Math 330 Section 6 - Spring 2024 - Homework 08

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Status - Reading Assignments:

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:
ch.2.1 – 2.7, ch.3, skim ch.4 (optional), ch.5 - 9.1

B/G (Beck/Geoghegan) Textbook:
ch.2 - 6.3

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Reading assignment for the break (optional):

a. Catch up on the topics you were not able to study thoroughly enough.
b. If you have not taken or are not currently taking linear algebra, this would be a good time to look at MF ch.11 through Example 11.11. That's nine pages of very easy material.
c. You will get a new perspective on limits of sequences and continuity of functions Review Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

Written assignments: See next page.

The written assignments are about proving MF thm.6.9 (Division Algorithm for Integers): Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. There exists a unique combination of two integers q ("quotient") and r ("remainder") such that

$$m \;=\; n \cdot q \;+\; r \quad \text{and } 0 \leq r < n.$$

Do not use induction for assignments 2 and 3. It would only make your task more difficult!

Written assignment 1:

Prove uniqueness of the "decomposition" $m = qn + r$ such that $0 \leq r < n$: If you have a second such decomposition $m = \tilde{q}n + \tilde{r}$ then show that this implies $q = \tilde{q}$ and $r = \tilde{r}$. Start by equating $qn + r \;=\; m \;=\; \tilde{q}n + \tilde{r}$ and see what can be said about $|r - \tilde{r}|$ if $q \neq \tilde{q}$. More hints further down!

Written assignment 2:

Much harder than #1: Prove the existence of q and r.

Hints for #2: Review the Extended Well–Ordering principle MF thm.6.8. Its use will give the easiest way to prove this assignment: Let

$$A \;:=\; A(m,n) \;:=\; \{r' \in [0,\infty[_{\mathbb{Z}} : \; \exists\, q' \in \mathbb{Z} \text{ such that } r' = m - q'n\}\,.$$

Show that $A \neq \emptyset$ by separately examining the cases

- $m \geqq 0$ (easy)
- $m < 0$ (probably the hardest part of the proof!)

Now you can apply the Extended Well–Ordering principle to the set A. How is $\min(A)$ related to your problem? (It is!) To better see what is going on, this may help:

- What is $m = nq + r$ for $n = 10$ and $m = 43,\; m = -43,\; m = -3$?
- What is $\min(A(m,n))$ in those three cases? Draw a picture!

Hint for both #1 and #2: MF prop. 3.61 and cor.3.5 at the end of ch.3.5 will come in handy in connection with using or proving $0 \leqq r < n$. They assert for the ordered integral domain $(\mathbb{Z}, +, \cdot, \mathbb{N})$ the following.

If $a, b \in [0,n[_{\mathbb{Z}}$, then $|a - b| \;\leqq\; \max(a,b)$, i.e., $\;-n \;<\; a - b \;<\; n$.