MATROID THEORY

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Errata and Update on Conjectures, Problems, and References
Latest update: February 20, 2003

The table below contains more than just a list of typing errors. Some statements and proofs have been corrected, simplified, or clarified. Moreover, the current status has been given for all the unsolved problems or conjectures that appear in Chapter 14. For those changes that simply involve the insertion of extra words, the corrected text is given with the inserted words underlined. It is planned to update this table at regular intervals and, eventually, these changes should be incorporated into the next printing of the book. The reader is encouraged to send the author <oxley@math.lsu.edu> corrections that do not appear in the table below.

Page	**Line**	**Change**
6	2	"a union of disjoint trees is a *forest.*"
27	5	Replace with "Then $\mathcal{D}$ is the set of circuits of a paving matroid on E if and only if there is a positive integer k with $k \le \lvert E\rvert$ and a subset $\mathcal{D}'$ of $\mathcal{D}$ such that"
27	10	Replace "$\mathcal{D}''$" by "$\mathcal{D} - \mathcal{D}'$".
32	16	"*X is a circuit if and only if X is a minimal non-empty set with the property that, for all x in X, $x \in \mathrm{cl}(X - x)$.*"
66	1	The caption on Figure 1.33 should be "Worker y_j can do job x_i if x_iy_j is an edge."
66	7	Replace "$\{x_1, x_2, x_4, x_6\}$" by "$\{x_1, x_2, x_3, x_6\}$".
77	-6	Replace Exercise 7 (which is probably correct but seems to have no elementary proof) with the following: "For matroids M_1 and M_2, their direct sum $M_1 \oplus M_2$ was defined in Exercise 1.1.7 (p.16). Show that: (i) $(M_1 \oplus M_2)^* = M_1^* \oplus M_2^*$; and (ii) $M_1 \oplus M_1^*$ is self-dual but need not be identically self-dual."
78	-4	"A matroid on a fixed set is uniquely determined by a list of its cyclic flats and their ranks."
79	12	Omit **(b)**. It implies that $\mathcal{D} \subseteq b(\mathcal{A})$ but not that $\mathcal{D} = b(\mathcal{A})$.
79	17	Replace the hint by: "Show that (a) and (c) are equivalent, and then use the symmetry of (c)."
85	-13	Replace the sentence beginning "Clearly" and the sentence following that by: "Clearly $(W^{\perp})^{\perp} \supseteq W$. In fact, equality holds here but the proof of this fact will use the next result, which establishes the link between matroid duality and orthogonality in vector spaces. This link is stated in terms of row spaces of matrices."

Page	**Line**	**Change**
86	15	With the given definition of dot product, the third sentence of the corollary is wrong. For example, it fails when $F = \mathbb{C}$. Replace it by: "*Moreover,* $(W^{\perp})^{\perp} = W$."
86	17	Replace the first two sentences of this paragraph by: "By this corollary, if $[I_r\|D]$ is an $r \times n$ matrix over F where $1 \leq r \leq n-1$, then $\mathcal{R}[I_r\|D]$ and $\mathcal{R}[-D^T\|I_{n-r}]$ have the right dimensions to be complementary subspaces of $V(n,F)$. Moreover, if $F = \mathbb{R}$, then these subspaces are indeed complementary. However, in general, these row spaces may have non-trivial intersection."
87	-2	"partition $(\{e\}, X, Y)$ of E <u>where X and Y may be empty,</u> the element e is in exactly one"
93	6	Replace the first sentence of the proof by: "Let G' be a connected graph such that $M(G') \cong M(G)$. Let G_0 be a planar embedding of G', and G^* be the geometric dual of G_0."
93	10	Replace "$G_0 \cong G$" by "$M(G_0) \cong M(G)$".
93	-1	Replace the sentence beginning "Let" and the next two sentences by: "Since X is a cocircuit of $M(G^*)$ and $x \in X \cap C_x$, it follows, by Proposition 2.1.11, that $\|X \cap C_x\| \geq 2$."
111	-11	Insert the following at the end of the paragraph: "Moreover, G/T is well-defined since one can easily check that $(G/e)/f = (G/f)/e$ for all edges e and f."
118	-9	"the following lemma <u>where, in the lemma and its proof, the blocks of a partition may be empty</u>."
137	-3	This exercise can also be obtained from a result of Z. Tuza [On two intersecting set systems and k-continuous Boolean functions, *Discrete Appl. Math* **16**(1987), 183–185]. Moreover, Manoel Lemos and James Oxley [A sharp bound on the size of a connected matroid, *Trans. Amer. Math. Soc.* **353**(2001), 4039–4056] have sharpened the bound here to the best-possible bound $\|E(M)\| \leq \frac{1}{2}mn$.
140	13	Proposition 5.1.3 has a much shorter proof. It suffices to show that if a matrix A has all its entries in $\{0,1,-1\}$ and each column has at most one $+1$ and at most one -1, then A is totally unimodular. This can be proved as follows (see p. 222 of William J. Cook, William H. Cunningham, William R. Pulleyblank, and Alexander Schrijver, *Combinatorial Optimization*, Wiley, New York, 1998). Let A' be a $k \times k$ submatrix of A. We argue by induction on k that $\det A'$ is in $\{0,1,-1\}$. This is clearly true if $k=1$. Now suppose that $k = n \geq 2$ and that the assertion holds for $k < n$. If A' has a column with at most one non-zero entry, then, by expanding the determinant along this column, we get, using the induction assumption, that $\det A'$ is in $\{0,1,-1\}$. We may now assume that every column of A' contains exactly one $+1$ and exactly one -1. Then the rows of A' sum to zero. Hence $\det A' = 0$ and the result follows.

