How to Calculate Duals in Linear Algebra
Math 507, Fall 2026

The ultimate goal is to find a formula for $T' : W' \to V'$, the dual mapping of a linear map $T : V \to W$. That enables us to use T'. We get a formula in terms of the **matrices!!** of vectors and maps.

I'll set up the background picture and then do a calculation in general and in a small example.

Setup

1. A field F, usually $\mathbb{R}$ or $\mathbb{C}$.
2. V, W are vector spaces, $\dim V = n$ and $\dim W = m$, and we have a linear mapping $T : V \to W$.
3. Dual spaces $V' = \mathcal{L}(V, F)$ and $W' = \mathcal{L}(W, F)$.
4. Chosen bases $B_V = (v_1, \ldots, v_n)$ of V and $B_W = (w_1, \ldots, w_m)$ of W.
5. Dual bases $B_{V'} = (\varphi_{v_1}, \ldots, \varphi_{v_n})$ of V' and $B_{W'} = (\varphi_{w_1}, \ldots, \varphi_{w_n})$ of W'.
   E.g., φ_{v_i} is defined on the basis of V by $\varphi_{v_i}(v_j) = 1$ if $j = i$ and 0 if $j \neq i$.
6. $M(T)$ is the matrix of T with respect to the chosen bases B_V and B_W.
7. The dual linear mapping $T' : W' \to V'$.
8. Since $T'(\psi) \in V'$, it is a linear functional that operates on elements of V, by the rule $[T'(\psi)](v) := (\psi \circ T)(v) \in F$. The matrix version of this formula is $M\big([T'(\psi)](v)\big) = M(\psi \circ T)M(v) = M(\psi)M(T)M(v)$.

General computation

This has many steps. Each step involves working with the expressions of objects (vectors and functions) in terms of the bases we're using, which means using their matrices.

I'll give the general method first and then a specific example, both presented in exactly the same way so you can look back and forth to compare them.

Vectors.

V1. A vector $v \in V$ has a basis expansion $v = \sum_{i=1}^{n} \alpha_i v_i$. The matrix of v (Definition 3.73) is $M(v) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}^T$ (in transpose notation).

V2. A vector $w \in W$ has the basis expansion $w = \sum_i \gamma_i w_i$. The matrix of w is $M(w) = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix}^T$.

Linear functionals.

All computations are done in terms of bases of the vector spaces and matrices of maps and vectors. I'll do this for V. I calculate the action of a linear functional on a vector, using matrices.

L1. A functional $\varphi \in V'$ has a basis expansion for which we use the dual basis: $\varphi = \sum_{j=1}^{n} \beta_j \varphi_{v_j}$. The matrix of φ is $M(\varphi) = \begin{pmatrix} \beta_1 & \cdots & \beta_n \end{pmatrix}$, a *row matrix*. It's a row matrix because the map is from V (dimension n) to F (dimension 1).

This is a crucial difference between vectors and functionals, because $\varphi(v)$ in terms of matrices is $M(\varphi(v)) = M(\varphi)M(v) = \sum_i \beta_i \alpha_i$. (This looks like a dot product. It isn't a dot product, it's a row times a column.)

L2. A functional $\psi \in W'$ has a basis expansion $\psi = \sum_j \delta_j \varphi_{w_j}$, so $M(\psi) = \begin{pmatrix} \delta_1 & \cdots & \delta_m \end{pmatrix}$.
Then, for $w \in W$, we have $M(\psi(w)) = M(\psi)M(w) = \sum_i \delta_i \gamma_i$.

Dual mappings.

We'll calculate T' in terms of its action on a linear functional $\psi \in W'$, that is, we'll calculate the matrix of $T'(\psi)$.

Since $T' \in \mathcal{L}(W', V')$, it operates on elements $\psi \in W'$. It is defined by $T'(\psi) := \psi \circ T \in V'$. The matrix of a composition is the product of matrices, so,

$$M(T'(\psi)) = M(\psi \circ T) = M(\psi)M(T). \tag{0.1}$$

Our calculation has proved a theorem.

Theorem 1. *The matrix formula for T' is the matrix product*

$$M\big([T'(\psi)](v)\big) = M(\psi)M(T)M(v)$$

for any $\psi \in W'$ and $v \in V$.

