. Solve the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &= 1 \\ x_1 + x_3 + x_4 + 3x_5 + 2x_6 + 3x_7 &= 2 \\ 3x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 + 2x_6 + 3x_7 &= 2 \\ 2x_1 - x_2 + 2x_3 - x_4 + 2x_5 - x_6 + 2x_7 &= -1 \end{aligned}$$

**Solution:** The augmented matrix of this system is

$$\left( \begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 \\ 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \end{array} \right)$$

The reduced row-echelon form of this matrix is

$$\left( \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & -1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The last column is not a pivot column so the system is consistent. The free variables are  $x_3, x_5, x_6$  and  $x_7$  and the system takes form:

$$\begin{aligned}
x_1 &= -x_3 - x_5 - x_7 \\
x_2 &= 2x_5 + x_6 + 2x_7 - 1 \\
x_3 &= x_3 \\
x_4 &= -2x_5 - 2x_6 - 2x_7 - 2 \\
x_5 &= x_5 \\
x_6 &= x_6 \\
x_7 &= x_7
\end{aligned}$$

We see that a basis of solutions to the associated homogeneous system is  $\mathbf{u}_1 = (-1, 0, 1, 0, 0, 0, 0)$ ,  $\mathbf{u}_2 = (-1, 2, 0, -2, 1, 0, 0)$ ,  $\mathbf{u}_3 = (0, 1, 0, -2, 0, 1, 0)$ ,  $\mathbf{u}_4 = (-1, 2, 0, -2, 0, 0, 1)$ .

A particular solution to the non-homogeneous system is obtained by setting  $x_3 = x_5 = x_6 = x_7 = 0$ . We get  $(0, -1, 0, 2, 0, 0, 0)$ .

The general solution to the system can be written as:

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0, -1, 0, 2, 0, 0, 0) + a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4$$

where  $a_1, a_2, a_3, a_4$  are independent parameters.

**III.** Let  $\mathbf{w}_1 = (0, 1, 1, 0, 1, 1, 0)$ ,  $\mathbf{w}_2 = (1, 1, 1, 0, -1, -1, 0)$ ,  $\mathbf{w}_3 = (2, 3, 3, 0, -1, -1, -2)$ ,  $\mathbf{w}_4 = (2, 1, 2, 0, 0, -1, 2)$ ,  $\mathbf{w}_5 = (1, 2, 2, 0, 0, 0, 1)$ . Find a basis of  $\text{span}(S)^\perp$ , where  $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ .

**Solution:** This problem simply asks for a basis of solutions to the homogeneous system of equations whose coefficient matrix

$$\begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & -1 & -1 & 0 \\
2 & 3 & 3 & 0 & -1 & -1 & -2 \\
2 & 1 & 2 & 0 & 0 & -1 & 2 \\
1 & 2 & 2 & 0 & 0 & 0 & 1
\end{pmatrix}$$

has rows equal to vectors in  $S$ . The reduced row echelon form of this matrix is

$$\begin{pmatrix}
1 & 0 & 0 & 0 & -2 & -2 & 0 \\
0 & 1 & 0 & 0 & -2 & -1 & 0 \\
0 & 0 & 1 & 0 & 3 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We see that  $x_4, x_5$  and  $x_6$  are free variables and

$$\begin{aligned}
x_1 &= 2x_5 + 2x_6 \\
x_2 &= 2x_5 + x_6 \\
x_3 &= -3x_5 - 2x_6 \\
x_4 &= x_4 \\
x_5 &= x_5 \\
x_6 &= x_6 \\
x_7 &= 0
\end{aligned}$$

A basis of solutions is  $(0, 0, 0, 1, 0, 0, 0)$ ,  $(2, 2, -3, 0, 1, 0, 0)$ ,  $(2, 1, -2, 0, 0, 1, 0)$ . This is a basis of  $\text{span}(S)^\perp$ .

**IV.** Let  $\mathbf{v}_1 = (1, 1, 1, 1, 1, 1, 1)$ ,  $\mathbf{v}_2 = (1, 0, 1, 1, 3, 2, 3)$ ,  $\mathbf{v}_3 = (3, 2, 3, 2, 3, 2, 3)$ ,  $\mathbf{v}_4 = (2, -1, 2, -1, 2, -1, 2)$ . Set  $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ . Find a system of homogeneous equations whose solution space is  $\text{span}(T)$ .

**Solution:** First we find a basis of  $\text{span}(T)^\perp$  using III. We get  $(-1, 0, 1, 0, 0, 0, 0)$ ,  $(-1, 2, 0, -2, 0, 1, 0)$ ,  $(0, 1, 0, -2, 0, 1, 0)$ ,  $(-1, 2, 0, -2, 0, 0, 1)$ . (Note that we did the computations in II.) Thus the system of equations for  $\text{span}(T)^\perp$  is:

$$\begin{array}{rcccccc}
-x_1 & & +x_3 & & & = 0 \\
-x_1 + 2x_2 & & & -2x_4 + x_5 & & = 0 \\
& & x_2 & -2x_4 & + x_6 & = 0 \\
-x_1 + 2x_2 & & & -2x_4 & & + x_7 = 0
\end{array}$$

**V.** Let  $S$  and  $T$  be the sets from III and IV respectively. Set  $V = \text{span}(T)$  and  $W = \text{span}(S)$ . Find a basis of  $V + W$  and  $V \cap W$ .