Page	Line	Change
148	2	"a graph can only be cleft at a cut-vertex unless one of the components produced by the cleaving has a single vertex."
155	-1	Insert the following at the end of the paragraph: "Elements e and f are *in parallel* in a matroid if $\{e, f\}$ is a circuit, and are *in series* if $\{e, f\}$ is a cocircuit."
157	12	Omit "without isolated vertices".
159	5	Replace this sentence by: "The next proposition follows by a straightforward induction argument."
161	-8	"the maximum number of chordal paths of length one that C can have."
165	16	Insert the following after "respectively.": "A finite projective space of dimension one consists of a single line containing k points where k is an arbitrary integer exceeding two."
165	22	"*Every finite projective space of dimension greater than two is isomorphic to $PG(n, q)$ for some integer n exceeding two and some prime power q.*"
195	-10	Replace "$1/t$" by "t^{-1}".
204	-2	J. F. Geelen, A. M. H. Gerards, and A. Kapoor have proved that the list becomes complete if one adds the matroid that is obtained from P_8 by relaxing its unique pair of disjoint circuit-hyperplanes. For more details, see the comment below for p.463 l.14.
208	17	After "$1 \leq i < j \leq p+1$", insert "and all the sets $\{x_1, x_2, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_{p+1}\}$ for $1 \leq k \leq p+1$."
208	-3	Omit the first row of the matrix after the partition, leaving intact the identity matrix I_p before the partition.
209	-1	"Suppose $[I_r \| D_1]$ is an F-representation of a binary matroid M where F has characteristic different from two."
210	12	"it must also be 0 over F; a contradiction."
212	5	"induced by the vertices of C'_d"
213	3	Replace this paragraph by: "Now suppose that $\det D' \neq 0$ when calculated over $\mathbb{R}$. Then, over F, we can pivot in D on a non-zero entry d'_{st} of D' to reduce column t of D' to a unit vector. By Lemma 6.6.2, this pivot produces another matrix with every entry in $\{0, 1, -1\}$. Moreover, it is straightforward to check that if this pivot is done over $\mathbb{R}$ instead of F, it produces the same matrix. By repeated pivots, we eventually obtain a matrix representing M over F in which every column of D' is a unit vector. Since exactly the same matrix is obtained when these pivots are done over $\mathbb{R}$ and $\|\det D'\|$ is unchanged by these operations, we conclude that, in the final matrix, $\det D'$ is still non-zero. But the form of this matrix now implies that $\det D' \in \{1, -1\}$, as required."
226	-2	Change the last entry in row 2 of this matrix from "0" to "1".
230	-10	Add at the end of this paragraph: "Recall that a finite projective space of rank two is isomorphic to $U_{2,n}$ for some $n \geq 3$."

