ADDITIONAL ERRATA AND COMMENTS ABOUT OXLEY, *MATROID THEORY*

http://www.math.binghamton.edu/zaslav/580.F01/my-errata.ps

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Page 35: Running head: The 1.5 should be 1.4.

Page 35: Problem 12: The correct hypothesis is that $C_j \not\subseteq \bigcup_{i \neq j} C_i$.

Page 44: Fig. 1.17 The line segment 6–8 should cross in front of 3–10, not the reverse.

Page 57: line 8: It appears that "by 1.7.4" should read "by the chain condition". (Since $\bigvee X \vee y \geq \bigvee X$ and equal height imply equality.)

Page 57: The standard symbol for the partition lattice is Π_n. $\mathcal{P}_n$ is rare, at best.

Page 58: proof of 1.7.8 There is no need for different cases. The second-case argument covers both cases.

Page 72: corank: This is an unfortunate choice of definition. (Not due to Oxley.) Not only is it common in matroid theory to define $\operatorname{corank}(S) = r(M) - r(S)$, but the latter definition is not confined to matroids but is also used in the more general theory of ranked posets (because of order duality). We will always use the definition: $\operatorname{corank}(S) = r(M) - r(S)$. We will call r^* the *dual rank* function.

Page 85: line -12*:* "vector space" (no plural).

Page 85: The link is much more than is stated in the book (Theorem 2.2.8 and Proposition 2.2.23), as we have seen in class. (Oxley chose not to go into greater detail about this, obviously because he couldn't do everything.)

Page 91: There is a minor confusion of terminology here. Strictly speaking, a particular embedding of a graph in the plane is called a *plane graph* but is not a graph (i.e., not an abstract graph). A *planar graph* is an abstract graph that *can be* embedded in the plane; whether it is or is not embedded at the moment is irrelevant. Every plane graph has a unique planar dual (as discussed in class) which is a plane graph, not a planar graph. A planar graph gets a dual graph by being embedded in the plane and then dualized; there can be more than one dual, from choosing different embeddings.

Page 91: Probably by an oversight, Oxley neglected to state the basic theorem:

Theorem 2.3.A. If G is a planar graph and G^* is a dual of G, then $M(G^*) = M^*(G)$.

Page 104: Terminology for contraction: Also called the contraction "of M by T" or "of M from T".

Page 125: line 14, end: Eliminate space before ; .

Page 125: def. of "block": This is not the usual definition in graph theory, although usage varies. The usual definition is: a 2-connected graph (this ignores loops). When we need to distinguish the two types, we might call Oxley's an *edge block* and the usual one a *vertex block*. (There is nothing wrong with Oxley's definition at all. One should just be aware of the other meaning.)

Page 138: line -6*:* The -1 should be -1.

Page 145: line 7: This is an unfortunate definition. The natural terminology, that will never confuse anyone and scarcely needs to be defined, is: *planar matroid* for a matroid of rank 3 and *planar-graphic matroid* for the matroid of a planar graph. These are the generally used terms as well. In the course we will always say "planar" and "planar-graphic" as defined here, not as in the book.

Page 149: line 15, etc.: Instead of "generalized cycle" I suggest the term *necklace*. The "parts" might then be called "beads".

Page 152: line 17: I think the $\theta(H_i)$ and $\theta(H_{-i})$ should not have θ's.

Page 164: We will not make a distinction between "projective geometry" and "projective space". I believe most writers, if pressed, would use the opposite convention to Oxley's ("geometry" for the axiomatic definition, "space" for the construction from coordinates), but most likely they wouldn't care. For consistency with most writings on projective geometry and matroids, we will call an abstract system (P, L, ι) as defined at the bottom of page 164 either a *projective geometry* or *projective space* (synonymously). The projective geometry $PG(n, F)$ over a field F will be called an *F-coordinatizable projective geometry (space)* or simply a *coordinatizable projective geometry (space)*. (These are also called, depending on the exact hypotheses, "pappian" and "desarguesian projective geometries" in the technical literature of projective geometry, for reasons that we need not go into.)

Page 164: Abstract projective space: The definition admits any simple matroid of rank at most 2, with the exception of the 2-point line $U_{2,2}$.

Page 165: Theorem 6.1.1 The statement must say "dimension > 2" and "integer $n > 2$". The conclusion is false, not only for $n = 2$, but also for $n = 1$. Any simple matroid of rank 2 consisting of $q+1$ points where $q < 2$ or q is not a prime power is a counterexample. (Oxley's errata have this correction.)

Page 170: Prop. 6.1.10: The proof actually shows that the Vamos matroid is not a submatroid (therefore not a minor) of any projective geometry.

Page 175: lines 23–24: By the above remark on 6.1.10, the Vamos matroid is such a matroid.

