

- Show all your work in a blue book.
- **Show enough work to justify your answer.**
- No books, notes, electronics or other aids.
- Good Advice: Do easy problems first. Do as many problems as you can if you do a good job on each one. Don't rush to give poor answers to all problems; it's more important to do a good job.

In all questions, V is a finite-dimensional vector space over a field F , with $\dim V = n$, and V' is its dual space. (The question will tell you when F is specifically the reals or complexes.)

(1) (3+3+4 points)

- Define V' in terms of V .
- Define the dual basis in V' of a basis $v_1, v_2, \dots, v_n$ in V .
- Prove that $\dim V' = n$.

Solution. (a) $V' = \{\varphi : V \rightarrow F\}$, i.e., the vector space of all linear functionals on V .

Solution. (b) For each $i = 1, 2, \dots, n$, define φ_i by $\varphi_i(v_j) = 1$ if $j = i$ and 0 if $j \neq i$. Then the dual basis is $\varphi_1, \varphi_2, \dots, \varphi_n$. (Note 1: A basis is a list; the order matters. Note 2: I have only defined the list but I haven't proved it is a basis.)

Solution. (c) Long proof: I have to prove my list in (b) is a basis. Then I know $\dim V' = n$.

(1) The functionals φ_i are linearly independent, because if a linear combination $\sum_i \alpha_i \varphi_i = 0$ (where $\alpha_i \in F$), then applying this combined functional to v_j , I get $0 = \sum_i \alpha_i \varphi_i(v_j) = \alpha_j \cdot 1$ for every j . Therefore, all $\alpha_j = 0$. That is, the only linear combination that equals 0 is the trivial one.

(2) The functions span because any $\varphi \in V'$ can be written as the linear combination $\sum_i \varphi(v_i) \varphi_i$.

Solution. (c) Short (and not complete) proof: There are n functionals φ_i and they are a basis, therefore $\dim V' = n$.

I will increase the credit for the long proof of (c).

(2) (5+5 points) Suppose U and W are subspaces of V , and $\dim V = n$, $\dim U = p$, $\dim W = q$. What can you say about:

(a) $\dim(U + W)$?

Solution. It satisfies $\max(p, q) \leq \dim(U + W) \leq \min(n, p + q)$.

(b) $\dim(U \oplus W)$ (assuming $U \oplus W$ is defined)?

Solution. It is $p + q$.

(3) (10 points) Let m be a positive integer and define polynomials $B_k(x) = x^k(1-x)^{m-k}$ for $k = 0, 1, \dots, m$. Prove that $B_0, B_1, \dots, B_m$ is a basis for $\mathcal{P}_m(\mathbb{C})$.

Solution. Let's call the list $\mathcal{B}_m$. We know $\dim \mathcal{P}_m(\mathbb{C}) = m + 1$, so we have the right number of basis vectors (yes, they are "vectors"). We should prove they are linearly independent.

Suppose not, then there exist numbers $a_0, \dots, a_m$ such that $\sum_{k=0}^m a_k B_k = 0$. The first thing I think of is to set $x = 1$, since $B_k(1) = 0$ if $k < m$ and $B_m(1) = 1^m \neq 0$. That implies $a_m = 0$. That means any linear dependence doesn't involve B_m .

Rewriting the linear dependence, $\sum_{k=0}^{m-1} a_k x^k (1-x)^{m-k} = 0$. Now $(1-x)$ is a common factor, so factor it out: $\sum_{k=0}^{m-1} a_k x^k (1-x)^{(m-1)-k} = 0$. The polynomials here are those in $\mathcal{B}_{m-1}$. We've proved that if $\mathcal{B}_m$ is linearly dependent, then so is $\mathcal{B}_{m-1}$ (for any $m > 0$). By this reasoning, if $\mathcal{B}_m$ is dependent, then $\mathcal{B}_0$ is dependent. But $\mathcal{B}_0$ contains only one polynomial, $x^0(1-x)^0 = 1$, which is not 0. Therefore, $\mathcal{B}_m$ can't be dependent. It follows that $\mathcal{B}_m$ can't be dependent. Done!

- (4) (5 points) Prove that there is no map $T \in \mathcal{L}(\mathbb{R}^5)$ such that $\text{range}(T) = \text{null}(T)$.

Solution. We know that $\dim T + \dim \text{Im } T = 5$. The isomorphism implies that $\dim T = \dim \text{Im } T$. This is impossible (because 5 is odd).

- (5) (5 points) Let A, B, C be $n \times n$ real matrices. Prove, without calculating any matrix product, that $A(B + C) = AB + AC$.

Solution. Let $V = \mathbb{R}^n$ with the standard basis. Then each of A, B, C is the matrix (with respect to the standard basis) of a linear operator $T_A : V \rightarrow V$, so that $A = M(T_A)$, etc. We know that operators satisfy $T_A(T_B + T_C) = T_A T_B + T_A T_C$, and that the matrices satisfy the same algebraic relations as the operators; therefore $A(B + C) = M(T_A(T_B + T_C)) = M(T_A T_B + T_A T_C) = AB + AC$. Done.

