MATH 304-01
Fall 2019
Exam 2
October 30, 2019
Time Limit: 90 Minutes

Name (Print): ____________________

This exam contains 3 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page. Question 1 is a **True/False** question. Clearly CIRCLE your correct answer. You are required to show your work on Questions 2 to 7 on this exam.

Instruction:

- **All solutions must be written on the blue book.**
- **At the end of the exam, please turn in both this exam and the blue book.**
- **Turn off and put away your cell phone.**
- **Notes, the textbooks, and digital devices are not permitted.**
- **Discussion or collaboration is not allowed.**
- **Justify your answers, and write clearly.**
- **Mysterious or unsupported answers will not receive full credit.**

Do not write in the table to the right.

Question	Points	Score
1	10	
2	15	
3	10	
4	15	
5	20	
6	25	
7	10	
Total:	105	

1. (10 points) In each question circle either True or False. No justification is needed.
   (a) True False Let A be a square matrix. If the columns of A are linearly dependent, then $\det(A) = 0$.
   (b) True False If A is a 5×5 matrix and $\mathrm{Col}A = \mathbb{R}^5$, then for each $\mathbf{b} \in \mathbb{R}^5$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.
   (c) True False Let A be a 5×8 matrix. If the null space of A has a basis consisting of 2 vectors. Then $\mathrm{rank}(A) = 3$.
   (d) True False If A and B are 3×3 matrices, then $\det(A + B) = \det(A) + \det(B)$.
   (e) True False The column space of A is the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.
   (f) True False Let A be an $m \times n$ matrix. Then $\dim \mathrm{Col}A + \dim \mathrm{Row}A = n$.
   (g) True False If two matrices A and B are row equivalent, then their row spaces are the same and if B is in row echelon form, then the nonzero rows of B form a basis for the row space of A.
   (h) True False If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.
   (i) True False Let V be a vector space. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is a set of vectors in V and S spans V, then some subset of S is a basis for V.
   (j) True False Let V be a finite-dimensional vector space. If every set of p elements in V fails to span V, then $\dim V > p$.

2. (15 points) Which of the following sets is a subspace of the given vector space? If it is not a subspace, find a specific example (two vectors or a vector and a scalar) to show that it is not a subspace.
   (a) The set $H = \left\{ \begin{bmatrix} 2a+b \\ a+b \\ a-3b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ in the vector space $\mathbb{R}^3$.
   (b) The set $H = \{ \begin{bmatrix} a \\ b \end{bmatrix} : a^2 + 4b^2 \leq 1, a, b \text{ in } \mathbb{R}\}$ in the vector space $\mathbb{R}^2$.
   (c) The set H of all polynomials $p(t)$ in $\mathbb{P}_2$ satisfying $p(0) = 1$, where $\mathbb{P}_2$ is the vector space of all polynomials of degree at most 2.

3. (10 points) Compute the following determinants.
   (a) $\begin{vmatrix} 1 & -2 & -2 \\ 2 & 1 & 2 \\ -1 & 3 & 3 \end{vmatrix}$
   (b) $\begin{vmatrix} 0 & 1 & 0 & 2 & -6 \\ 4 & 10 & 30 & 20 & 19 \\ 0 & 2 & -1 & -2 & -9 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -6 & 0 \end{vmatrix}$

4. (15 points) Let A, B and C be 5×5 matrices. Assume that $\det(A) = 2, \det(B) = -3$ and $\det(C) = -4$. Compute
   (a) $\det(ABC^2)$.
   (b) $\det(A^{-2}B^2C^T)$.
   (c) $\det(2A)$.
   (d) Let P be an invertible 5×5 matrix. What is $\det(P^TAP^{-1})$?
   (e) Assume that Q is a 5×5 matrix such that $Q^TAQ = A$. Show that $\det Q = \pm 1$.

5. (20 points) Assume that the matrix A is row equivalent to the matrix B, where

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 3 & 6 & -3 & 3 & 6 \\ 2 & 4 & -1 & 5 & 3 \\ 1 & 2 & 0 & 4 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Find Rank$A$, dim Nul A and dim Row$A$.
   (b) Find a basis for Col$A$.
   (c) Find a basis for Row$A$.
   (d) Find a basis for Nul$A$.

6. (25 points) Let $p_1(t) = t + 1$, $p_2(t) = t^2 + t$ and $p_3(t) = t^2 + t - 2$ be polynomials in $\mathbb{P}_2$, the vector space of all polynomials in the variable t of degree at most 2. Let $\mathcal{B} = \{1, t, t^2\}$ be a basis for $\mathbb{P}_2$.
   (a) Show that the set $\{p_1(t), p_2(t), p_3(t)\}$ is linearly independent (Hint. Use $\mathcal{B}$-coordinate vectors or definition of linear independence).
   (b) Without further calculation explain why $\mathcal{C} = \{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\mathbb{P}_2$?
   (c) Find the change-of-coordinates matrix $P(\mathcal{B} \leftarrow \mathcal{C})$ from $\mathcal{C}$ to $\mathcal{B}$.
   (d) Let $p(t) = t^2 + 4t + 5$. Find the coordinate vector $[p(t)]_{\mathcal{C}}$.
   (e) Suppose $q(t) \in \mathbb{P}_2$ and $[q(t)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Find the polynomial $q(t)$.

7. (10 points) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for a vector space V. Show that for each $\mathbf{x} \in V$, there exists a unique set of scalars $c_1, c_2, \cdots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$.