

# Math 330 Section 1 - Spring 2026 - Homework 14

Published: Monday, April 6, 2026

Running total: 49 points

Last submission: Friday, April 24, 2026

## Status - Reading Assignments:

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:

ch.1 – ch.3, skim ch.4, ch.5.1 – ch.12.8.

B/G (Beck/Geoghegan) Textbook:

ch.1 – 7.1, ch.8 – 13.4

Other material (optional):

B/K lecture notes ch.1.1 and ch.1.2, except ch.1.2.4

Stewart Calculus 7ed - ch.1.7: “The Precise Definition of a Limit”

## Reading assignment 1 - due Monday, April 13.

- a. Carefully read MF ch.12.9. Draw plenty of pictures!
- b. Carefully read MF ch.12.10. The Cauchy criterion is particularly useful when examining the convergence behavior of infinite series. Understand the connection AND DIFFERENCE between the completeness axiom and completeness of  $(X, d) = (\mathbb{R}, d_{\|\cdot\|_2})$ !

## Reading assignment 2 - due Wednesday, April 15:

- a. Review convergence and continuity in  $(X, d) = (\mathbb{R}, d_{\|\cdot\|_2})$  (Ch.9.3).
- b. Extra carefully read MF ch.13.1.1. Be aware that some of the Chapter 9.3 material does not hold in general metric spaces!
- c. Carefully read MF ch.13.1.2. (Very brief.)

## Reading assignment 3 - due Friday, April 17:

- a. Carefully read MF ch.13.1.3 (quite brief) through Theorem 13.4. The stronger students are encouraged to study Theorem 13.5.
- b. Read carefully MF ch.13.2.1. Understand the DIFFERENCE between convergence and uniform convergence, continuity and uniform continuity, convergence and uniform continuity. Definitely on quizzes and/or the final exam and one of the homework sets!
- c. Review your Calc II knowledge about series, in particular the concept of absolute convergence.
- d. Read carefully MF ch.13.2.2 through Remark 13.14. Be sure to understand why a permutation (rearrangement) of the indices of a (infinite) sequence corresponds to a bijection  $\mathbb{N} \xrightarrow{\sim} \mathbb{N}$ .

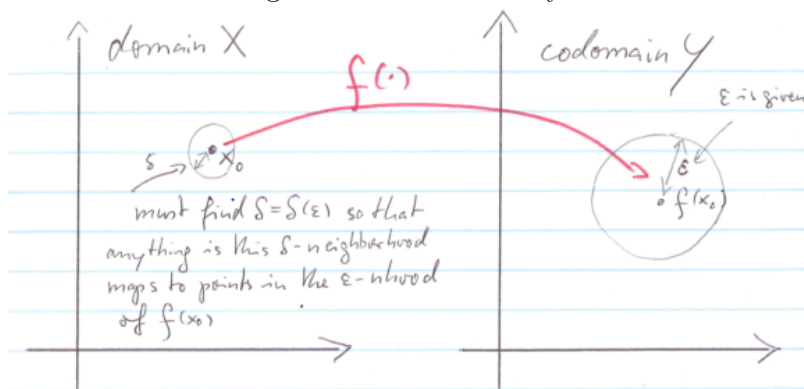
**Be sure to read pages 2 and 3!**

## Supplementary instructions for reading MF ch.12:

When you read or reread any topics in those chapters then the following is good advice:

- a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces  $(\mathbb{R}^2, d_{\|\cdot\|_2})$  and  $(\mathcal{B}(X, \mathbb{R}), d_{\|\cdot\|_\infty})$  for this. Do these drawings in particular for
  - open sets and neighborhoods (ch.12.3)
  - convergence, expressed with nhoods (the end of def.12.10 in ch.12.4)
  - metric and topological subspaces (ch.12.7): draw an irregular shaped subset  $A \subseteq \mathbb{R}^2$  in two pieces  $A = A_1 \uplus A_2$  which do not overlap. Draw some points  $x_j \in A$  with  $\varepsilon$ -nhoods (circles with radius  $\varepsilon$  about  $x_j$ ) so that some circles are entirely in  $A$ , one with  $x_j \in A_1$  which reaches into  $A^\circ$  but not into  $A_2$ , and one with  $x_j \in A_2$  which reaches both into  $A^\circ$  and  $A_1$ . What is  $N_\varepsilon^A(x_j)$ ?
  - Contact points, closed sets and closures (ch.12.8): Draw subsets  $B \subseteq \mathbb{R}^2$  with parts of their boundary (periphery) drawn solid to indicate that points there belong to  $B$  and other parts drawn dashed to indicate that those boundary points belong to the complement. What is  $\bar{B}$ ?  
 Draw points “completely inside”  $B$ , others “completely outside”  $B$ , and others on the solid and dashed parts of the boundary. Which ones can you approximate from within  $B$  by sequences? Which ones can you surround by circles that entirely stay within  $B$ , i.e., which ones are interior points of  $B$ ? Which ones can you surround by circles that entirely stay outside the closure of  $B$ , i.e., which ones are entirely within  $\bar{B}^c$ ? Use those pictures to visualize the definitions in this chapter and thm 12.7 and thm.12.8.
  - Now repeat that exercise with an additional set  $A$  which is meant to be a metric subspace of  $\mathbb{R}^2$ .
- b. MF ch.13.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 0.1:  $\varepsilon$ - $\delta$  continuity



