

EXERCISE 3F # 14

$$T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z) = x(4, 7) + y(5, 8) + z(6, 9).$$

$$\text{Thus, } M(T) = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

$$M(T') = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

I don't use these matrices in the solution.

(a).

Note that $\varphi_1(a, b) = \varphi_1(ae_1 + be_2) = a\varphi_1(e_1) + b\varphi_1(e_2) = a$ and similarly $\varphi_2(a, b) = b$. Using these formulas,

$$T'(\varphi_1)(x, y, z) = \varphi_1(T(x, y, z)) = \varphi_1(4x + 5y + 6z, 7x + 8y + 9z) = 4x + 5y + 6z.$$

$$T'(\varphi_2)(x, y, z) = \varphi_2(T(x, y, z)) = \varphi_2(4x + 5y + 6z, 7x + 8y + 9z) = 7x + 8y + 9z.$$

(b).

Note that $\psi_1(x, y, z) = \psi_1(xe_1 + ye_2 + ze_3) = x$, etc. Therefore,

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3 \text{ because } T'(\varphi_1)(v) = 4\psi_1(v) + 5\psi_2(v) + 6\psi_3(v).$$

$$T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3 \text{ because } T'(\varphi_2)(v) = 7\psi_1(v) + 8\psi_2(v) + 9\psi_3(v).$$

EXERCISE 3F # 5

The best solution is to use the dual basis, $\psi_1, \dots, \psi_m$, of the standard basis of $\text{range}(T)$. Then

$$\psi_i \circ T(v) = \psi_i\left(\sum_j \varphi_j(v)w_j\right) = \sum_j \varphi_j(v)\psi_i(w_j) = \sum_j \varphi_j(v)\delta_{ij} = \varphi_i(v).$$

Thus, φ_i is a composition of linear functions, so it is linear.

I used the very handy Kronecker delta function, $\delta_{ij} := 1$ if $i = j$, and 0 if $i \neq j$.

Congratulations to those who noticed this proof!