Page	Line	Change
236	9	Replace "both by a path in X and" by "by".
243	-5	The proof given can be shortened by replacing from "We show next" onwards by: "But, from above, $r(S(M_1, M_2)) = r(M_1) + r(M_2)$. Since $\lvert B\rvert = r(M_1) + r(M_2)$, we deduce that B is indeed a basis of $S(M_1, M_2)$."
246	2	This proof can be shortened by replacing all of the first paragraph except the first two sentences by: "First, we observe that, as $\lvert E(M)\rvert \geq 2$ and M is connected, p is not a coloop of M. Thus p is not a coloop of $M/E(M_1)$ or of $M/E(M_2)$."
266	-9	"Evidently, $\underline{\text{if } r(M) = 0, \text{ then } T(M) = M, \text{ while if } r(M) > 0,}$ $\underline{\text{then}}$ $r(T(M)) = r(M) - 1$."
269	-17	Replace the first two lines of Exercise 7 by "Suppose M_1 is a matroid and M_2 is an elementary quotient $(M_1 +_{\mathcal{M}} e)/e$ of M_1. Prove that if $M_1 +_{\mathcal{M}} e$ is graphic, then there are"
282	16	"let $E_i = E(G[V' \cup V(H_i)]) - E(G[V'])$."
283	11	The last sentence of this proof should be: "Hence u and v are connected in G_2; a contradiction."
283	-14	"We remark that $\underline{\text{both Cunningham (1981) and}}$ Inukai and Weinberg (1981)"
302	-9	"then M/f is minimally 3-connected."
309	-13	"there is a hyperplane containing $[E - (H_1 \cup H_2)] \cup [H_1 \cap H_2]$."
325	-15	Replace "$\{1, 2, \ldots, m + k\}$" by "$\{1, 2, \ldots, m + k + 1\}$".
339	17	"If $\underline{r \geq 3, \text{ then}}$ M is not graphic."
339	-13	Replace "five" by "six".
376	-11	Omit "and suppose that $e \in E(M)$."
380	6	Replace "$\mathcal{C}_f$" by "$\mathcal{C}(f)$".
382	8	The summation here should be over "$v \in V(C)$".
388	-19	"*Let $\mathcal{A}$ be a family $(A_j : j \in J)$ of* $\underline{\textit{nonempty}}$ *subsets*"
389	-6	It should be "$f(X) = r(X) + d$".
394	4	This lemma holds for all functions $\sigma : J \to S$. When σ is not a surjection, all the elements of $S - \sigma(J)$ are loops of $\sigma(M)$.
406	10	Replace with "an intricate algebraic argument. R. Rado [Abstract linear dependence, *Colloq. Math.* **14**(1966), 257–264] has pointed out that this argument generalizes to matroids in the natural way. In particular, a vector that is a linear combination of some set of vectors corresponds to a matroid element that is in the closure of some set."
424	-3	Replace with "$\mathrm{cl}_M(X) = \mathrm{cl}_1([X \cup \mathrm{cl}_2(X \cap E_2)] \cap E_1) \cup \mathrm{cl}_2([X \cup \mathrm{cl}_1(X \cap E_1)] \cap E_2)$."
424	-2	Replace with "$r(X) = r([X \cup \mathrm{cl}_2(X \cap E_2)] \cap E_1) + r([X \cup \mathrm{cl}_1(X \cap E_1)] \cap E_2) - r(T \cap [\mathrm{cl}_1(X \cap E_1) \cup \mathrm{cl}_2(X \cap E_2)])$."
432	3	In Lemma 13.1.7(ii): "*its determinant is* $\alpha \det(Z - \alpha^{-1} \underline{y}\underline{x}^T)$."
435	11	Replace "$\left[\begin{smallmatrix} -1 & d \\ 1 & 1 \end{smallmatrix}\right]$" with "$\left[\begin{smallmatrix} -1 & 1 \\ d & 1 \end{smallmatrix}\right]$".
435	-16	Replace the first line of Exercise 1 with "Prove that a real matrix $[I_r \vert D]$ is totally unimodular if and"