Written assignments on page 3

## Written assignments:

**Written assignment 1:** (3 points!) Prove MF prop.11.13 (Properties of the sup norm):  $h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$  defines a norm on  $\mathcal{B}(X, \mathbb{R})$

This assignment is worth three points: **One point each** for pos.definite, absolutely homogeneous, triangle inequality!

**Hint:** Go for a treasure hunt in ch.9.2 (Minima, Maxima, Infima and Suprema), and look at the properties of  $\sup(A)$  ( $A \subseteq \mathbb{R}$ ) to prove absolute homogeneity and the triangle inequality.

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove that  $\|\cdot\|_\infty$  satisfies the triangle inequality (11.29c) of a norm you will have to write something along the following lines:

**c.** Triangle inequality.

NTS:  $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$  for all  $f, g \in \mathcal{B}(X, \mathbb{R})$ .

Proof:

$$\begin{aligned}\|f + g\|_\infty &= \sup\{|f(x)| + |g(x)| : x \in X\} \quad (\text{definition of } \|\cdot\|_\infty) \\ &= \dots \quad (\dots) \\ &\leq \dots \quad (\dots) \\ &= \|f\|_\infty + \|g\|_\infty \quad (\dots)\end{aligned}$$

(There may be fewer or more steps in your proof. The above only serves as an illustration!)

No need to justify properties of the absolute value  $|\alpha|$  of a real number  $\alpha$ , but you will need to justify why

$$\sup\{|\alpha f(x)| : x \in X\} = |\alpha| \sup\{|f(x)| : x \in X\},$$

and why

$$\sup\{|f(x) + g(x)| : x \in X\} \leq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}.$$

|                                                                                                                                                        |
|--------------------------------------------------------------------------------------------------------------------------------------------------------|
| An aside: <b>DO NOT</b> write $\ f(x)\ _\infty$ when you deal with the real number $f(x)$ (and <b>you probably mean</b> the absolute value $ f(x) $ ). |
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|                                                                            |
|----------------------------------------------------------------------------|
| $\ \cdot\ _\infty$ is defined for functions $f$ , NOT for numbers $f(x)$ ! |
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**Written assignment 2 (3 points):**

Prove MF thm.12.1 (Norms define metric spaces): Let  $(V, \|\cdot\|)$  be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}_{\geq 0}; \quad (x, y) \mapsto d_{\|\cdot\|}(x, y) := \|y - x\|$$

defines a metric space  $(V, d_{\|\cdot\|})$ .

This assignment is worth three points: **One point each** for pos.definite, symmetry, triangle inequality!

**Hint:** You will have to show for each one of (12.1a), (12.1b), (12.1c) how it follows from def. 11.17: Which one of (11.31a), (11.31b), (11.31c) do you use at which spot?

Careful with symmetry: What is the reason that  $\|a - b\| = \|b - a\|$ ?

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove, e.g., that  $d_{\|\cdot\|}(\cdot, \cdot)$  satisfies the triangle inequality (12.1c) of a metric you will have to write something along the following lines:

**c.** Triangle inequality.

NTS:  $d_{\|\cdot\|}(x, z) \leq d_{\|\cdot\|}(x, y) + d_{\|\cdot\|}(y, z)$  for all  $x, y, z \in X$ .

Proof:

$$\begin{aligned} d_{\|\cdot\|}(x, z) &= \|z - x\| && \text{(definition of the metric } d_{\|\cdot\|}) \\ &= \dots && (\dots) \\ &\leq \dots && (\dots) \\ &= d_{\|\cdot\|}(x, y) + d_{\|\cdot\|}(y, z) && (\dots) \end{aligned}$$

(There may be fewer or more steps in your proof. The above only serves as an illustration!)