MATH 304-01
Fall 2019
Exam 3
November 25, 2019
Time Limit: 90 Minutes

Name (Print): ______________________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page. Question 1 is a **True/False** question. Clearly CIRCLE your correct answer. You are required to show your work on Questions 2 to 6 on this exam.

Instruction:

- **All solutions must be written on the blue book.**
- **At the end of the exam, please turn in both this exam and the blue book.**
- **Turn off and put away your cell phone.**
- **Notes, the textbooks, and digital devices are not permitted.**
- **Discussion or collaboration is not allowed.**
- **Justify your answers, and write clearly.**
- **Mysterious or unsupported answers will not receive full credit.**

Do not write in the table to the right.

Question	Points	Score
1	10	
2	10	
3	20	
4	20	
5	25	
6	25	
Total:	110	

1. (10 points) In each question circle either True or False. No justification is needed.

   (a) **False** Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Then $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A.

   (b) **True** If A is a 3×3 matrix and $\lambda_1 = 1, \lambda_2 = -1$ and $\lambda_3 = 2$ are distinct eigenvalues of A. Then $\det(A) = -2$.

   (c) **True** An $n \times n$ matrix A is invertible if and only if 0 is not an eigenvalue of A.

   (d) **True** Let V be an inner product space with inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ for $\mathbf{u}, \mathbf{v} \in V$. For $\mathbf{x}, \mathbf{y} \in V$, we have $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$.

   (e) **False** If a square $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.

   (f) **False** Let A be an $n \times m$ matrix. Then $\mathrm{Col}(A)^{\perp} = \mathrm{Nul}(A)$.

   (g) **True** If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{x} \in W \cap W^{\perp}$, then $\mathbf{x} = \mathbf{0}$.

   (h) **False** If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with its usual dot product, then

   $$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

   (i) **False** Let A be an $n \times n$ matrix. If the columns of A are linearly independent, then 0 is an eigenvalue of A.

   (j) **True** An $n \times n$ matrix P is called an orthogonal matrix if $P^T P = I$. If A and B are $n \times n$ orthogonal matrices, then AB is also an orthogonal matrix.

2. Let $T : \mathbb{P}_2[t] \to \mathbb{P}_3[t]$ be the linear transformation defined by $T(p(t)) = (t+1)p(t)$ for all $p(t) \in \mathbb{P}_2[t]$.

   (a) (4 points) Find $T(q(t))$ with $q(t) = 1 - 2t + t^2$.

   (b) (6 points) Find the matrix for T relative to the bases $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{1, t, t^2, t^3\}$.

Solutions.

(a) $T(q(t)) = (t+1)q(t) = (t+1)(1 - 2t + t^2) = 1 - t - t^2 + t^3$.

(b) We have

$$\begin{aligned} T(1) &= (t+1)1 = t + 1 = 1 \cdot 1 + 1 \cdot t + 0 \cdot t^2 + 0 \cdot t^3 \\ T(t) &= (t+1)t = t + t^2 = 0 \cdot 1 + 1 \cdot t + 1 \cdot t^2 + 0 \cdot t^3 \\ T(t^2) &= (t+1)t^2 = t^2 + t^3 = 0 \cdot 1 + 0 \cdot t + 1 \cdot t^2 + 1 \cdot t^3. \end{aligned}$$

So the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

3. (20 points) Let $\mathbb{P}_2[t]$ be an inner product space with the inner product given by

$$\langle f(t), g(t)\rangle = \int_0^1 f(t)g(t)dt$$

for $f(t), g(t) \in \mathbb{P}_2[t]$.

(a) Compute $\langle 6t-1, 12t^2\rangle$.

(b) Compute $||6t-1||$.

(c) Let L be the subspace of $\mathbb{P}_2[t]$ spanned by $2t-1$. Compute $\text{proj}_L(12t^2)$, the projection of $12t^2$ into L.

(d) Let $p(t), q(t) \in \mathbb{P}_2[t]$ with $||p(t)|| = ||q(t)|| = 1$ and $\langle p(t), q(t)\rangle = 0$. Show that

$$||p(t) - q(t)|| = \sqrt{2}.$$

Solutions.

(a)

$$\langle 6t-1, 12t^2\rangle = \int_0^1 (6t-1)(12t^2)dt = \int_0^1 (72t^3 - 12t^2)dt = (18t^4 - 4t^3)|_0^1 = 14.$$

(b) $||6t-1|| = \sqrt{\langle 6t-1, 6t-1\rangle} = \sqrt{7}$ since

$$\langle 6t-1, 6t-1\rangle = \int_0^1 (6t-1)^2 dt = \int_0^1 (36t^2 - 12t + 1)dt = (12t^3 - 6t^2 + t)|_0^1 = 7.$$

(c) We have

$$\text{proj}_L(12t^2) = \frac{\langle 12t^2, 6t-1\rangle}{\langle 6t-1, 6t-1\rangle}(6t-1) = \frac{14}{7}(6t-1) = 12t-2.$$

(d) Since $\langle p(t), q(t)\rangle = 0$ and $||p(t)|| = ||q(t)|| = 1$, we deduce that

$$\langle p(t), -q(t)\rangle = -\langle p(t), q(t)\rangle = 0 \text{ and } ||-q(t)|| = ||q(t)|| = 1.$$

By Pythagorean theorem, we have

$$\begin{aligned} ||p(t) - q(t)||^2 &= ||p(t) + (-q(t))||^2 \\ &= ||p(t)||^2 + ||-q(t)||^2 \\ &= ||p(t)||^2 + ||q(t)||^2 = 2 \end{aligned}$$

Hence $||p(t) - q(t)|| = \sqrt{2}$.