Proof. First step: We have a function $T'(\psi)$ that acts on the vector V; the matrix formula for this is $M\big([T'(\psi)](v)\big) = M(T'(\psi))M(v)$. Second step: We have a composition of mappings as in Equation (0.1), so we apply that equation to get $[M(\psi)M(T)]M(v)$. Then apply associativity of multiplication. That's the proof. □

(Warning about weird notation: The product $M(\psi)M(T)M(v)$ is technically a 1×1 matrix, but we think of a 1×1 matrix as being the same as a scalar. This is not my fault.)

An example of a dual mapping

We choose $V = \mathbb{R}^2$ with basis B_V: $v_1 = (1,3)$, $v_2 = (2,1)$ and $W = \mathbb{R}^2$ with basis B_W: $w_1 = (1,0)$, $w_2 = (0,1)$. The only hard parts are to find the matrix of $v \in V$ in terms of B_V and that of $T(v) \in W$ in terms of B_W.

The linear map $T : V \to W$ is defined by $T(x_1, x_2) = (2x_1 - x_2, x_1)$. We need to find this in terms of B_V and B_W, because we need the matrix $M(T; B_V, B_W)$ of T with respect to those bases. First, let's rewrite T in terms of matrices:

$$T(x_1, x_2) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{0.2}$$

That 2×2 matrix is the matrix $M(T; E_V, E_W)$ of T with respect to the standard bases E_V and E_W, not our chosen bases, so we have to do a conversion using the basis-change formula. Unfortunately, Formula 3.84 is for a mapping $V \to V$, so let's think about how to solve this. Fortunately,

$$M(T; B_V, B_W) = M(I; B_V, E_V)M(T; E_V, E_W)M(I; E_W, B_W). \tag{0.3}$$

That is a special case of composition of mappings, $T = I \circ T \circ I$ (where I is the identity map). Remember that every matrix of a map is in terms of specific bases.

Our example has $B_W = E_W$ so $M(I; E_W, B_W) = I_2$, the identity matrix. We know

$$M(T; E_V, E_W) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

because this is the matrix in our formula for T in Equation (0.2).

It remains to find $M(I;B_V,E_V)$, which is the matrix that converts $M(v;E_V)=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ to $M(v;B_V)$; that is,

$$M(I;B_V,E_V)\begin{pmatrix}x_1\\x_2\end{pmatrix}=M(v;B_V).$$

(To simplify notation I'll write $A:=M(I;B_V,E_V)$.) In particular,

$$A\begin{pmatrix}1&2\\3&1\end{pmatrix}=A\begin{pmatrix}v_1&v_2\end{pmatrix}=\begin{pmatrix}Av_1&Av_2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

because we're converting v_1 and v_2 from the standard basis to the basis B_V in which they themselves are the basis vectors. So

$$M(I;B_V,E_V)=\begin{pmatrix}1&2\\3&1\end{pmatrix}^{-1}=\begin{pmatrix}-\frac{1}{5}&\frac{1}{5}\\\frac{3}{5}&-\frac{1}{5}\end{pmatrix}. \tag{0.4}$$

That gives us the conversion of any $v\in V$ from the standard E_V basis to the B_V basis, namely,

$$M(v;B_V)=\begin{pmatrix}-\frac{1}{5}&\frac{1}{5}\\\frac{3}{5}&-\frac{1}{5}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}. \tag{0.5}$$

Now we can compute the matrix we wanted all along.

$$M(T;B_V,B_W)=\begin{pmatrix}-\frac{1}{5}&\frac{1}{5}\\\frac{3}{5}&-\frac{1}{5}\end{pmatrix}\begin{pmatrix}2&-1\\1&0\end{pmatrix}I_2=\begin{pmatrix}-\frac{1}{5}&\frac{1}{5}\\1&-\frac{4}{5}\end{pmatrix}. \tag{0.6}$$

Success!, because in Theorem 1, $M(T)=M(T;B_V,B_W)$. We put this into Theorem 1; in our special example, Theorem 1 says:

Corollary 2.

$$M\big([T'(\psi)](v)\big)=M(\psi;B'_W)\begin{pmatrix}-\frac{1}{5}&\frac{1}{5}\\1&-\frac{4}{5}\end{pmatrix}M(v;B_V). \tag{0.7}$$

Even better, if we plug Equation (0.5) into Equation (0.7) we have a straighforward formula in terms of standard coordinates in V:

$$M\big([T'(\psi)](v)\big)=M(\psi;B'_W)\begin{pmatrix}-\frac{1}{5}&\frac{1}{5}\\1&-\frac{4}{5}\end{pmatrix}\begin{pmatrix}-\frac{1}{5}&\frac{1}{5}\\\frac{3}{5}&-\frac{1}{5}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}. \tag{0.8}$$