- (6) (5 points) Let U, W be subspaces of V . Suppose $a, b \in V$ are such that $a + U = b + W$. Prove that $U = W$.

Solution. $a + U = \{a + u : u \in U\}$ and $b + W = \{b + w : w \in W\}$. Now, $b + 0 \in b + W = a + U$ so $b = a + u_0$ for some $u_0 \in U$. Therefore, $a + U = a + u_0 + W$. That means each $a + u \in a + U$ can be written as $a + u_0 + w$ for some $w \in W$, in other words, for each $u \in U$, there is some $w \in W$ such that $u = u_0 + w$. Rewrite: $w = u - u_0 \in U$. Conclusion: $W \subseteq U$. The same argument shows that $U \subseteq W$. Thus, $U = W$.

- (7) (10 points) **Corrected: I intended to assume V is 1-dimensional.** Let $\varphi, \psi \in V'$ be such that $\text{null } \varphi \subseteq \text{null } \psi$. Prove there is some $c \in F$ such that $\psi = c\varphi$.

Solution. For the case $\dim V = 1$: If $\dim \text{null } \psi = 1$, then $\psi = 0$; then let $c = 0$. If $\dim \text{null } \psi = 0$, then $\text{null } \psi = \{0\}$ so also $\text{null } \varphi = \{0\}$. Let $v_1 \in V$ be not 0. Then $\varphi(v_1) \neq 0$, so let $c = \psi(v_1)/\varphi(v_1)$. Now let $w \in V$; since $V = \text{span}(v_1)$, $w = av_1$ for some $a \in F$; thus, $\psi(w) = a\psi(v_1) = ac\varphi(v_1) = c\varphi(av_1) = c\varphi(w)$, so $\psi = c\varphi$.

Solution. This is solvable for any dimension of V ; I'll give credit for a correct solution.

- (8) (5+10 points) For U a subspace of V :

- (a) Define the annihilator U^0 .

Solution. $U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$.

- (b) Let $i : U \rightarrow V$ be the inclusion mapping, $i(u) := u$. Prove that $\text{null } i' = U^0$.

Solution. We have $i' : V' \rightarrow U'$ defined by $i'(\psi)(u) = \psi(i(u))$ for $\psi \in V'$ and $u \in U$, but since $U \subseteq V$, this says $i'(\psi)(u) = \psi(u)$; that is $i'(\psi) = \psi|_U$. It follows that $i'(\psi) = 0$ iff $\psi(u) = 0$ for all $u \in U$; that is, iff $\psi \in U^0$. Thus, $\text{null } i' = U^0$.

- (9) (5 points) State the fundamental theorem of *algebra* (not “of linear algebra”).

Solution. *Theorem.* Let p be any monic polynomial with complex coefficients (that includes real coefficients) and let $n = \deg p$. Then there exist complex numbers $z_1, z_2, \dots, z_n$ such that $p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$.

- (10) (10 points) Let $T : F^3 \rightarrow F^3$ be defined by $T(x, y, z) = (y, z, x)$. Find an invariant subspace under T that is neither F^3 nor $\{0\}$, or prove there is none.

Solution. Let U be an invariant subspace that is not $\{0\}$. Then there exists a vector $(a, b, c) \neq (0, 0, 0)$ in U . Then $T(a, b, c) = (b, c, a) \in U$ and $T^2(a, b, c) = (c, a, b) \in U$. Then $(a + b + c)(1, 1, 1) \in U$. Then there are two cases.

Case 1: $(a + b + c) = 0$. This linear equation defines a proper subspace $U_0 := \{(x, y, z) \mid x + y + z = 0\}$ of V , and it is obviously invariant. Therefore, U_0 is a legitimate answer.

Case 2: $(a + b + c) \neq 0$. Then $U_1 := \text{span}\{(1, 1, 1)\}$ looks like an invariant subspace, and it obviously is, with $\dim U_1 = 1$. That is another legitimate answer.

- (11) (15 points) Let V be a complex vector space. Find the eigenvalues and eigenvectors of the operator $T \in \mathcal{L}(V \times V)$ defined by $T(u, w) = (w, u)$ for $(u, w) \in V \times V$.

Solution. The basic equation is $T(u, w) = \lambda(u, w)$, so $(w, u) = (\lambda u, \lambda w)$. Assume (u, w) is an eigenvector. Since $w = \lambda u$ and $u = \lambda w$, we conclude that $\lambda^2 w = w$ and $\lambda^2 u = u$ for any eigenvector. Because (u, w) is an eigenvector, u or w (or both) $\neq 0$; it follows that $\lambda^2 = 1$ so $\lambda = 1$ or -1 .

Suppose $\lambda = 1$; what does that say about the eigenvector? It says $u = w$. Therefore, there is an eigenspace $\{(u, u) : u \in V\}$ for eigenvalue 1.

Suppose $\lambda = -1$. This says $u = -w$, so there is an eigenspace $\{(u, -u) : u \in V\}$ for eigenvalue -1 .

(Extra thought (optional): Is $V \times V$ the direct sum of the eigenspaces?)