Page	Line	Change
463	-20	The conjecture has been proved when $q = 4$ by J. F. Geelen, A. M. H. Gerards, and A. Kapoor [The excluded minors for $GF(4)$-representable matroids, *J. Combin. Theory Ser. B* **79**(2000), 247–299]. The complete set of excluded minors for $GF(4)$-representability is $\{U_{2,6}, U_{4,6}, F_7^-, (F_7^-)^*, P_6, P_8, P_8''\}$. The last matroid is obtained from P_8 by relaxing the unique pair of disjoint circuit-hyperplanes of P_8. For $q \geq 5$, the conjecture remains open. James Oxley, Charles Semple, and Dirk Vertigan [Generalized $\Delta - Y$ exchange and k-regular matroids, *J. Combin. Theory Ser. B* **79**(2000), 1–65] have proved that there are at least 2^{q-4} excluded minors for $GF(q)$-representability.
463	-12	This conjecture has been settled by James Oxley, Dirk Vertigan, and Geoff Whittle [On inequivalent representations of matroids over finite fields, *J. Combin. Theory Ser. B* **67** (1996), 325–343]. The conjecture holds for $q = 5$. A $GF(5)$-representable matroid has at most six inequivalent $GF(5)$-representations with equality being attained, for example, by $U_{3,5}$. However, the conjecture fails for all larger values of q. It is still open whether Kahn's conjecture would be true if "3-connected" were replaced by "4-connected".
464	7	This problem has been answered affirmatively by Geoff Whittle in the same paper in which he settled Conjecture 14.1.11 (see the comment below for p.464 l.-2).
464	15	Let $\mathcal{M}$ be the class of matroids that is the union of the classes of binary and ternary matroids. Bogdan Oporowski, James Oxley, and Geoff Whittle [On the excluded minors for the matroids that are either binary or ternary, Preprint 1997-7, http://www.math.lsu.edu/˜preprint/] have conjectured that the set of excluded minors for $\mathcal{M}$ consists of $U_{2,4} \oplus F_7$, $U_{2,4} \oplus F_7^*$, $U_{2,4} \oplus_2 F_7$, $U_{2,4} \oplus_2 F_7^*$, $U_{2,5}$, $U_{3,5}$, and the unique matroids that are obtained by relaxing a circuit-hyperplane in each of $AG(3,2)$ and T_{12}. The matroid T_{12} is a 12-element rank-6 self-dual binary matroid that is represented over $GF(2)$ by the 15×12 matrix whose rows are indexed by the edges of the Petersen graph P_{10} and whose columns are the incidence vectors of the 5-cycles of P_{10} (see S. R. Kingan [A generalization of a graph result of D. W. Hall, *Discrete Math.* **173** (1997), 129–135]). Oporowski, Oxley, and Whittle showed that the conjectured list contains all excluded minors for $\mathcal{M}$ with at most 23 elements and that, corresponding to every remaining excluded minor, there is a binary matroid whose ground set is the disjoint union of two circuit-hyperplanes such that relaxing both circuit-hyperplanes produces a ternary matroid. The excluded minor is obtained by relaxing exactly one of the distinguished pair of circuit-hyperplanes.

Page	Line	Change
464	-2	This conjecture has been settled by Geoff Whittle [On matroids representable over $GF(3)$ and other fields, *Trans. Amer. Math. Soc.* **349**(1997), 579–603]. He proved that the following are equivalent for a matroid M. (i) M is representable over $GF(p)$ for all odd primes p. (ii) M is representable over $GF(3)$ and $\mathbb{Q}$. (iii) M is representable over $GF(3)$ and $\mathbb{R}$. (iv) M is representable over $GF(3)$ and $GF(5)$. (v) M is representable over $GF(3)$ and $GF(q)$ where q is an arbitrary but fixed odd prime power that is congruent to 2 mod 3. (vi) M can be represented over $\mathbb{Q}$ by a matrix all of whose non-zero subdeterminants are in $\{\pm 2^i : i \in \mathbb{Z}\}$.
469	19	Joseph P. S. Kung [Critical exponents, colines, and projective geometries, *Combin. Probab. Comput.* **9**(2000), 355–362] has proved that this conjecture fails spectacularly: for a given prime power q, it holds for only finitely many ranks r.
469	-6	This conjecture fails trivially for $U_{1,1}$ but otherwise has been proved by Winfried Hochstättler and Bill Jackson [Large circuits in binary matroids of large cogirth: I, II, *J. Combin. Theory Ser. B* **74** (1998), 35–52, 53–63]. The conjecture follows from their result that if M is a simple binary matroid that is not isomorphic to F_7^* and has no F_7-minor, then M has a circuit of size $r(M)+1$ provided every cocircuit of M has size at least $\max\{3, \frac{1}{2}(r(M)+1)\}$. Observe that this result does not require M to be connected. Examples of Hochstättler and Jackson show that, contrary to the speculation on p.470, l.3, the conjecture does not hold for all binary matroids with no F_7^*-minor. Indeed, it fails for the matroid that is the parallel connection of three copies of F_7 across a common basepoint.