4. (20 points) Let $V = \mathbb{P}_3[t]$ be an inner product space with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$$

for $f, g \in V$. Use the Gram-Schmidt process to find an orthogonal basis for the subspace W of V spanned by $\{1, t, t^2\}$. (Do not normalize).

Solutions.

Let $p_1(t) = 1, p_2(t) = t$ and $p_3(t) = t^2$.

We have

$$\begin{aligned} q_1(t) &= p_1(t) = 1 \\ q_2(t) &= p_2(t) - \frac{\langle p_2(t), q_1(t)\rangle}{\langle q_1(t), q_1(t)\rangle} q_1(t) \\ q_3(t) &= p_3(t) - \frac{\langle p_3(t), q_1(t)\rangle}{\langle q_1(t), q_1(t)\rangle} q_1(t) - \frac{\langle p_3(t), q_2(t)\rangle}{\langle q_2(t), q_2(t)\rangle} q_2(t) \end{aligned}$$

Now

$$\langle q_1(t), q_1(t)\rangle = \int_{-1}^{1} q_1(t)^2 dt = \int_{-1}^{1} dt = t|_{-1}^{1} = 2$$

and

$$\langle p_2(t), q_1(t)\rangle = \int_{-1}^{1} p_2(t)q_1(t)dt = \int_{-1}^{1} tdt = \frac{1}{2}t^2|_{-1}^{1} = 0.$$

So $q_2(t) = p_2(t) = t$. Now

$$\langle p_3(t), q_1(t)\rangle = \int_{-1}^{1} p_3(t)q_1(t)dt = \int_{-1}^{1} t^2 dt = \frac{1}{3}t^3|_{-1}^{1} = \frac{2}{3}$$

$$\langle q_2(t), q_2(t)\rangle = \int_{-1}^{1} q_2(t)^2 dt = \int_{-1}^{1} t^2 dt = \frac{1}{3}t^3|_{-1}^{1} = \frac{2}{3}$$

$$\langle p_3(t), q_2(t)\rangle = \int_{-1}^{1} p_3(t)q_2(t)dt = \int_{-1}^{1} t^3 dt = \frac{1}{4}t^4|_{-1}^{1} = 0.$$

Thus

$$\begin{aligned} q_3(t) &= p_3(t) - \frac{\langle p_3(t), q_1(t)\rangle}{\langle q_1(t), q_1(t)\rangle} q_1(t) - \frac{\langle p_3(t), q_2(t)\rangle}{\langle q_2(t), q_2(t)\rangle} q_2(t) \\ &= t^2 - \frac{\frac{2}{3}}{2} 1 - \frac{0}{\frac{2}{3}} t \\ &= t^2 - \tfrac{1}{3}. \end{aligned}$$

Therefore $\{1, t, t^2 - \frac{1}{3}\}$ is an orthogonal basis for $\mathbb{P}_2[t]$.

5. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

(a) (7 points) Find the characteristic polynomial of A. (Show your work).

(b) (4 points) Verify that A has three distinct eigenvalues $\lambda_1 = 1, \lambda_2 = -2$ and $\lambda_3 = 2$.

(c) (9 points) For each λ_i, find a basis for the eigenspace corresponding to eigenvalue λ_i.

(d) (5 points) Determine whether A is diagonalizable or not. If it is, find matrices P and D such that $A = PDP^{-1}$, where D is a diagonal matrix. If not, explain why.

Solutions.

(a) Characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + 4\lambda - 4.$$

(b) It is easy to check that $\lambda_1 = 1, \lambda_2 = -2$ and $\lambda_3 = 2$ are solutions of the characteristic polynomial of A since

$$-\lambda^3 + \lambda^2 + 4\lambda - 4 = \lambda^2(1-\lambda) - 4(1-\lambda) = (1-\lambda)(\lambda^2 - 4) = (1-\lambda)(\lambda - 2)(\lambda + 2).$$

