Math 330 Section 6 - Spring 2024 - Homework 11

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Status - Reading Assignments:

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:
ch.2.1 – 2.7, ch.3, skim ch.4 (optional), ch.5 - 9.9, ch.11 (linear algebra) through Example 11.11

B/G (Beck/Geoghegan) Textbook:
ch.2 - 12

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Other:
Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

New reading assignments:

Reading assignment 1 - due Monday, March 25, 2024:

- Carefully read the remainder of MF ch.11.2.1 All of this should be an easy read for those who have taken linear algebra or currently are doing so.

Reading assignment 2 - due: Wednesday, March 27, 2024:

a. Carefully read MF ch.11.2.2. Be sure to understand for $p = 2$ why $\|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$ is a measure for the size of f. This will be easier if you draw a picture for $p = 1$!

b. The strong students are invited to peruse the optional chapter MF 11.2.3.

Reading assignment 3 - due Friday, March 29, 2024:

a. Read carefully MF ch.12.1 and ch.12.2. It is crucial that you become familiar with the examples given there You will have massive problems with metric and topological spaces if you did not study MF ch.9.3.

b. Read carefully MF ch.12.3. What does an open set in $\mathbb{R}$ with $d(x,y) = |y - x|$ look like?

Be sure to read pages 2 and 3!

Supplementary instructions for reading MF ch.12:

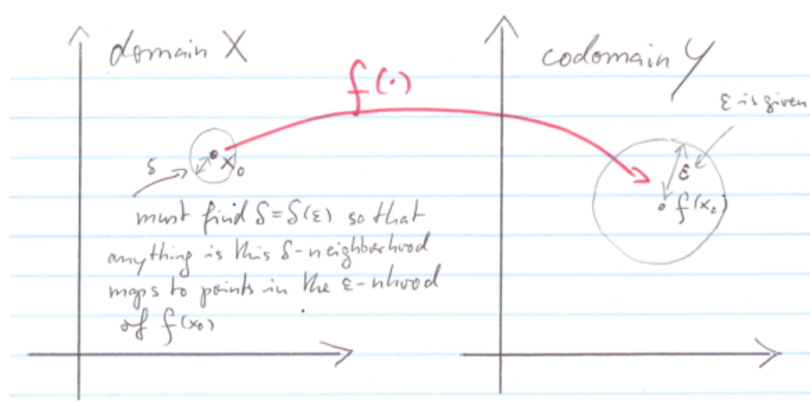
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \biguplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^{\complement}$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^{\complement}$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^{\complement}$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 0.1: ε-δ continuity



Written assignments on page 3

Written assignment 1:

Let $f(x) = x^2$. Prove by use of "ε-δ continuity" that f is continous at $x_0 = 1$. You MUST work with ε-δ continuity (thm.9.7) **NOT WITH SEQUENCE CONTINUITY**, and you cannot use any "advanced" knowledge such as the product of continuous functions being continuous, etc.

This assignment is worth $\boxed{\text{up to 3 points.}}$ Partial credit will be given and you can turn it in repeatedly.

Special instructions for assignment 1: Turn in your scratchpaper where you solve for δ (see the hints below).

Hints:

a. What does $d(x, x_0) < \delta$ and $d(f(x), f(x_0) < \varepsilon$ translate to?

b. $x^2 - 1 = (x+1)(x-1)$.

c. Do the following on scratch paper: Work your way backward by establishing a relationship between $\varepsilon > 0$ and δ and then "solving for δ" That part should not be in your official proof.

c1. Only small neighborhoods matter (see Proposition 9.24 at the end of the chapter on convergence and continuity.): Given $0 < \varepsilon < 1$ try to find δ that works for such ε. Restrict your search to $\delta < 1$. What kind of bounds do you get for $|\,x^2 - 1|$, $|x+1|$, $|x-1|$ if $0 < \delta < 1$? In particular what kind of bounds to you get for $|x+1|$?

c2. Put all the above together. Show that you obtain $|f(x) - f(x_0)| \leq 3\delta$?. How then do you choose δ when you consider ε as given? You'll get the answer by "solving $|f(x) - f(x_0)| \leq 3\delta$ for δ".

c3. All of the above was done under the assumption that $\delta < 1$. Satisfy it by replacing δ with $\delta' := \min(\delta, 1)$

d. Only now you are ready to construct an acceptable proof: Let $\varepsilon > 0, \delta := \ldots$, and $\delta' := \min(\delta, 1)$. Then ....

Written assignment 2: Prove MF Thm. 9.8: If $m \in [0, \infty[_{\mathbb{Z}}$ is not a perfect square then $\sqrt{m}$ is irrational.

Hint: Work with lowest term representations. See the proof that $\sqrt{2}$ is irrational.

No partial credit for this one!