Page 185: Definition of vector representation: This is a good place to introduce a more fundamental definition. A *vector representation of M over F* is a mapping $f : E(M) \to F^m$ (for some $m \geq 0$) such that a set $S \subseteq E(M)$ is independent in M if and only if $f(S)$ is independent in F^m. We count multiplicity; thus if $f(x) = f(y)$, then $\{f(x), f(y)\}$ is dependent in F^m. (That is, $f(S)$ is a multiset of vectors.) The matrix representations previously defined are merely a particular way to present the mapping f; for some purposes a very valuable way, as we have seen.

A more general definition replaces F^m by any abstract vector space over F. This is perfectly valid but not especially necessary.

Page 185: *Equivalence of representations:* I believe the notion of equivalence in the book, which I shall call *matroid equivalence*, is not the best. A better one is that of projective equivalence. We call two representations of M over F *projectively equivalent* if one is obtained from the other by a combination of 6.3.1–6 (with no exception in dimension 2). (This definition is based on the fundamental paper of Brylawski and Lucas [1]. Welsh [3, p. 281] defines an equivalence that is projective equivalence without the field automorphisms.)

A *projective automorphism* of $\mathrm{PG}(n, F)$ is the projection of an invertible semilinear transformation T of F^{n+1}. That is, it is the effect of T on the simplification $\mathrm{PG}(n, F)$ of $M(F^{n+1})$. A *matroid automorphism* is any permutation that preserves the matroid structure (equivalently, it preserves collinearity). The fundamental theorem of projective geometry tells us, among other things, that every matroid automorphism of $\mathrm{PG}(d, F)$ is a projective automorphism, *provided that* $d \geq 2$. That is, for projective dimension 2 or more, matroid and projective automorphisms are the same. *If* $d = 1$*, this is not true.* A projective line $\mathrm{PG}(1, F)$, whose matroid is $U_{2,q+1}$ where $q = |F|$ (see Exercise 6.1.3), has as matroid automorphisms the full symmetric group of its points, but a projective automorphism is determined by its values on any 3 points (as I showed in class for the 3 particular points $[1, 0], [1, 1], [0, 1]$). There are more matroid automorphisms than projective automorphisms. (We can say that the projective structure of a projective line is not entirely determined by the matroid structure.)

Restating the two definitions of equivalence: *Oxley* calls two representations of M in a projective space $\mathrm{PG}(n, F)$ equivalent if one equals the other followed by a *matroid* automorphism of $\mathrm{PG}(n, F)$. They are *projectively* equivalent only if the matroid automorphism is a *projective* automorphism.

I will give some reasons for thinking that the correct automorphisms are the projective ones. (I might summarize as follows: Oxley thinks of $\mathrm{PG}(n, F)$ as a matroid, but I think of it as a coordinatized projective geometry, which therefore has more structure than just its matroid. I am following the tradition of projective geometry. It was not clear to me for a long time that the projective-geometry notion is superior, but I have finally concluded, after dealing with various representation problems, that it is, and that we are missing something important about representation of matroids if we ignore the projective structure on $\mathrm{PG}(n, F)$.)

There are several reasons for preferring projective to matroid equivalence. The simplest is that it eliminates special exceptions for (or exclusions of) rank 2 or dimension 2 in many results, e.g., Proposition 6.3.13 and (less obviously) Theorem 6.3.10. The more important reason is that it has good consistency properties.

To explain this, consider representing a matroid M of rank 2 (let's say, $U_{2,n}$ with $n > 3$) over a field F. (Assume F is large enough that a representation exists. That is, $|F| \geq n - 1$.) We could represent M in $\mathrm{PG}(1, F)$. Then, all representations of M are matroid equivalent. Now think of the $\mathrm{PG}(1, F)$ as a line in the plane $\mathrm{PG}(2, F)$. Suddenly, not all representations of M in that line are matroid equivalent. The reason is that not all matroid automorphisms of the line can be extended to matroid automorphisms of the plane. In fact, the automorphisms that can be extended are precisely the projective automorphisms of the line. This seems to me to be a serious difficulty. Oxley gets around it by requiring representations to be in projective spaces of rank equal to $r(M)$, but that is not such a good idea: it is artificial; worse, it is incompatible with some constructions, such as the lift construction (which we haven't studied).

(If you like, you can replace the $\mathrm{PG}(1, F)$ and $\mathrm{PG}(2, F)$ in the previous paragraph by F^2 and F^3. The same difficulty appears.)

Thus, if we want the property of unique representability over F to be independent of which dimension of projective or vector space we use for the representation, we have to prefer projective equivalence over matroid equivalence.

Page 186: line 6: "same size" means same dimension. (That is, the dimension of the vector space.)

Page 186: line −6: Properly speaking, what Oxley is defining is an *invertible* semilinear transformation.

Page 186: line −5: "there is" should be "there are".

Page 189: line 10: "there is" should be "there are".

Page 190 ff.: If this was supposed to be "D-sharp", then the symbol should be $\sharp$ (musical notation), not # (pound (weight) sign).