Page	Line	Change
471	-9	This problem has been solved by Dirk Vertigan [On the intertwining conjecture for matroids, in preparation]. He showed that, provided M_1 and M_2 satisfy some relatively weak conditions, the set of minor-minimal matroids having both M_1- and M_2-minors is infinite. A positive result has been proved by J. F. Geelen [On matroids without a non-Fano minor, *Discrete Math.* **203**(1999), 279–285]. He considered the class $\mathcal{N}$ of matroids that contain neither the non-Fano matroid nor its dual as a minor and showed that, for every connected matroid M, there are only finitely many minor-minimal members of $\mathcal{N}$ that have both M- and $U_{2,4}$-minors.
472	10	The result here that is credited to B. Jackson was first proved by W. Mader [Kreuzungfreie a, b-Wege in endlichen Graphen, *Abh. Math. Sem. Univ. Hamburg* **42**(1974), 187–204]. Mader showed that every k-connected simple graph G with minimum degree at least $k+2$ has a cycle C such that $G\backslash C$ is k-connected. In the special case $k = 2$, Jackson independently strengthened Mader's result by proving that if e is an edge of a simple 2-connected graph G with minimum degree k and $k \geq 4$, then G has a cycle C of length at least $k-1$ such that $e \notin C$ and $G\backslash C$ is 2-connected.
472	14	A negative answer to this problem is given in Manoel Lemos and James Oxley [On removable circuits in graphs and matroids, *J. Graph Theory* **30**(1999), 51–66]. An example of Lemos is given there of a simple connected cographic matroid M in which every cocircuit has at least four elements but there is no circuit C such that $M\backslash C$ is connected. Some positive results have been obtained in the absence of the requirement that the matroid be simple. L. A. Goddyn, J. van den Heuvel, and S. McGuinness [Removable circuits in multigraphs, *J. Combin. Theory Ser. B* **71** (1997), 130–143] verified a conjecture of Jackson (1980) by proving that if G is a 2-connected graph with minimum degree at least four and no minor isomorphic to the Petersen graph, then G has a cycle C for which $G\backslash C$ is 2-connected. Moreover, Luis A. Goddyn and Bill Jackson [Removable circuits in binary matroids, *Combin. Probab. Comput.* **8**(1999), 539–545] proved that if M is a connected binary matroid that does not have both F_7 and F_7^* as minors, and every cocircuit of M has at least five elements, then M has a circuit C such that $M\backslash C$ is connected and $r(M\backslash C) = r(M)$.
472	-13	Haidong Wu [Contractible elements in graphs and matroids, *Combin. Probab. Comput.*, to appear] has answered this question affirmatively under the extra assumption that M is regular.