(c) Finding a basis for the eigenspace corresponding to $\lambda_1 = 1$. We need to solve the homogeneous linear system $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$.
We have

$$\begin{aligned} (A - \lambda_1 I|\mathbf{0}) &= \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & -2 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \\ &\xrightarrow{R_2 + R_1} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So we have the system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

and so

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \end{aligned}$$

hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus $\{\mathbf{v}_1\}$ is a basis for the eigenspace corresponding to $\lambda_1 = 1$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Finding a basis for the eigenspace corresponding to $\lambda_1 = -2$. We need to solve the homogeneous linear system $(A - \lambda_2 I)\mathbf{x} = \mathbf{0}$.
We have

$$
\begin{aligned}
(A - \lambda_2 I|\mathbf{0}) \quad &= \begin{pmatrix} 3 & -1 & 1 & 0 \\ -1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \overset{R_1 \leftrightarrow R_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 3 & 1 & 0 \\ 3 & -1 & 1 & 0 \end{pmatrix} \\
&\overset{R_2+R_1}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{pmatrix} \overset{R_3-3R_1}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -4 & -2 & 0 \end{pmatrix} \\
&\overset{R_3+R_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_2/4}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\overset{R_1-R_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}.
$$

So we have the system

$$
\begin{aligned}
x_1 + \tfrac{1}{2}x_3 &= 0 \\
x_2 + \tfrac{1}{2}x_3 &= 0
\end{aligned}
$$

and so

$$
\begin{aligned}
x_1 &= -\tfrac{1}{2}x_3 \\
x_2 &= -\tfrac{1}{2}x_3
\end{aligned}
$$

hence

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix} = -\frac{1}{2}x_3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.
$$

Thus $\{\mathbf{v}_2\}$ is a basis for the eigenspace corresponding to $\lambda_2 = -2$ where $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Finding a basis for the eigenspace corresponding to $\lambda_1 = 2$. We need to solve the homogeneous linear system $(A - \lambda_3 I)\mathbf{x} = \mathbf{0}$.
We have

$$
\begin{aligned}
(A-\lambda_3 I|\mathbf{0}) &= \begin{pmatrix} -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -3 & 0 \end{pmatrix} \xrightarrow{-R_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -3 & 0 \end{pmatrix} \\
&\xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & 0 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \\
&\xrightarrow{R_2 \leftrightarrow} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2/-2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}.
$$

So we have the system

$$
\begin{aligned}
x_1 + x_2 &= 0 \\
x_3 &= 0
\end{aligned}
$$

and so

$$
\begin{aligned}
x_1 &= -x_2 \\
x_3 &= 0
\end{aligned}
$$

hence

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.
$$

Thus $\{\mathbf{v}_3\}$ is a basis for the eigenspace corresponding to $\lambda_3 = 2$ where $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Hence A is diagonalizable and

$$
P = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{pmatrix}
$$

and

$$
D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
$$

6. (25 points) Let $V = \mathbb{R}^4$ with the usual dot product $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in V$. Let

$$
\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{z} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 4 \end{bmatrix}.
$$

Let $S = \{\mathbf{x}, \mathbf{y}\}$.

(a) Verify that S is an orthogonal basis for the subspace W spanned by S.

(b) Find the unit vector $\mathbf{u}$ in the direction of $\mathbf{x}$.

(c) Find the distance between $\mathbf{x}$ and $\mathbf{y}$.

(d) Find $\text{proj}_W(\mathbf{z})$, the projection of $\mathbf{z}$ into $W = \text{span}(S)$.

(e) Write $\mathbf{z} = \mathbf{w} + \mathbf{u}_1$, where $\mathbf{w} = \text{proj}_W(\mathbf{z}) \in W$ and $\mathbf{u}_1 \in W^\perp$.

Solutions.

(a) We have that $\mathbf{x} \cdot \mathbf{y} = 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot 1 + (-1) \cdot 1 = 0$ so S is an orthogonal set of non-zero vectors and since S spans $W = \text{span}(S)$, S is an orthogonal basis for W.

(b) The unit vector in the direction $\mathbf{x}$ is

$$\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{\mathbf{x}}{2} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix},$$

where $||\mathbf{x}||^2 = \mathbf{x} \cdot \mathbf{x} = 1^2 + (-1)^2 + 1^2 + (-1)^2 = 4$, hence $||\mathbf{x}|| = \sqrt{4} = 2$.

(c) We have $\mathbf{x} - \mathbf{y} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -2 \end{bmatrix}$ and so

$$\text{dist}(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})} = \sqrt{0^2 + (-2)^2 + 0^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}.$$

(d) Since S is an orthogonal basis for W, the projection of $\mathbf{z}$ into the subspace W can be determined by

$$\mathbf{w} = \text{proj}_W(\mathbf{z}) = \frac{\mathbf{z} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}\mathbf{x} + \frac{\mathbf{z} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\mathbf{y} = \frac{4}{4}\mathbf{x} + \frac{8}{4}\mathbf{y} = \mathbf{x} + 2\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}.$$

Thus $\mathbf{u}_1 = \mathbf{z} - \mathbf{w} = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 3 \end{bmatrix}$.

We see that

$$\mathbf{u}_1 \cdot \mathbf{x} = 1(-1) + (-1)(-3) + 1 \cdot 1 + (-1) \cdot 3 = 0$$

and

$$\mathbf{u}_1 \cdot \mathbf{y} = 1(-1) + 1(-3) + 1 \cdot 1 + 1 \cdot 3 = 0.$$

So $\mathbf{u}_1 \in W^\perp$ as S spans W.