Math 330 Section 6 - Spring 2024 - Homework 15

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Status - Reading Assignments:

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:
ch.2.1 – 2.7, ch.3, skim ch.4 (optional), ch.5 - 15.3

B/G (Beck/Geoghegan) Textbook:
ch.2 - 13

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Other:
Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

New reading assignments:

None of the material starting with Ch.15.4 will be on any exam or quiz.

Reading assignment 1 - due Monday, April 22, 2024:

a. Read MF ch.15.4 and ch.15.5.1.

Reading assignment 2 - due: Thursday, April 25, 2024:

a. Skim the intro to MF ch.16 and ch.16.3. The strong students may want to follow any link to material that is treated in the earlier parts of ch.16.

Reading assignment 3 - due Friday, April 26, 2024:

a. Review ch.11.2.2 (Normed Vector Spaces) and the early parts of ch.12 (ch.12.1 – 12.3).

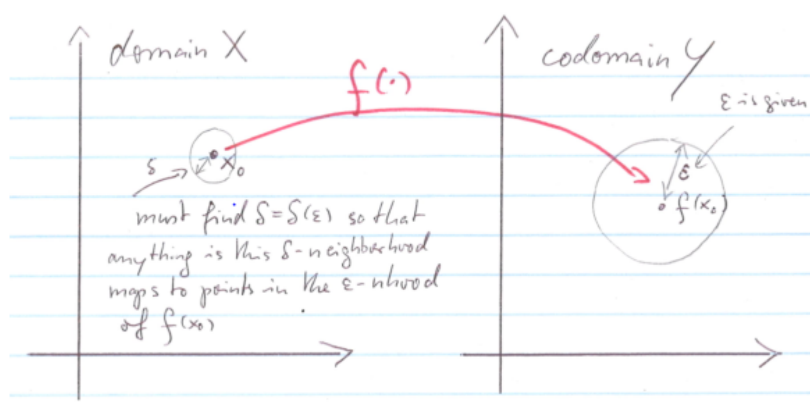
Be sure to read pages 2 and 3!

Supplementary instructions for reading MF ch.12:

When you read or reread any topics in those chapters then the following is good advice:

- **a.** MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for
- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \biguplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^\complement$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^\complement$ and A_1. What is $N^A_\varepsilon(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^\complement$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.
- **b.** MF ch.12.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 0.1: ε-δ continuity



Written assignments on page 3

Written assignment 1: Let (X, d) be a metric space and $A \subseteq X$, $A \neq \emptyset$. Let

$$\gamma := \gamma(A) := \inf\{d(x,y) : x, y \in A \text{ and } x \neq y\}.$$

(a) Prove that if $\gamma > 0$ and $(x_n)_n$ is Cauchy, then $(x_n)_n$ is constant, eventually.

(b) Use **(a)** to prove that if $\gamma > 0$, then A is complete.

One point each for (a) and (b)!

Written assignment 2 (2 points):
Let $X := \mathbb{R}$ equipped with the standard Euclidean metric $d(x, x') = |x - x'|$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the following sequence of functions:

$$f_n(x) \ := \ \begin{cases} 0 & \text{if } |x| > \frac{1}{n}, \\ nx + 1 & \text{if } \frac{-1}{n} \leqq x \leqq 0, \\ -nx + 1 & \text{if } 0 \leqq x \leqq \frac{1}{n}, \end{cases}$$

i.e., the point $\big(x, f_n(x)\big)$ is on the straight line between $(-\frac{1}{n}, 0)$ and $(0, 1)$ for $\frac{-1}{n} \leqq x \leqq 0$, it is on the straight line between $(0, 1)$ and $(\frac{1}{n}, 0)$ for $0 \leqq x \leqq \frac{-1}{n}$, and it is on the x–axis for all other x. Draw a picture! Let $f(x) := 0$ for $x \neq 0$ and $f(0) := 1$.

a. Prove that f_n converges pointwise to f on $\mathbb{R}$. In other words, prove that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$.

b. Prove that f_n does not converge uniformly to f on $\mathbb{R}$. You may use without proof that each of the functions f_n is continuous on $\mathbb{R}$.

One point each for (a) and (b)!

Hint for part **(a)**: For $\boxed{\text{fixed } x \neq 0}$, what happens eventually, i.e., for big enough n? Transform the inequalities $\cdots \leq x \leq \ldots$ into inequalities for n and you should see what happens.

Example (NOT legit as a proof): If $x = 0.01$, what happens if $n > 1000$? Thus $\lim\limits_{n\to\infty} f_n(0.01)$ = WHAT?

No need to submit those hints as part of your HW. Just use them!

Hint for part **(b)**: Look at the (very few propositions and theorems of Ch.13.2.1).