Page	Line	Change
474	6	Marcel Wild [The asymptotic number of inequivalent binary codes and nonisomorphic binary matroids, *Finite Fields Appl.* **6** (2000), 192–202] has asserted that $a(n) \sim \frac{1}{n!}\sum_{k=0}^{n}\left[{n \atop k}\right]_2$. Moreover, if $\beta(n) = 2^{n^2/4 - n\log_2 n + n\log_2 e - (1/2) log_2 n}$ for all positive integers n, then there are constants d_1 and d_2 such that $a(2n+1) \sim d_1\beta(2n+1)$ and $a(2n) \sim d_2\beta(2n)$. Rounded to 6 decimal places, $d_1 = 2.940982$ and $d_2 = 2.940990$. Wild sent a correction to the argument in this paper to *Mathematical Reviews* and this appears in the review of the paper, **MR2001i:94077**. Robert F. Lax [On the character of S_n acting on subspaces of $\mathbb{F}_q^n$, in preparation] has found a different error in Wild's argument and it is not clear how this should be corrected. Nevertheless, it is believed that Wild's assertion is correct.
479	-20	The final reference for Akkari, S. (1991) is: "Akkari, S. (1992), A minimal 3-connectedness result for matroids, *Discrete Math.* **103**, 221–232. [10.2]"
483	5	The final reference for Coullard, C. R. and Oxley, J. G. (1992) is: "Coullard, C. R. and Oxley, J. G. (1992), Extensions of Tutte's wheels-and-whirls theorem, *J. Combin. Theory Ser. B* **56**, 130–140. [11.3]"
489	9	The final reference for Kung, J. P. S. (1991) is: "Kung, J. P. S. (1993), Extremal matroid theory. In *Graph structure theory* (eds. N. Robertson and P. Seymour), Contemp. Math. **147**, pp. 21–61, Amer. Math. Soc., Providence. [14.8]"
489	-12	The final reference for Lemos, M. (1990) is: "Lemos, M. (1994), Matroids having the same connectivity function, *Discrete Math.* **131**, 153–161. [8.1]"
495	3	An update on Robertson, N. and Seymour, P. D. (1990) is: "Robertson, N. and Seymour, P. D. (1990), Graph minors XX. Wagner's conjecture, *J. Combin. Theory Ser. B*, to appear."
496	21	The final reference for Seymour, P.D. (1992) is: "Seymour, P. D. (1995), Matroid minors. In *Handbook of combinatorics* (eds. R. Graham, M. Grötschel, L. Lovász), pp. 527–550, Elsevier, Amsterdam; M.I.T. Press, Cambridge. [11.1, 11.2, 11.3]"
496	-6	The page numbers for Truemper, K. (1982a) are "112–139."
497	16	The final reference for Truemper, K. (1992a) is: "Truemper, K. (1992a), A decomposition theory for matroids. VI. Almost regular matroids, *J. Combin. Theory Ser. B* **55**, 253–301. [13.2]"
497	18	The final reference for Truemper, K. (1992b) is: "Truemper, K. (1992b), A decomposition theory for matroids. VII. Analysis of minimal violation matrices, *J. Combin. Theory Ser. B* **55**, 302–335. [13.2]"

Page	Line	Change
504	-3	The correct statement here is: "A connected $GF(4)$-representable matroid is uniquely $GF(4)$-representable if and only if it has no 2-separation $\{X, Y\}$ such that $X \supseteq X'$ and $Y \supseteq Y'$ where $\{X', Y'\}$ is the unique 2-separation of R_6 (10.1.11)."
512	10	"The matroid J is self-dual but is not identically self-dual."
512	-6	The entry in the bottom right-hand corner of the matrix representing P_8 over $GF(3)$ should be "0" instead of "1".
514	-1	Add to the list of properties: • $M^*(K_{3,3})$ is the complement of $U_{2,3} \oplus U_{2,3}$ in $PG(3,2)$.
515	4	Add "Equivalently, $GF(q)$-representable if and only if $q \not\equiv 2 \pmod 3$."
516	-5	"The Pappus matroid is F-representable if and only if $\|F\| = 4$ or $\|F\| \geq 7$."
517	-11	"$M(K_5)$ is an excluded minor for the class of cographic matroids (6.6.5, 13.1.1) and $M^*(K_5)$ is an excluded minor for the class of graphic matroids (6.6.5)."
519	-1	Add the following to the list of properties of $S(5,6,12)$: • Connectivity is 5. • Has no spanning circuits. • Every contraction of two elements and deletion of two elements is isomorphic to P_8.
526	18	In column 2, the entry should be "cosimple matroid, 347".
528	18	In column 1, the entry should be "Higgs lift, 28".

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