**Solution:** Since  $V + W = \text{span}(T \cup S)$  we find a basis of  $V + W$  by applying I. Thus we start with a matrix whose columns are the vectors from  $T \cup S$ :

$$\left( \begin{array}{cccc|cccc}
1 & 1 & 3 & 2 & 0 & 1 & 2 & 2 & 1 \\
1 & 0 & 2 & -1 & 1 & 1 & 3 & 1 & 2 \\
1 & 1 & 3 & 2 & 1 & 1 & 3 & 2 & 2 \\
1 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 2 & 1 & -1 & -1 & 0 & 0 \\
1 & 2 & 2 & -1 & 1 & -1 & -1 & -1 & 0 \\
1 & 3 & 3 & 2 & 0 & 0 & -2 & 2 & 1
\end{array} \right)$$

Its reduced row-echelon form is

$$\left( \begin{array}{cccc|cccc}
1 & 0 & 0 & -7 & 0 & 0 & -2 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \\
0 & 0 & 1 & 3 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right)$$

The pivot columns of this matrix tell us that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2$  is a basis for  $V + W$ .

In order to find a basis of  $V \cap W$  we may use one of the following methods.

**Method 1.** Find equations for  $V$  and  $W$  using IV. We see that the equations for  $V$  are

$$\begin{array}{rcccccc}
-x_1 & & +x_3 & & & = 0 \\
-x_1 + 2x_2 & & & -2x_4 + x_5 & & = 0 \\
& & x_2 & -2x_4 & + x_6 & = 0 \\
-x_1 + 2x_2 & & & -2x_4 & & + x_7 = 0
\end{array}$$

(we found them in IV).

In III we found a basis for  $\text{span}(W)^\perp$ . Thus  $W$  is a solution space to the system

$$\begin{array}{rcccccc}
& & & x_4 & & = 0 \\
2x_1 + 2x_2 - 3x_3 & & + x_5 & & & = 0 \\
2x_1 + x_2 - 2x_3 & & & + x_6 & & = 0
\end{array}$$

Thus  $V \cap W$  is the solution space to the combined system

$$\begin{array}{rcccccc}
-x_1 & & +x_3 & & & = 0 \\
-x_1 + 2x_2 & & & -2x_4 + x_5 & & = 0 \\
& & x_2 & -2x_4 & + x_6 & = 0 \\
-x_1 + 2x_2 & & & -2x_4 & & + x_7 = 0 \\
& & & x_4 & & = 0 \\
2x_1 + 2x_2 - 3x_3 & & + x_5 & & & = 0 \\
2x_1 + x_2 - 2x_3 & & & + x_6 & & = 0
\end{array}$$

To get a basis of  $V \cap W$  we just need to find a basis of solutions to this system. The coefficient matrix of this system is

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ -1 & 2 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & -3 & 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The reduced row-echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the method of II we see that a basis of solutions is  $(-2, -1, -2, 0, 0, 1, 0)$ ,  $(1, 0, 1, 0, 1, 0, 1)$ . This is a basis for  $V \cap W$ .

**Method 2.** This method works best if  $S$  and  $T$  are linearly independent. We could first apply I to find a basis of  $V$  and  $W$ . But it is more efficient to do everything at once. We start with a matrix whose columns are the vectors from  $T$  followed by the vectors from  $S$ :

$$\begin{pmatrix} 1 & 1 & 3 & 2 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 2 & -1 & 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 3 & 2 & 1 & 1 & 3 & 2 & 2 \\ 1 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 2 & 1 & -1 & -1 & 0 & 0 \\ 1 & 2 & 2 & -1 & 1 & -1 & -1 & -1 & 0 \\ 1 & 3 & 3 & 2 & 0 & 0 & -2 & 2 & 1 \end{pmatrix}$$

The reduced row-echelon form of this matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 & -7 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the left part we remove all the non-pivot columns (they are combinations of pivot columns and are not needed; it corresponds to the fact that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is a basis of  $V$ ). The right part is not in the reduced row-echelon form. Its reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This tells us that  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$  form a basis of  $W$  and we can remove the last (non-pivot) column of

the right hand side of  $A$ . Thus we get a matrix

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The homogeneous system with this matrix as coefficient matrix has the following basis of solutions:  $(-2, -2, 2, 1, 0, 1, 0)$ ,  $(0, 2, -1, 0, 3, 0, 1)$ . This implies that  $-2\mathbf{v}_1 - 2\mathbf{v}_2 + 2\mathbf{v}_3 = (2, 2, 2, 0, -2, -2, -2)$  and  $2\mathbf{v}_2 - \mathbf{v}_3 = (-1, -2, -1, 0, 3, 2, 3)$  form a basis of  $V \cap W$ .

**General Remark.** In order to know the pivot columns of a matrix we do not need to get reduced row-echelon form. Any row-echelon form (not necessarily reduced) will work.