Page 193: Prop. 6.4.5: It might be desirable to elucidate this proposition by the following lemmas, whose proofs are obvious. If the rows of a matrix A are labelled $1, \ldots, r$ and the columns are labelled $1, \ldots, n$ (corresponding to the matroid points $e_1, \ldots, e_n$), then $A[I, J]$ denotes the minor of A consisting of the elements in rows $i \in I$ and columns $j \in J$.

Lemma 1. *Let A be an $r \times n$ matrix representing a matroid M of rank r. Let $S \subseteq [n]$ with $|S| = r$. Then S corresponds to a basis of M if and only if* $\det A\big[[r], S\big] \neq 0$.

Lemma 2. *Let $A = [\, I \mid D \,]$ in Lemma 1. Then S corresponds to a basis of M if and only if $D\big[[r] \setminus S, S \setminus [r]\big] \neq 0$.*

Page 195: line −4: In fact, Theorem 6.4.7 is not used in the proof of Proposition 6.4.8. Where it is needed is in the proof of Theorem 6.6.3.

Page 209: Lemma 6.6.2: The statement is incorrect. It is not sufficient that M be binary; it is necessary to assume that $[\, I_r \mid D_1 \,]$ itself is a GF(2)-representation matrix of M.

Furthermore, it should be made clear that the entries of D_1 are in $\{0, \pm 1\}$ when viewed in characteristic 0 (e.g., over $\mathbb{R}$) and the entries of D_2 will then be in $\{0, \pm 1\}$ when viewed in characteristic 0. (This is necessary because $\operatorname{char} F$ may not be 0.)

Another comment: it seems only necessary that $[\, I_r \mid D_1 \,]$ represent M over some field of characteristic not 2 and some field of characteristic 2.

Oxley has a correction for this lemma, but I don't think it captures everything.

Page 210: line 21: I would rather say that (i) $\Rightarrow$ (ii) due to the obvious fact that a totally unimodular matrix representing M over $\mathbb{R}$ (or $\mathbb{Q}$) also represents M over any field.

Page 212: line −7: "over F, <u>and also over GF(2) because, viewing D and $D^\sharp$ over $\mathbb{R}$, $D \equiv D^\sharp$ mod 2.</u>"

Page 213: para. 2: Oxley's Errata pages give a correction for this paragraph. The following is my commentary on the original paragraph, which supplements Oxley's correction.

The pivoting is done over F, not $\mathbb{R}$ since we don't know the matrix represents M over $\mathbb{R}$. We have to know from the proof of Lemma 6.6.2 that the entries in D_2 are still all in $\{0, \pm 1\}$ even when viewed in characteristic 0, because the only way a pivot can produce a different value of an entry is to produce a 2, which is ruled out by the fact that it would be $0 \mod 2$ but not in F, therefore contradicting the assumption that M is represented by $[\, I_r \mid D_1 \,]$ over both GF(2) and F. The conclusion is that the determinant of D' is in $\{0, \pm 1\}$ in characteristic 0. Thus, $[\, I_r \mid D_1 \,]$ is totally unimodular and represents M in characteristic 0.

Page 213: line 21: Just a thought: Perhaps, strictly speaking, these results are matroid *analogs*, not generalizations, of Kuratowski's theorem. Kuratowski's theorem does not follow from Tutte's results, except by use of Whitney's 2-isomorphism theorem, which is rather complicated.

Page 230: top equation: This is called the "modular law". Contrast with the semimodular law.

Page 314: Ex. 2(i): In the matrix, the A_4 should be I_4.

Page 314: Ex. 3(ii): "all non-zero chains <u>in N</u>".

Page 314: Chain-groups: It is a pity that the only careful and modern treatment of chain-groups is that in Tutte's textbook [2, Ch. VIII], and that this treatment avoids mentioning matroids. It would be very desirable to have a development of the main points in a matroid book, specifically, a section in this book. (Tutte's book is "modern" in that it has the notion of a primitive chain-group, that generalized chain-groups over integral domains and clarifies some of the basic ideas.)

Page 524: The operation of complementary dualization of $\mathcal{S} \subseteq 2^E$ is omitted from the list. Its definition, $\mathcal{S}' = \{E \setminus S : S \in \mathcal{S}\}$, is implicit on page 69, proof of Corollary 2.1.5.

Page 529: Index: modular short circuit axiom, p. *234*.

References

[1] T.H. Brylawski and D. Lucas, Uniquely representable combinatorial geometries. In: *Colloquio Internazionale sulle Teorie Combinatorie* (Proc., Rome, 1973), Vol. I, pp. 83–104. Atti dei Convegni Lincei, 17. Accademia Nazionale dei Lincei, Rome, 1976. MR 55 #113 (book). Zbl. 392.51007.

[2] W.T. Tutte, *Graph theory.* Encyc. Math. Appl., 21. Addison-Wesley, Reading, Mass., 1984. MR 87c:05001. Zbl. 554.05001. Reprinted by Cambridge University Press, Cambridge, 2001. MR 2001j:05002. Zbl. 964.05001.

[3] D.J.A. Welsh, *Matroid Theory.* Academic Press, London, 1976. MR 55 #148. Zbl. 343.05002.